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Large Deviations Estimates of Escape Time for Lagrangian Systems

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Abstract— **This paper is concerned with analysis of the asymptotic behavior of a Lagranian system with small noise effects. The domain of the system operation is supposed to be within the domain of attraction of an asymptotically stable point of the unperturbed system. If noise is weak, escape from the reference domain is a rare event associated with large deviations in the system. This paper uses an extension of large deviations theory to the degenerate systems to develop the escape time asymptotics for a weakly perturbed Lagrangian system. Estimation of the statistical quantities is reduced to minimization of an associated action functional. It is shown that, in the case of the Lagrangian system, the solution of the associated variational problem can be found in a closed form, as a function of the system and noise parameters. As an example, motion of a 2*n*-dimensional linear system in an ellipsoidal domain is studied. Application of the theory to the nonlinear systems is illustrated by estimation of the lifetime of the Henon-Heiles system.**

I. INTRODUCTION

Stationary statistics of the Lagrangian dynamics can be found as a stationary solution of a relevant Kolmogorov equation [1]. However, calculation of the statistical quantities, associated with escape processes, encounters serious technical difficulties. The present paper focuses attention on estimation of escape time for a weakly perturbed Lagrangian system. In the problem of interest the unperturbed system is asymptotically stable, the noise effects are small and the escape time is large.

Consider the unperturbed model

$$\dot{X} = f(X), \; X \in R^m \tag{1}$$

The point $X = 0$ is presumed to be asymptotically stable fixed point of system (1) with the domain of attraction G_0, an admissible domain of motion $G \subseteq G_0$ is an open bounded set in R^m with smooth boundary Γ.

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Consider random perturbation of system (1)

$$\dot{X} = f(X) + \varepsilon\Delta\,\dot{w}\,(t), \; X(0) = 0 \tag{2}$$

where $0 < \varepsilon << 1$ is a small parameter, $w(t)$ is standard Wiener process R^r, Δ is an $m\times r$ - matrix.

Whatever small the noise perturbation may be, it induces deviations from the unperturbed equilibrium $X = 0$ and escape of the process (2) from the reference domain with a non-zero probability.

A performance criterion of interest is the mean time $\mathbf{E}T^\varepsilon$ where $T^\varepsilon = \inf\,(t : X(t) \notin G)$ is the first moment the process $X(t)$ escapes the reference domain G. The direct calculation of $\mathbf{E}T^\varepsilon$ in the small noise limit requires consideration of a singular Dirichlet problem for an associated Fokker-Plank equation. Both analytic and computational approaches to the solution of a singular PDE are prohibitively difficult, see [2] and references therein. The appropriate mathematical basis to study the long-term behavior of system (2) is large deviations theory. We employ the main issues of the theory in the form presented by Kushner [3] and Freidlin and Wentzell [4].

Large deviations theory provides an alternative approach to the analysis of the weakly perturbed dynamics, opposed to the PDE solution; see [5] for a review and discussion of recent advances in theory and applications. Essentially, the estimates of $\mathbf{E}T^\varepsilon$ and related quantities are obtained as the solution of a minimization or variational deterministic problem. Large deviations principle provides the cost (action) functional that must be minimized by the "most likely" exit path. The Hamilton-Jacobi equation associated with minimization of the action functional was derived. Formally, the mean escape time and escape probability can be found from this equation.

Note that the diffusion model is only an approximation of more intricate phenomena. Freidlin and Wentzell [4] have considered large deviations principle for Markov processes; Kushner [4] has introduced the cost functional for systems excited by degenerate fast noise and examined Gaussian excitations. Gulinskii and Liptser [6], Liptser, Spokoiny and Veretennikov [7] have interpreted large deviations principle for the diffusion model as limiting for systems with wide-band ergodic noise.

A similar approach is useful in escape control. Large deviations principle has been applied to minimization of escape probability [8] and risk-sensitive escape control [9], [10]. A combination of the large deviations and stochastic averaging methods was used in the problem of controlling large deviations for a quasi-Hamiltonian system in the plane [11], [12].

Despite the well-developed theory, explicit solutions for multidimensional systems are few in numbers. The asymptotic estimates of escape time have been found for linear systems [4], [13] and nonlinear non-degenerate diffusion systems with the drift coefficient $f(x) = -\nabla U(x) + l(x)$ provided $(\nabla U(x), l(x)) = 0$ [4]. The function $U(x)$ is called quasipotential of the system.

The Lagrangian or Hamiltonian-type systems do not allow separation of the quasipotential part. Wu [14] has established large deviations principle for stochastic Hamiltonian-type systems but the closed-form estimates of escape time and probability have not been derived.

The objective of this paper is to investigate the escape processes in the multidimensional systems governed by the Lagrangian equations. For the sake of brevity, we consider small white noise model as an approximation. We derive the action functional for the stochastic Lagrangian system and construct the Hamilton-Jacobi equation for the associated variational problem. It is shown that, under some assumptions, this equation can be resolved analytically. The solution is found as a function of the system and noise parameters.

The paper is organized as follows. Section 2 contains some background results concerning large deviation theory. We make use of Kushner's definition of the action functional for a degenerate system and transform the mean escape time problem into a deterministic variational problem. The Hamilton-Jacobi equation is derived for the variational problem associated with calculation of the mean escape time for a degenerate diffusion.

In Section 3 we discuss the basic idea and derive the Hamilton-Jacobi equation associated with escape problem for the Lagrangian systems. We demonstrate that the properties of the solution are closely tied with the properties of the Hamiltonian function of the conservative part of the system. The systems allowing the closed-form solution are examined.

Section 4 considers the illustrating examples. We estimate the meant time until escape of a multidimensional linear system from an n-dimensional ellipsoid. The boundary of the admissible domain is defined by the constraints to the system coordinates but there are no restrictions to the system velocities. The Henon-Heiles model illustrates an application of the theory to nonlinear systems. The mean time until escape through the potential barrier is estimated.

II. Basic Methodology

Consider systems (1) and (2) and assume that

- (i) function $f(X)$ in the right-hand part of systems (1) and (2) is continuous and satisfies the Lipschitz condition in $G\times\Gamma$;

- (ii) system (1) has a unique asymptotically stable fixed point $X = 0$ in G, and all trajectories originating in the domain $G\times\Gamma$ tend to $X = 0$.

- (iii) the right hand sides of system (2) can be presented in the form

$$f(X) = \begin{vmatrix} f_1(X) \\ f_2(X) \end{vmatrix}, \quad \Delta = \begin{vmatrix} 0 & 0 \\ 0 & \Delta_2 \end{vmatrix} \tag{3}$$

the partition $X = (X_1, X_2)$ is of the same dimension as $f = (f_1, f_2)$, dimensionalities of the vectors f_2, w and the matrix Δ_2 are compatible.

Following [3], we introduce a stochastic counterpart of the action functional. At the first stage, we define an asymptotic cumulant function

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln\{\mathbf{E} \exp \int_0^{T/\varepsilon^2} [\alpha'(f(X) + \Delta\dot{w}(t))]dt\ \} = \int_0^T F(X,\alpha)\, ds \tag{4}$$

where the partition $\alpha = (\alpha_1, \alpha_2)$ is of the same dimension as $f = (f_1, f_2)$. Here and below the prime denotes transpose of a vector or a matrix.

The function F takes the form

$$F(X,\alpha) = \alpha_1' f_1(X) + \alpha_2' f_2(X) + \alpha_2' \Sigma_{22}\alpha_2/2 \tag{5}$$

The function F is to be assumed absolutely continuous, the matrix $\Sigma_{22} = \Delta_2'\Delta_2$ is positive definite.

Introduce the dual Cramer transformation (also called the Legendre transformation) C

$$C(X, \beta) = \sup_{\alpha} [\beta'\alpha - F(X, \alpha)]$$

where $\beta = (\beta_1, \beta_2)$, and partition β_1, β_2 have the same dimension as f_1, f_2. This yields

$$C(X,\beta) = (\beta_2 - f_2(x))'\Sigma_{22}^{-1}(\beta_2 - f_2(x)), \text{ if } \beta_1 = f_1(x) \tag{6}$$

$C(X,\beta) = \infty$ otherwise.

Introduce the action functional

$$S(T,\varphi)=\frac{1}{2}\int_0^T C(\varphi,\dot{\varphi})\,\mathrm{d}t\,,\ \ \dot{\varphi}_1=f_1(\varphi) \tag{7}$$

where $C(\varphi,\dot{\varphi})$ is defined by formula (6), $\varphi(t)\in C_{[0,T]}(R^m)$. If $\varphi(t)$ is not absolutely continuous, then we take $S(T,\varphi)=\infty$.

Let $\varphi(t)$ be an extremal of the functional (7) leading from an origin 0 to a point X, that is

$$S(X)=\inf\{S(T,\varphi)\colon \varphi(0)=0,\ \varphi(T)=X\} \tag{8}$$

In a deterministic system, an extremal depicts a trajectory, leading from an origin to a given point X. In a weakly perturbed stochastic system this extremal approximates, with probability close to 1, the trajectory of exit from the reference domain [3], [4].

The mean time needed to reach the boundary Γ of the domain G from an initial point $X = 0$ satisfies the estimate [3], [4]

$$\lim_{\varepsilon\to 0}\ \varepsilon^2\ln(\mathbf{E}T^{\varepsilon})=\inf_{X\in\Gamma}\ S(X)=S_0 \tag{9}$$

Estimation of escape time is thus reduced to minimization of criterion (7) for system (6). The variational problem (6) – (8) can be resolved in a standard way.

The Hamilton–Jacobi equation associated with the variational problem (6) – (8) takes the form

$$(f_1(X),\frac{\partial S}{\partial X_1})+(f_2(X),\frac{\partial S}{\partial X_2})+\frac{1}{2}(\Delta_2\frac{\partial S}{\partial X_2},\Delta_2\frac{\partial S}{\partial X_2})=0$$

$$S(0)=0 \tag{10}$$

where the brackets (a, b) denote scalar product.

We assume that

- (iv) equation (10) has a unique continuous and continuously differentiable solution $S(X) > 0$.

It follows from the previous consideration that, under assumptions (i) – (iv), estimate (9) holds for the solution $S(x)$ of Eq. (10).

*Remark 1***.** Formally, exit from an arbitrary initial point $X(0)=x\in G$ should be examined. However, it is reasonable to consider exit from a small neighborhood of the origin. If assumption (ii) holds, a trajectory of system (3) starting at $X(0) = x$ is trapped into a small neighborhood of the origin $X = 0$, and motion evolves within this locality until the burst-like exit from the admissible domain. The time of attraction to the origin, as well as the time of exit along the exit paths are of $O(1)$ and negligibly small compared with the period of motion near the asymptotically stable the origin. The last-mentioned time interval is estimated as the time until escape.

III. Escape time estimates for Lagrangian systems

Write Eq. (10) for a Lagrangian system. Here and below we make use of the notations and definitions accepted in classic mechanics.

The equation of motion is

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial\dot{q}}-\frac{\partial L}{\partial q}+B\dot{q}=\varepsilon\sigma\dot{w}(t) \tag{11}$$

where $L(q,\dot{q})$ is the Lagrangian of the system, $w(t)\in R^r$ is standard Wiener process, σ is such that the matrix $A=\sigma'\sigma$ is symmetric and positive definite. The parameter $\varepsilon << 1$ is the small parameter of the system.

The Lagrangian $L(q,\dot{q})$ is defined as

$$L(q,\dot{q})=T(q,\dot{q})-\Pi(q) \tag{12}$$

where $T(q,\dot{q})=(\dot{q},M(q)\dot{q})/2$ and $\Pi(q)$ are kinetic and potential energy of the system, respectively. The matrices $M(q)$ and B are symmetric positive definite square matrix. An admissible region G is an open domain in R^{2n} with boundary Γ. The point $q = 0$ is assumed to be a unique minimum of the function $\Pi(q)$ in G such that $\Pi(0) = 0$. Under these assumptions, the point $q = 0$ is the asymptotically stable fixed point of the unperturbed system $(\sigma = 0)$. The reference domain G is assumed to be within the domain of attraction of the stable equilibrium.

Reduce system (11) to the form (3). Using the transformation

$$p=\partial L(q,\dot{q})/\partial\dot{q} \tag{13}$$

we define the impulse p. Construct the function

$$H(q,p)=(\dot{q},p)-L(q,\dot{q}),\ \dot{q}(q,p)=M^{-1}(q)p \tag{14}$$

It follows from formulas (12) - (14) that

$$H(q,p)=T(q,\dot{q})+\Pi(q)=\widetilde{T}(q,p)+\Pi(q) \tag{15}$$

where $\widetilde{T}(q,p)=T(q,M^{-1}(q)p)$. Using formulas (13) - (15), we transform Eq. (11) into the system

$$\dot{q}=\frac{\partial H}{\partial p}$$
$$\dot{p}=-\frac{\partial H}{\partial q}-B\frac{\partial H}{\partial p}+\varepsilon\sigma\dot{w}(t) \tag{16}$$

Let the assumptions of Section 2 hold for system (16). Making use of the notations

$$X_1 = q,\ \ X_2 = p, f_1(X) = \frac{\partial H}{\partial p},\ f_2(X) = -\frac{\partial H}{\partial q} - B\frac{\partial H}{\partial p},\ \Delta_2 = \sigma$$

we transform Eq.(10) into the Hamilton-Jacobi equation associated with system (16)

$$\{H,S\} - (B\frac{\partial H}{\partial p},\frac{\partial S}{\partial p}) + \frac{1}{2}(\sigma\frac{\partial S}{\partial p},\sigma\frac{\partial S}{\partial p}) = 0$$

$$S(0,0) = 0 \tag{17}$$

where the Poisson bracket $\{H, S\}$ is

$$\{H, S\} = (\frac{\partial H}{\partial p},\frac{\partial S}{\partial q}) - (\frac{\partial H}{\partial q},\frac{\partial S}{\partial p})$$

It is easy to prove that Eq. (17) is satisfied if

$$\frac{\partial S}{\partial p} = K\frac{\partial H}{\partial p},\ \frac{\partial S}{\partial q} = K\frac{\partial H}{\partial q}$$

$$K = 2A^{-1}B,\ \ A = \sigma'\sigma \tag{18}$$

Since potential energy Π is independent of p, the first equation of systems (18) can be rewritten in the form

$$\frac{\partial S}{\partial p} = K\frac{\partial \widetilde{T}}{\partial p} = KM^{-1}(q)p$$

that is

$$S(q, p) = (p, KM^{-1}(q)p)/2 + Q(q) \tag{19}$$

From Eqs. (18) and (19) we find

$$\frac{\partial Q}{\partial q} = K\frac{\partial \Pi}{\partial q},\ Q(0) = 0 \tag{20}$$

Formulas (19) and (20) define the solution of Eq. (17) and the associated estimate (9). Consider some special cases allowing the closed-form solution.

1. The matrix $K = \kappa I_n$, where I_n is the identity matrix of the *n-th* order, κ is a scalar. We obtain from Eqs. (18)

$$S(q,p) = \kappa H(q,p) \tag{21}$$

2. The matrix $K = \mathrm{diag}\{k_1,\ldots,k_n\} = I_n k$, where the vector $k = (k_1,\ldots, k_n)$. Let potential energy $\Pi(q)$ be in the form

$$\Pi(q) = \sum_{i=1}^{n} f_i(q_i),\ f_i(0) = 0 \tag{22}$$

This yields

$$Q(q) = \sum_{i=1}^{n} k_i f_i(q_i) = (k, f(q)) \tag{23}$$

where the vector $f = (f_1,\ldots, f_n)$. Formulas (19), (20) and (23) yield

$$S(q,p) = (p,KM^{-1}p)/2 + (k, f(q)) \tag{24}$$

3. In a number of applications, the boundary of the admissible domain is identified through the limiting values of the variable $q \in G_q$ where G_q: $\{q \in G_q\}$ is a bounded open set in R^n with boundary Γ_q. There are no particular restrictions to the impulse p. However, assumptions of Section II imply some constraints on the impulse vector p. Then, it is implicitly supposed that an unperturbed orbit (q, p) starting within the domain $G_q \times G_p$ does not leave the sub-domain G_q.

From our consideration it follows that the admissible cylindrical domain G and its boundary Γ can be defined as

$$G\!: G_q \times G_p,\quad \Gamma\!: \Gamma_q \times G_p \tag{25}$$

where G_p: $\{p \in G_p\}$ is an open set in R^n. The domain G satisfies conditions of Section II and the additional assumption

− (ii –*a*) all trajectories originating in $G \times \Gamma$ tend to the stable point *not leaving the sub-domain* G_q.

In general, the domains G_p and G cannot be found in an explicit form. However, the low bound (9) can be calculated. Since the first term of function (19) is a positive definite quadratic form, and the limiting values of p are not fixed, the lowest value of function (19) is achieved if $p = 0$ for any function $Q(q)$. This yields

$$\inf_{\Gamma} S(q,p) = \inf_{\Gamma_q \times G_p} S(q,p) = \inf_{\Gamma_q} S(q,0) = \inf_{\Gamma_q} Q(q)$$

Substitution of the last equality into formula (9) gives

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\mathbf{E}T^{\varepsilon}) = \inf_{\Gamma} S(q,p) = \inf_{\Gamma_q} Q(q) = S_0 \tag{26}$$

Estimate (22) implies that kinetic energy of the system is unsubstantial in the leading order term of the logarithmic asymptotics.

IV. Applications and Examples

A. Motion of a linear system in an ellipsoidal domain

There arise a number of problems where a system should be kept in a given domain G until some particular job is finished. For example, in the problem of pointing of a telescope on a satellite, the domain G and the duration of the process are determined by the object to be photographed and the time required. The problem of controlled pointing of a telescope on a point moving within a circular or rectangular domain in the plane has been discussed in [15] and [16], respectively.

Now we estimate the mean time until escape from an n-dimensional ellipsoid for a MDF linear system.

The equations of motion are written in the form

$$M\ddot{q} + B\dot{q} + Cq = \varepsilon\sigma\,\dot{w}(t) \tag{27}$$

where $q \in R^n$, M, B, C are symmetric positive definite $n\times n$ matrices, $w(t)$ is r-dimensional standard Wiener process, σ is an $n\times r$ matrix such that the matrix $A = \sigma'\sigma$ is positive definite. Kinetic and potential energy of the system are, respectively,

$$T(\dot{q}) = (\dot{q}, M\dot{q})/2\,,\quad \Pi(q) = (q, Cq)/2 \tag{28}$$

We identify the admissible domain of motion G_q and the boundary Γ_q by formulas

$$G_q : (q, Lq) < 1\,,\ \Gamma_q : (q, Lq) = 1 \tag{29}$$

where L is a symmetric positive definite $n\times n$ matrix. This implies that the admissible domain is independent of the system's velocity.

Equations (22) – (28) determine the action functional

$$S(q,p) = (p, KM^{-1}p)/2 + (q, KCq)/2 \tag{30}$$

Substituting functions (29) and (30) into formula (22), we find

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\mathbf{E}T^{\varepsilon}) = \min_{(q,Lq)=1} (q, Qq)/2\,,\quad Q = KC \tag{31}$$

Minimization of the quadratic form in the left-hand side of equality (31) can be performed in the standard way.

As an example, we estimate escape time for a system in the plane. Equations of motion are

$$\ddot{q}_i + b_i\dot{q}_i + k_i^2 q_i = \varepsilon\sigma_i\,\dot{w}_i(t)\,,\ i = 1, 2 \tag{32}$$

where $w_i(t)$ are standard Wiener processes. The domain G_q and the boundary Γ_q are defined as

$$\begin{aligned} &\frac{q_1^2}{\alpha_1^2} + \frac{q_2^2}{\alpha_2^2} < 1,\ q_{1,2} \in G_q \\ &\frac{q_1^2}{\alpha_1^2} + \frac{q_2^2}{\alpha_2^2} = 1,\ q_{1,2} \in \Gamma_q \end{aligned} \tag{33}$$

Equations (32) are mutually independent but the variables q_1, q_2 are interconnected through the boundary condition (33). From Eqs. (32) we find the matrix $Q = 2\mathrm{diag}\{\lambda_i^2\}$, $\lambda_i^2 = b_i k_i^2/\sigma_i^2$. Then, by formula (31) we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\mathbf{E}T^{\varepsilon}) = \min_{q_1,q_2\in\Gamma_q} [(\lambda_1^2 q_1^2 + \lambda_2^2 q_2^2)] = l^2 \tag{34}$$

where $l^2 = \min[(\alpha_1\lambda_1)^2, (\alpha_2\lambda_2)^2]$. If $\lambda_1 = \lambda_2 = \lambda$, then $l = \lambda\min(\alpha_1, \alpha_2)$. This implies that the exit path reach the boundary at the point of intersection with the minimal semi-axis of ellipsis (33) (see [4] for discussion).

B. Motion of a particle in the Henon-Heiles potential

The Henon-Heiles potential was introduced to describe dynamics of a star in a gravitational potential of a galaxy. It has been shown [17] that escape from the potential well is associated with the passage from regular to irregular motion. From this viewpoint, estimation of the mean escape time can be interpreted as estimation of the lifetime of the system.

The equations of motion have the form

$$\ddot{q}_i + b\dot{q}_i + \frac{\partial\Pi(q_1,q_2)}{\partial q_i} = \varepsilon\sigma\dot{w}_i(t)\,,\ i = 1, 2 \tag{35}$$

with the potential

$$\Pi(q_1, q_2) = \frac{1}{2}\left(q_1^{\,2} + q_2^{\,2} + 2q_1^{\,2}q_2 - \frac{2}{3}q_2^{\,3}\right) \tag{36}$$

Potential (36) has the equipotential curves $\Pi(q_1,q_2) = Const < 1/6$; the equality $\Pi(q_1,q_2) = 1/6$ determines the separatrix [17].

We consider the admissible domain G_q as the domain encircled by the separatrix, that is

$$\begin{aligned} &G_q: \{\Pi(q_1, q_2) < 1/6\} \\ &\Gamma_q: \{\Pi(q_1, q_2) = 1/6\} \end{aligned} \tag{37}$$

Formulas (21), (36) and (37) entail the estimate

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\mathbf{E}\,T^{\varepsilon}) = \min_{\Gamma_q} k\Pi(q_1,q_2) = \frac{bc}{6\sigma^2} \tag{38}$$

V. CONCLUSIONS

Theory of large deviations is applied to the problem of escape from the reference domain for a weakly perturbed Lagrangian system. Formally, the system is interpreted as a nonlinear degenerate diffusion. The techniques employed involve the reduction of the escape problem to a deterministic variational problem for the action functional, and the solution of an associated Hamilton-Jacobi equation. It is shown that, in some special cases, the solution can be found in the closed form, and the mean escape time can be defined as a function of the system and noise parameters.

The approach proposed is of potential use in the dynamics and control problems for a wide class of mechanical and physical systems. As an illustration, the pointing problem for an *n*-degrees-of-freedom linear system is discussed. The time until escape from an ellipsoidal domain is estimated.

The lifetime of a number of physical systems is related to the time until escape through the potential barrier. Estimation of this quantity for the system with the Henon-Heiles potential is presented as an example.

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