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Equivalence of AR-Representations in the Light of the Impulsive-Smooth Behavior

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***Abstract*— The paper presents a new notion of equivalence of non-regular AR- representations, based on the coincidence of the impulsive-smooth behaviors of the underlying systems. The proposed equivalence is characterized by a special case of the usual unimodular equivalence and a restriction of the matrix transformation of full equivalence [21].**

I. INTRODUCTION

An equivalence relation preserving the structures of matrices, in a systems theory context, first appears in [22], as *strict system equivalence* and its modification known as *Fuhrmann system equivalence* [2]. These equivalences guarantee the systems have the same finite frequency structure. Thus systems with the same finite structure exhibit the same smooth behavior, while in [17] it is shown that strict system equivalence implies the existence of an isomorphism between the smooth solution spaces of the systems. To analyze simultaneously the finite and infinite structures of the system matrices Verghese [24] proposed, in the case of generalized state space systems, the notion of *strong equivalence* which took on a closed form description in [20] as *complete system equivalence*. In [14] ([19]) an interpretation of these equivalences as an isomorphism of the corresponding behaviors was given.

Behaviors were introduced in [25], [26] and have since been extensively studied. In this context, two AR-representations are equivalent if they represent the same smooth-behavior [18] or if they represent isomorphic smooth-behaviors ([3], [4], [5]). In the first case, the polynomial matrices that describe the AR-representations are shown to be *unimodular equivalent*, whereas in the second case they are *Fuhrmann system equivalent* [4]. The behavioral approach, at least in its original form, is not concerned with the infinite frequency (impulsive) behavior. During recent studies (see [1], [7], [8], [9], [10]) the importance of impulsive behavior in "switched" or "multimode" systems is recognized, and relevant questions about minimality and equivalence, in a behavioral framework, are addressed.

Our approach to the problem of equivalence of non-regular AR-representations has, as a starting point, the relation between the smooth-impulsive behaviors of such systems. We should notice that the notion of *fundamental equivalence* proposed here ensures the smooth-impulsive behaviors of the two systems are identical. We establish that the matrix conditions guaranteeing fundamental equivalence coincide with those of *full unimodular equivalence* (presented in section 3), which is a special case of *full matrix equivalence* presented in [13]. This provides a natural connection between the behavioral setting and the theory of polynomial matrix transformations.

II. PRELIMINARY RESULTS

In what follows $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers, respectively, $\mathbb{R}[s]$ the ring of polynomials with real coefficients and $\mathbb{R}(s)$ the field of real rational functions. Following [11], [12], [1], [7]-[10] we adopt the class of *impulsive-smooth* distributions (ℓ_{imp}), as the "function space" for our purposes. The class ℓ_{imp} consists of distributions that are linear combinations of a smooth function and a purely impulsive distribution. The purely impulsive part is essentially any finite linear combination of the Dirac delta distribution δ, and its (distributional) derivatives $\delta^{(i)}$, $i \geq 1$. A smooth distribution corresponds to a function that is smooth on $\mathbb{R}^+$ and 0 elsewhere.

Consider a *non-regular* linear time-invariant system described by the AR - representation

$$\Sigma : A(\partial)\xi(t) = 0, t \in [0, +\infty) \tag{1}$$

where $\partial = d/dt$ is the differential operator (interpreted as right-hand differentiation at the origin), $\xi(t) \in \mathbb{R}^m$ and $A(\partial) = A_q\partial^q + \ldots + A_1\partial + A_0 \in \mathbb{R}^{k\times m}[\partial]$ has $rank_{\mathbb{R}(s)}A(s) = r$ and $A_q \neq 0$. *Non-regular* is used here either for non-square, or square but not invertible, polynomial matrices (and AR- representations accordingly). In order to conform with the distributional framework introduced in [10], [1] we give the distributional version of (1)

$$A(p)\xi = S_{q-1}(p)X_A\bar{\xi}_0 \tag{2}$$

where $S_{q-1}(p) = \left[\begin{array}{cccc} Ip^{q-1} & \cdots & Ip & I \end{array}\right]$,

$$\bar{\xi}_0 = \left[\begin{array}{c} \xi_{0,0} \\ \vdots \\ \xi_{0,q-1} \end{array}\right] \text{and } X_A = \left[\begin{array}{ccc} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{array}\right] \tag{3}$$

while $\xi \in \ell_{imp}^m$ are vector distributions in ℓ_{imp} and $\xi_{0,j} \in \mathbb{R}^m$ are arbitrary real vectors which have to be interpreted as the initial values of equations (2), i.e. the values of the jth derivative of $\xi(t)$ "at $t = 0^-$ ", immediately before starting the dynamic process given by (1). The vector $\bar{\xi}_0$

will be termed the *initial value* of ξ, while $X_A\bar{\xi}_0$ will be termed the *initial condition* for ξ, for reasons which will become apparent subsequently. Accordingly the real vector space $\mathcal{X} = \mathcal{R}(X_A)$ will be termed the *initial condition space* of (2). For each $\bar{\xi}_0 \in \mathbb{R}^{qm}$ we define the solution set

$$B\left(\bar{\xi}_0\right) := \left\{\xi \in \ell_{imp}^m : A(p)\xi = S_{q-1}(p)X_A\bar{\xi}_0\right\}$$

and every $\xi \in B\left(\bar{\xi}_0\right)$ is called a solution of (2) for $\bar{\xi}_0$. An important feature of non-regular systems of the form (2) is that they are not in general *solvable* for every initial value (and thus initial condition) of ξ. This can be seen in the following example.

Definition 1: [9]-[10] (1) is C-solvable (control-solvable) for $\xi \in \ell_{imp}^m$ if

$$\forall\bar{\xi}_0 \in \mathbb{R}^{qm} \text{ (and thus } X_A\bar{\xi}_0) : B\left(\bar{\xi}_0\right) \neq \varnothing$$

The solvability requirement is reasonable in our case where we study equivalence of systems through their solution spaces (behaviors). Let

$$\ell_f := \left\{f \in \ell_{imp} : f = f_1 f_2^{-1}, f_1, f_2 \in \ell_{p-imp}, f_2 \neq 0\right\}$$

be the subalgebra of *fractional impulses*. Then we have the following basic result [6] :

Lemma 2: Let $T(s) \in \mathbb{R}^{k_1\times k_2}(s), \eta(s) \in \mathbb{R}^{1\times k_1}(s), w(s) \in \mathbb{R}^{k_2\times 1}(s)$ and let $T(p), \eta(p), w(p)$ be the corresponding distributional matrices,then

$$\eta(s)T(s) = 0 \Leftrightarrow \eta(p)T(p) = 0$$
$$T(s)w(s) = 0 \Leftrightarrow T(p)w(p) = 0$$

A characterization of C-solvable systems is the following

Theorem 3: The non-regular AR- representation (2) is C-solvable iff all the left minimal indices of $A(s)$ are zero.

Proof: (if) Assume that (2) is *C-solvable* and there exists a left minimal polynomial basis of $A(s)$, $\{v_1(s), v_2(s), \ldots, v_{k-r}(s)\}$, where the row vectors $v_j(s) = v_{j\eta_j}s^{\eta_j} + \ldots + v_{j0} \in \mathbb{R}^{1\times k}[s]$ and η_j are the left minimal indices of $A(s)$. Furthermore we assume that the vectors $v_j(s)$ are ordered in descending order, i.e. $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_{k-r}$. Then the polynomial matrix $V(s) = \begin{bmatrix} v_1^T(s) & \cdots & v_{k-r}^T(s) \end{bmatrix}^T$ has full normal row rank, no finite zeros, is row proper and

$$V(s)A(s) = 0 \quad (4)$$

Assume now that $\eta_1 > 0$, i.e. that there exists at least one left minimal index of $A(s)$ of order greater than zero. We look for a contradiction. Since (2) is C-solvable for every initial condition it holds for every $X_A\bar{\xi}_0$. From Lemma 2, we have that $V(s)A(s) = 0$ is equivalent to $V(p)A(p) = 0$. Premultiplying (2) by $V(p)$ we have $V(p)S_{q-1}(p)X_{A_i}\bar{\xi}_0 = 0$, $\forall\bar{\xi}_0$ or equivalently by Lemma 2,

$$V(s)S_{q-1}(s)X_{A_i} = 0 \quad (5)$$

Equating the coefficients of s^0 in (4)-(5) implies $V_0A_0 = 0$ and $V_0A_j = 0$, $j = 1, 2, ..q$. Thus $V_0A_j = 0$, for all $j = 0, 1, 2, ...q$. Equating successively the coefficients of $s^1, s^2, ..., s^{\eta_1}$ in (4),(5) and making use of the corresponding relations for coefficients of $V(s)$ of lower order, we get

$$V_iA_j = 0, \ \forall i = 0, 1, ...\eta_1, \forall j = 0, 1, ..., q \quad (6)$$

Now since $V(s)$ is row proper we can write $V(s) = diag\{s^{\eta_1}, s^{\eta_2}, ..., s^{\eta_{k-r}}\}[V]_r^h + \{\text{lower order terms}\}$, where $[V]_r^h$ is a constant full row rank matrix, with its i^{th} row being the i^{th} row of V_{η_i}. In view of (6) it is obvious that $[V]_r^hA(s) = 0$. The matrix $[V]_r^h$ thus satisfies all the properties of a left minimal polynomial basis of $A(s)$ and its row orders are obviously less than the corresponding ones of $V(s)$. This is a contradiction since $V(s)$ is assumed to be minimal. Thus $\eta_1 = 0$ and $\eta_j = 0$ for every $j = 0, 1, ..., k-r$, which completes the proof of the (if) part.

(only if) Assume that $\eta_j = 0$ for every $j = 0, 1, ..., k-r$. Then $\exists$ a constant left minimal polynomial basis of $A(s)$, $V \in \mathbb{R}^{(k-r)\times k}$. Thus $\exists$ a constant square, invertible matrix W having as its last $k-r$ rows the rows of V such that

$$WA(s) = \begin{bmatrix} \bar{A}(s) \\ 0 \end{bmatrix} \quad (7)$$

where $\bar{A}(s) \in \mathbb{R}^{r\times m}[s]$ is a full row rank rational matrix. We can also write

$$A(s) = \bar{W}\bar{A}(s) \Leftrightarrow A_j = \bar{W}\bar{A}_j, j = 0, 1, ..., q \quad (8)$$

where $\bar{W} \in \mathbb{R}^{p\times r}$ is the matrix consisting of the first r columns of W^{-1}. Consider now the equation

$$\bar{A}(p)\xi = \bar{S}_{q-1}(p)X_{\bar{A}}\bar{\xi}_0 \quad (9)$$

where $\bar{S}_{q-1}(p) = [p^{q-1}I_r, ..., I_r]$ and $X_{\bar{A}}, \bar{\xi}_0$ as in (3). The above equation is C-solvable simply because $\bar{A}(p)$ has full row rank and thus the fractional impulse space (rational vector space) spanned by $\bar{S}_{q-1}X_{\bar{A}}$ is always contained in the corresponding space spanned by $\bar{A}(p)$ (see Proposition 2.7, [10]). Premultiplying both sides of (9) by $\bar{W}$ and using (8), we get $A(p)\xi = \bar{W}\bar{S}_{q-1}(p)X_{\bar{A}}\bar{\xi}_0$. Now $\bar{W}\bar{S}_{q-1}(p) = S_{q-1}(p)diag\{\bar{W}, \bar{W}, ...\bar{W}\}$, where $S_{q-1}(p) = [p^{q-1}I_p, ..., I_p]$. Using again (8) we have

$$A(p)\xi = S_{q-1}(p)X_A\bar{\xi}_0 \quad (10)$$

Thus every solution of (9) is also a solution of (10). Since (9) is solvable for every initial condition so is (10). ■

The above result gives a characterization of solvability of non-regular AR- representations in terms of the structural invariants of the polynomial matrix $A(s)$, and is a generalization of the corresponding conditions appearing in [9], [10].

Corollary 4: Every C-solvable system of the form (2) can be replaced to an equivalent full row rank system, which has the same solution space as (2).

Proof: The proof is straightforward in view of the proof of the 'only if' part of theorem 3. ■

It follows that the full row rank assumption can be made without loss of generality for solvable systems. Thus we restrict ourselves in the sequel to the case, $rankA(s) = k =$

r. Following the terminology of [10] we denote by $\mathcal{B}$ the *solution space* or *behavior* of Σ, i.e.

$$\mathcal{B} = \{\xi \in \ell_{imp}^m : A(p)\xi = S_{q-1}(p)X_A\bar{\xi}_0, \quad (11)$$

$$\forall\bar{\xi}_0 = \begin{pmatrix} \xi_{00}^T & \xi_{01}^T & \cdots & \xi_{0(q-1)}^T \end{pmatrix}^T \in \mathbb{R}^{qm}\} \quad (12)$$

In the case where $A(p)$ is square and non-singular, $\mathcal{B}$ is finite dimensional and its dimension is equal to the total number of finite (n) and infinite zeros ($\hat{q}$) of $A(p)$ (multiplicities accounted for) [23] i.e.

$$\dim \mathcal{B} = n + \hat{q}$$

In the more general case, where $A(p)$ contains a right null space structure, $\mathcal{B}$ infinite dimensional. This is easy to see if we consider any *fractional impulse* lying in $\ker A(p)$. Obviously the *fractional impulse* vector distribution satisfies (2) and $\ker A(p)$ is an infinite dimensional vector space over ℓ_f. Let

$$\mathcal{Z} = \{\xi \in \mathcal{B} : X_A\bar{\xi}_0 = 0\} \quad (13)$$

i.e. the subspace of $\mathcal{B}$ which contains the solutions having zero *initial conditions*. The fact that there are solutions corresponding to zero initial conditions is somehow unnatural, since what is usually expected from a system of homogeneous differential equations is its non-trivial solutions to be triggered by non-zero initial conditions. An alternative interpretation to the question of what constitutes the solution space of non-regular systems, which overcomes this problem, has been proposed in [15]. According to this approach, the trajectory space $\mathcal{B}$ can be partitioned according to the relation

$$\xi \sim \xi' \Leftrightarrow X_A\bar{\xi}_0 = X_A\bar{\xi}_0' \quad (14)$$

It is easy to see that '$\sim$' is an equivalence relation and the resulting equivalence classes consist of distributional solutions of (2) that correspond to the same initial condition vector $X_A\bar{\xi}_0$. If $\xi \in \mathcal{B}$, write

$$[\xi] = \xi + \mathcal{Z} \quad (15)$$

where $\mathcal{Z} = [0_{\mathcal{B}}]$ and $[\xi]$ is the equivalence class of ξ. By $\mathcal{B}/\mathcal{Z}$ we denote the quotient space of $\mathcal{B}$ with $\mathcal{Z}$, i.e. the set of all equivalence classes of $\mathcal{B}$. It can be proved [15], that $\mathcal{B}/\mathcal{Z}$ is a finite dimensional vector space over $\mathbb{R}$ which can be decomposed as follows

$$\mathcal{B}/\mathcal{Z} = (\mathcal{B}^{\mathbb{C}} \oplus \mathcal{B}^{\infty} \oplus \mathcal{B}^{\varepsilon})/\mathcal{Z} = \mathcal{B}^{\mathbb{C}}/\mathcal{Z} \oplus \mathcal{B}^{\infty}/\mathcal{Z} \oplus \mathcal{B}^{\varepsilon}/\mathcal{Z} \quad (16)$$

where $\mathcal{B}^{\mathbb{C}}, \mathcal{B}^{\infty}, \mathcal{B}^{\varepsilon}$ are finite dimensional distributional spaces corresponding to the finite zero structure, the infinite zero structure and the right minimal indices of $A(s)$. Actually $\mathcal{B}/\mathcal{Z}$, is the finite dimensional sectional of the infinite dimensional vector space $\mathcal{B}$. The dimensions of the $\mathcal{B}^{\mathbb{C}}/\mathcal{Z}, \mathcal{B}^{\infty}/\mathcal{Z}, \mathcal{B}^{\varepsilon}/\mathcal{Z}$ spaces are $n, \hat{q}$ and ε respectively, where $n, \hat{q}$ and ε are the total number of finite zeros, infinite zeros and right minimal indices of $A(s)$ (multiplicities accounted for). Furthermore it can be shown that the dimension of $\mathcal{B}/\mathcal{Z}$ is

$$\dim \mathcal{B}/\mathcal{Z} \overset{[15]}{=} n + \hat{q} + \varepsilon \overset{[24]}{=} \delta_M (A(s)) \quad (17)$$

where $\delta_M (A(s)) = rank_{\mathbb{R}} X_A$ denotes the McMillan degree of $A(s)$. In the general case, where $A(s)$ has not necessarily full row rank then it is known [24] that $\delta_M (A(s)) = n + \hat{q} + \varepsilon + \eta$ where η denotes the total number of left minimal indices (multiplicities accounted for). As a result of this discussion we call $\mathcal{B}/\mathcal{Z} \triangleq \bar{\mathcal{B}}$ the *quotient solution space* of (2). Notice that according to the above definitions, there is a one to one correspondence between the initial condition vectors $X_A\bar{\xi}_0$ and the elements of $\bar{\mathcal{B}}$, and thus an isomorphism between the initial conditions space $\mathcal{X}$ and the quotient solution space. On this basis the quotient solution space is a finite dimensional view of the actual solution space (behavior) of the AR- representation (2). The behavior itself is of course, an infinite dimensional and complete view of the solution space.

III. Fundamental equivalence

The aim of this section is to establish a connection between the existing theory of matrix equivalence and the behavioral framework. In the case where both the *smooth-impulsive* solution set is of interest, we need a restriction of the unimodular equivalence transformation.

Definition 5: $P_1(s), P_2(s) \in \mathbb{R}[s]^{r\times m}$ are said to be *fully unimodular equivalent* (FE) if $\exists$ a unimodular matrix $U(s) \in \mathbb{R}[s]^{r\times r}$ such that

$$U(s) P_1(s) = P_2(s)$$

where the compound matrix $\begin{bmatrix} U(s) & P_2(s) \end{bmatrix}$

i) has no infinite zeros

ii) $\delta_M \begin{bmatrix} U(s) & P_2(s) \end{bmatrix} = \delta_M [P_2(s)]$ where $\delta_M(\cdot)$ indicates the McMillan degree of the indicated matrix.

Full unimodular equivalence is a special case of *full system equivalence* [21] and thus has the nice property of preserving the finite and infinite zero structure of polynomial matrices (see [13]) in contrast to unimodular equivalence which preserves only the finite aspects.

We now introduce a notion of equivalence using a solution space approach, and in this way we provide a direct dynamic interpretation of the conditions appearing in definition 5. We give the following definition

Definition 6: Let the systems be described by

$$\Sigma_i : A_i(\partial)\xi_i(t) = 0, t \in [0, +\infty), i = 1, 2$$

where $\partial = d/dt$ is the differential operator, $\xi_i(t) \in \mathbb{R}^m$ and $A_i(\partial) = A_{iq}\partial^q + \ldots + A_{i1}\partial + A_{i0} \in \mathbb{R}^{r\times m}[\partial], i = 1, 2$ is a polynomial matrix with $rank_{\mathbb{R}(s)} A_i(s) = r$ and A_{iq} not both identically zero, and let the distributional version of the above systems be the following [10]

$$\Sigma_i : A_i(p)\xi_i = S_{q-1}(p)X_{A_i}\bar{\xi}_0^i$$

where $\xi_i \in \ell_{imp}^{m_i}$ are vector distributions in ℓ_{imp}. Define also as

$$\mathcal{B}_i := \left\{\xi_i \in \ell_{imp}^{m_i} : A_i(p)\xi_i = S_{q-1}(p)X_{A_i}\bar{\xi}_0^i\right\}, i = 1, 2$$

The systems Σ_i are *fundamentally equivalent* iff $\mathcal{B}_1 = \mathcal{B}_2$.

Fundamental equivalence extends the notion of [18] to the "smooth-impulsive" solution set. We are interested in the conditions under which $\mathcal{B}_1 \subseteq \mathcal{B}_2$. We need the following

Lemma 7: If $U(s), A_1(s), A_2(s)$ are polynomial matrices of appropriate dimensions such that

$$U(s)A_1(s) = A_2(s) \tag{18}$$

then

$$X_U X_{A_1} = 0 \text{ and } \bar{U} X_{A_1} = X_{A_2} - X_U \bar{A}_1 \tag{19}$$

where

$$X_P = \begin{bmatrix} P_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ P_1 & \cdots & P_q \end{bmatrix}, \ \bar{P} = \begin{bmatrix} P_0 & \cdots & P_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_0 \end{bmatrix}$$

and $P(s) = P_q s^q + \ldots + P_0$ is any of the matrices $A_1(s), A_2(s), U(s)$ (assuming without loss of generality that all the above matrices have the same degree q).

Proof: The proof is straightforward by equating like powers of s in (18). ■

Theorem 8: The following statements are equivalent:

(i) $\mathcal{B}_1 \subseteq \mathcal{B}_2$

(ii) $\exists U(s) \in \mathbb{R}^{r\times r}[s] : U(s)A_1(s) = A_2(s)$ and $\delta_M[U(s), A_2(s)] = \delta_M(A_2(s))$.

Proof: Assume $\mathcal{B}_1 \subseteq \mathcal{B}_2$. Then the same inclusion property will hold for $\hat{\mathcal{B}}_1 \subseteq \hat{\mathcal{B}}_2$ where

$$\hat{\mathcal{B}}_i := \left\{ \xi_i \in \ell_{sm}^{m_i} : A_i(p)\xi_i = S_{q-1}(p) X_{A_i} \bar{\xi}_0^i \right\}, i = 1, 2$$

Thus from [16] (p. 36, lemma 3.7) we have $\exists U(s) \in \mathbb{R}^{r\times r}[s] : U(s)A_1(s) = A_2(s)$. Furthermore let $\xi_1 \in \mathcal{B}_1$. Then ξ_1 satisfies $A_1(p)\xi_1 = S_{q-1}(p)X_{A_1}\bar{\xi}_0^1$. Since $\mathcal{B}_1 \subseteq \mathcal{B}_2$ holds, ξ_1 must satisfy $A_2(p)\xi_2 = S_{q-1}(p)X_{A_2}\bar{\xi}_0^2$, which gives $A_2(p)\xi_1 = S_{q-1}(p)X_{A_2}\bar{\xi}_0^2$ or equivalently $U(p)A_1(p)\xi_1 = S_{q-1}(p)X_{A_2}\bar{\xi}_p^2$ and finally $U(p)S_{q-1}(p)X_{A_1}\bar{\xi}_0^1 = S_{q-1}(p)X_{A_2}\bar{\xi}_0^2$. Now equating powers of p in the last equation we obtain

$$(X_U X_{A_1})\bar{\xi}_0^1 = 0 \text{ and } X_{A_2}\bar{\xi}_0^2 = (\bar{U} X_{A_1})\bar{\xi}_0^1 \tag{20}$$

which in view of (19), reduce to $X_{A_2}\bar{\xi}_0^2 = (X_{A_2} - X_U \bar{A}_1)\bar{\xi}_0^1$. Since the last equation is solvable for $\bar{\xi}_0^2$, we conclude that

$$\mathcal{R}(X_U \bar{A}_1) \subseteq \mathcal{R}(X_{A_2}) \tag{21}$$

Notice that since $A_1(s)$ has full row rank the same will hold for $[\bar{A}_1, X_{A_1}]$ (see [23]). Thus $\mathcal{R}(X_U) = \mathcal{R}(X_U[\bar{A}_1, X_{A_1}])$ $= \mathcal{R}(X_U \bar{A}_1) + \mathcal{R}(X_U X_{A_1}) = \mathcal{R}(X_U \bar{A}_1)$. Thus, we obtain $\mathcal{R}(X_U) \subseteq \mathcal{R}(X_{A_2})$, which implies that $rank[X_U, X_{A_2}] = rank X$, i.e.

$$\delta_M[U(s), A_2(s)] = \delta_M(A_2(s)) \tag{22}$$

which proves that (i) implies (ii).

The converse follows easily from the above analysis. ■

An interesting map connects the initial condition spaces of the two systems as described by the following

Corollary 9: If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then an induced injective mapping between the initial conditions of the two systems is given by

$$X_{A_2}\bar{\xi}_0^2 = \bar{U}\left(X_{A_1}\bar{\xi}_0^1\right) \tag{23}$$

and thus

$$\bar{U}\mathcal{R}(X_{A_1}) \subseteq \mathcal{R}(X_{A_2}) \tag{24}$$

Proof: Equation (23) is simply the second equation of (20). Obviously (24) must hold since (23) is solvable with respect to $\bar{\xi}_0^2$. ■

According to the theorem 8 if additionally $\mathcal{B}_2 \subseteq \mathcal{B}_1$ then $\exists V(s) \in \mathbb{R}^{r\times r}[s] : V(s)A_2(s) = A_1(s)$ and $\delta_M[V(s), A_1(s)] = \delta_M(A_1(s))$ and thus $V(s)A_2(s) = A_1(s) \Leftrightarrow V(s)U(s)A_1(s) = A_1(s) \Leftrightarrow (V(s)U(s) - I)A_1(s) = 0$. Since $A_1(s)$ has full row rank then $U(s) \in \mathbb{R}^{r\times r}[s]$ defined in the above Theorem is unimodular (see also [16]). According to Corollary 9 we have from the above relation that $\mathcal{R}(X_{A_2}) \subseteq \mathcal{R}(X_{A_1})$ and thus $\mathcal{R}(X_{A_1}) = \mathcal{R}(X_{A_2})$ or otherwise $\delta_M(A_1(s)) = \delta_M(A_2(s))$. Therefore, an isomorphism exists between the initial condition spaces of the two systems. The inclusion (24) ensures that the image of every initial condition of Σ_1 is mapped through $\bar{U}$, to an initial condition of Σ_2. The following corollary guarantees that every solution of Σ_1 starting from the zero initial condition will be mapped to a zero initial condition solution of Σ_2.

Corollary 10: If $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then

$$\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \tag{25}$$

where $\mathcal{Z}_i = \left\{ \xi_i \in \mathcal{B}_i : X_{A_i}\bar{\xi}_0^i = 0 \right\}, i = 1, 2.$

IV. Induced Maps of Fundamental Equivalence

Suppose that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and thus from Corollary 10 $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then there exists a unique mapping between the quotient solution spaces

$$I^* : [\xi_1] \in \mathcal{B}_1/\mathcal{Z}_1 \mapsto [\xi_1] \in \mathcal{B}_2/\mathcal{Z}_2 \tag{26}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{I} & \mathcal{B}_2 \\ \downarrow_{I_{\mathcal{B}_1}} & & \downarrow_{I_{\mathcal{B}_2}} \\ \mathcal{B}_1/\mathcal{Z}_1 & \xrightarrow{I^*} & \mathcal{B}_2/\mathcal{Z}_2 \end{array}$$

is commutative, where I is the unit map and I^* is its restriction to $\mathcal{B}_i/\mathcal{Z}_i$. $I_{\mathcal{B}_1}, I_{\mathcal{B}_2}$ are the natural projections. Note that I^* is injective iff $\mathcal{Z}_1 = I^{-1}(\mathcal{Z}_2)$ and surjective iff $\mathcal{Z}_2 + imI = \mathcal{B}_2$. Furthermore $\mathcal{B}_1 \subseteq \mathcal{B}_2$ implies that restriction of the I to $\mathcal{Z}_1$, denoted I_*, maps $\mathcal{Z}_1$ into $\mathcal{Z}_2$. We need the conditions for I_* to be a bijection. Note that $\mathcal{B}_1, \mathcal{B}_2$ are infinite dimensional and it turns out to be easier to study the properties of I through the structures of I_* and I^*. The map I is injective by definition. We note the following.

Theorem 11: Let I^*, I_* be the maps defined above, then I^*, I_* are surjective $\Rightarrow I$ is bijective

Furthermore we have

Theorem 12: Let I^*, I_* be the maps defined above, then

i) If I is surjective then I^* is surjective.

ii) If I_* is surjective then I^* is injective.

iii) The map I_* is injective.

iv) If I is surjective and I^* is injective then I_* is surjective.

The above theorems give a complete picture of the conditions for I^*, I_* to be bijections. It is clear that the injectiveness of I^* (surjectiveness of I_*) is not a direct consequence of the injectiveness (surjectiveness) of the unit map I. Nevertheless the following is true.

Corollary 13: Let $I : \mathcal{B}_1 \to \mathcal{B}_2$ and I^*, I_* be the maps defined above, then

$$I \text{ is bijection } \Leftrightarrow \ I_* \text{ and } I^* \text{ are bijective}$$

We establish the properties of I, I^* and I_*, in terms of the matrices of the Σ_1 and Σ_2.

Theorem 14: For $\bar{U}$ of (23) the following hold

(i) I^* is injective iff $\ker \bar{U} \cap \mathcal{R}(X_{A_1}) = \{0\}$

(ii) I^* is surjective iff $\mathcal{R}(X_{A_2}) = \bar{U}\mathcal{R}(X_{A_1})$

Proof: (i) Injectiveness of I^* means that $I^*[\xi_1] = \mathcal{Z}_2 \Rightarrow [\xi_1] = \mathcal{Z}_1$ or equivalently $\xi_1 \in \mathcal{Z}_2 \Rightarrow \xi_1 \in \mathcal{Z}_1$. If $X_{A_2}\bar{\xi}_0^2$ are the initial conditions corresponding to $\xi_2 = \xi_1$ then the fact that $\xi_2 \in \mathcal{Z}_2$ means that $X_{A_2}\bar{\xi}_0^2 = 0$, which in view of (23) implies

$$\bar{U}X_{A_1}\bar{\xi}_0^1 = 0 \tag{27}$$

Now obviously the requirement $\xi_1 \in \mathcal{Z}_1$ gives $X_{A_1}\bar{\xi}_0^1 = 0$. Thus I^* is injective iff equation (27) implies $X_{A_1}\bar{\xi}_0^1 = 0$ for every ξ_1, which establishes (i).

(ii) We first notice that $\bar{U}\mathcal{R}(X_{A_1}) \subseteq \mathcal{R}(X_{A_2})$ holds from (24). We thus need the reverse inclusion. Now I^* is surjective iff for every $\xi_2 \in \mathcal{B}_2$ there exists $\xi_1 \in \mathcal{B}_1$ such that $[\xi_2] = I^*[\xi_1]$ or equivalently $\xi_2 - \xi_1 \in \mathcal{Z}_2$. Following the proof of theorem 8 we conclude that (23) should be solvable for every initial condition $X_{A_2}\bar{\xi}_0^2$, hence the condition in (ii). ■

If $U(s)$ is unimodular then $\ker \bar{U} = \{0\}$, since $\det U(0) = \det(U_0) \neq 0$. Thus (i) in Theorem 14 holds and so I^* is injective. For the surjectiveness of I_* (and thus bijectiveness, since it is an injective map), notice that $\mathcal{Z}_1, \mathcal{Z}_2$ are isomorphic to $\ker A_1(p)$, $\ker A_2(p)$ respectively, which are finite dimensional vector spaces over ℓ_f having the same dimension i.e. $m - r$. Thus I_* is always surjective. Now I is always injective according to its definition.

Lemma 15: Let $T(s) \in \mathbb{R}^{k\times(k+r)}[s], V(s) \in \mathbb{R}^{(k+r)\times r}$ be polynomial matrices with $rank_{\mathbb{R}(s)}T(s) = k$ and $rank_{\mathbb{R}(s)}V(s) = r$, such that $T(s)V(s) = 0$. Furthermore let $V(s)$ have no zeros in $\mathbb{C}\cup\{\infty\}$ and denote by $\varepsilon = \sum_{i=1}^{r} \varepsilon_i$ the sum of right minimal indices of $T(s)$. Then

$$\varepsilon = \delta_M(V(s)) \tag{28}$$

Proof: If $V(s)$ is column proper then it forms a minimal basis of $\ker T(s)$ and its column degrees are the orders of the poles at $s = \infty$ of $V(s)$ (see [23]). The theorem then holds since the only poles of $V(s)$ are those at $s = \infty$.

Assume that $V(s)$ is not column proper and let $S^{\infty}_{V(s)}(s) = diag\{s^{q_1}, s^{q_2}, \ldots, s^{q_r}\}$ be its Smith - McMillan form at $s = \infty$. It is known [23] that the orders of the poles at $s = \infty$ can be obtained from the formula $q_i = m_i - m_{i-1} \geq 0$, $i = 1, 2, \ldots r$ where $m_0 = 0$ and $m_i = \max\deg\{$minors of order i of $V(s)\}$, $i = 1, 2, \ldots r$. It is easy to see that $q = \sum_{i=1}^{r} q_i = m_r$ and since $V(s)$ is polynomial

$$\delta_M(V(s)) = q = m_r \tag{29}$$

Consider now the unimodular matrix $W(s)$ which reduces $V(s)$ to $\bar{V}(s)$, which is column proper, i.e.$\bar{V}(s) = V(s)W(s)$. Then $\bar{V}(s)$ is a minimal basis of $\ker T(s)$, with column degrees equal to the right minimal indices of $T(s)$. Moreover the column degrees of $\bar{V}(s)$ will be the orders of its poles at $s = \infty$. Thus if $\bar{q}$ is the total number of poles at $s = \infty$ then

$$\bar{q} = \varepsilon = \bar{m}_r \tag{30}$$

where $\bar{m}_r = \max\deg\{$minors of order r of $\bar{V}(s)\}$. Now the minors of order r of $V(s)$ remain invariant (up to multiplication by a non zero constant) in $\bar{V}(s)$. Thus $\bar{m}_r = m_r$ which in view of (29), (30) proves the lemma. ■

The main result is thus

Theorem 16: The systems in (2) are fundamentally equivalent iff there exists a unimodular matrix $U(s)$ satisfying

(i) $U(s)A_1(s) = A_2(s)$.

(ii) $\delta_M \begin{bmatrix} U(s) & A_2(s) \end{bmatrix} = \delta_M(A_2(s))$.

(iii) $\begin{bmatrix} U(s) & A_2(s) \end{bmatrix}$ have no zeros at $s = \{\infty\}$.

Proof: (if) Assume that there exists a the unit map $I : \mathcal{B}_1 \to \mathcal{B}_2$ satisfying (i)-(iii). Conditions (i)-(ii) guarantee (see theorem 8) that I is a well defined map $I : \mathcal{B}_1 \to \mathcal{B}_2$. The restriction of I to $\mathcal{Z}_1$, i.e. I_*, will be bijective since $\ker A_1(p)$, $\ker A_2(p)$ have the same dimension over ℓ_f. Using theorem 12 shows that I^* is injective. Now write (i) as

$$\begin{bmatrix} U(s) & A_2(s) \end{bmatrix} \begin{bmatrix} A_1(s) \\ -I \end{bmatrix} = 0 \tag{31}$$

$\begin{bmatrix} U(s) & A_2(s) \end{bmatrix}$ has full row rank and so the dimension of its kernel is $\dim\ker \begin{bmatrix} U(s) & A_2(s) \end{bmatrix} = r + m - r = m$. The matrices in (31) satisfy lemma 15, so

$$\delta_M \begin{bmatrix} A_1^T(s) & -I \end{bmatrix}^T = \varepsilon$$

where ε is the sum of the right minimal indices of $\begin{bmatrix} U(s) & A_2(s) \end{bmatrix}$. Now since $\begin{bmatrix} U(s) & A_2(s) \end{bmatrix}$ has no zeros in $\mathbb{C} \cup \{\infty\}$, $\delta_M(\begin{bmatrix} U(s) & A_2(s) \end{bmatrix}) = \varepsilon$. Using the McMillan degree conditions in (ii) we conclude $\delta_M(A_1(s)) = \delta_M(A_2(s))$ or equivalently $rankX_{A_1} = rankX_{A_2}$. Combining this with the fact that I^* is injective and using statement (i) of theorem 14 and equation (23) we obtain $rankX_{A_2} = rank(\bar{U}X_{A_1})$. From (24) we thus conclude that $\bar{U}\mathcal{R}(X_{A_1}) = \mathcal{R}(X_{A_2})$, which is the necessary and sufficient condition (theorem 14, (ii)) for surjectiveness of I^*. The surjectiveness of I follows from theorem 11, which proves the sufficiency of (i)-(iii).

(only if) Assume now that the systems in (2) are fundamentally equivalent. Then conditions (i)-(ii) holds since I is a well defined map from $\mathcal{B}_1$ to $\mathcal{B}_2$. Since I is surjective, I^* will be surjective, thus using (ii) of theorem 14 we have $\mathcal{R}(X_{A_2}) = \bar{U}\mathcal{R}(X_{A_1})$ or equivalently $rankX_{A_2} =$

$rank(\bar{U}X_{A_1}) = rankX_{A_1} - \dim(\ker \bar{U} \cap \mathcal{R}(X_{A_1}))$. Hence

$$\begin{aligned}\dim(\ker \bar{U} \cap \mathcal{R}(X_{A_1})) &= \delta_M A_1(s) - \delta_M A_2(s) \qquad (32)\\ &= \delta_M \left[\begin{array}{cc} A_1^T(s) & -I \end{array}\right]^T - \delta_M \left[\begin{array}{cc} U(s) & A_2(s) \end{array}\right]\end{aligned}$$

Following similar lines as the (if) part it is easy to see that $\delta_M \left[\begin{array}{cc} A_1^T(s) & -I \end{array}\right]^T = \varepsilon$ and $\delta_M \left[\begin{array}{cc} U(s) & A_2(s) \end{array}\right] = n + \hat{q} + \varepsilon$, where $n, \hat{q}$ is the total number of finite and infinite zeros of $\left[\begin{array}{cc} U(s) & A_2(s) \end{array}\right]$ respectively (note that $\left[\begin{array}{cc} U(s) & A_2(s) \end{array}\right]$ has full row rank since $A_2(s)$ has full row rank). Now (32) becomes

$$\dim(\ker \bar{U} \cap \mathcal{R}(X_{A_1})) = -n - \hat{q} \geq 0 \qquad (33)$$

which implies $n + \hat{q} = 0$, i.e. that $\left[\begin{array}{cc} U(s) & A_2(s) \end{array}\right]$ has no zeros in $\mathbb{C} \cup \{\infty\}$ which proves condition (iii). Moreover from (33) we get $\dim(\ker \bar{U} \cap \mathcal{R}(X_{A_1})) = 0$ or equivalently $\ker \bar{U} \cap \mathcal{R}(X_{A_1}) = \{0\}$, which by (i) of theorem 14 implies that I^* is injective. Now it easy to see that I_* is injective since I is injective and surjective because theorem 14 is satisfied. ■

Corollary 17: If Σ_1, Σ_2 are fundamentally equivalent then $I : \mathcal{B}_1 \to \mathcal{B}_2$ induces a bijective map I^* between the quotient solution spaces $\mathcal{B}_1/\mathcal{Z}_1, \mathcal{B}_2/\mathcal{Z}_2$ of the systems. Further the restriction of I to $\mathcal{Z}_1$, $I_* : \mathcal{Z}_1 \to \mathcal{Z}_2$, is bijective.

We have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_1/\mathcal{Z}_1 & \xrightarrow{\phi_1} & \mathcal{X}_{A_1} \\ \downarrow_I & & \downarrow_{I^*} & & \downarrow_{\bar{U}} \\ \mathcal{B}_2 & \xrightarrow{\pi_2} & \mathcal{B}_2/\mathcal{Z}_2 & \xrightarrow{\phi_2} & \mathcal{X}_{A_2} \end{array}$$

where I, I^* and $\bar{U}$ are the maps between the behaviors, the quotient solution spaces and the initial conditions spaces respectively. $\pi_i, i = 1, 2$ are the natural projections, ϕ_i, $i = 1, 2$ are the isomorphisms between the quotient solution spaces and the initial condition spaces, that is the map that takes equivalence classes to their corresponding initial conditions.

V. Conclusions

A characterization of C-solvability of a non-regular AR-representation in terms of the left minimal structure of the polynomial matrix that describes the AR-representation is given, and is a generalization of the corresponding conditions appearing in [9], [10]. The definition of equivalence between AR-representations that has been presented in the behavioral context by [18], [16] has been extended to the case where the smooth-impulsive solution sets are of interest. An alternative characterization of this equivalence, has been given in terms of a transformation between polynomial matrices, named full unimodular equivalence. Full unimodular equivalence is a special case of the known full matrix equivalence, appearing in [21], and has the nice property of preserving both the finite and infinite zero structure and the right minimal indices of the associated polynomial matrices. These invariants are strongly connected with the smooth-impulsive behavior of the AR-representation represented by such polynomial matrices. In this sense, fundamental equivalence of non-regular systems provides a dynamic interpretation of known algebraic results.

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