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Some Results on Practical Asymptotic Stabilizability of Switched Systems

Xuping Xu
Dept. of Electrical and Computer Engineering
Penn State Erie
Erie, PA 16563, USA
E-mail: Xuping-Xu@psu.edu

Guisheng Zhai
Dept. of Mechanical Engineering
Osaka Prefecture University
Sakai, Osaka 599-8531, JAPAN
E-mail: zhai@me.osakafu-u.ac.jp

Abstract— **In this paper, we report some new results on practical asymptotic stabilizability of switched systems consisting of time-invariant subsystems. After formally introducing the concept of practical asymptotic stabilizability, we propose some sufficient conditions based on energy functions. We then point out that the vector fields of a switched system can be decomposed into two parts, namely, vector fields corresponding to a common equilibrium and vector fields corresponding to integrator dynamics. Such a decomposition makes it possible to study the relationship between conventional asymptotic stabilizability and practical asymptotic stabilizability of switched systems. Based on the decomposition, we present methods for estimating the region of attraction for switched affine systems.**

I. Introduction

In our previous papers [8], [9], [10], we have pointed out that, under appropriate switching laws, switched systems whose subsystems have different or no equilibria may still exhibit interesting behavior similar to that of a conventional stable system near an equilibrium. Such behavior is defined as practical stabilizability in these papers. It is a natural extension of the traditional concept of practical stability [4], [5], which is concerned with bringing the system trajectories to be within given bounds.

It should be noted that the notion of practical stabilizability is mainly concerned with the local behavior of the system within given bounds around the origin. However, in many cases, we are interested not only in the local behavior of the system but also in its behavior in a larger region around the origin. For example, we may want to know whether the system can exhibit "convergent behavior" similar to that of a conventional asymptotically stable system. Therefore, it is important to extend the practical stabilizability notion so that such behavior can be studied. In [11], we formally introduced the notion of practical asymptotic stabilizability and presented some preliminary sufficient conditions.

In this paper, we further our studies on practical asymptotic stabilizability of switched systems. The contributions include the followings. We first formally introduce the concept of practical asymptotic stabilizability for switched systems consisting of time-invariant subsystems and propose some sufficient conditions based on energy functions. We then point out that the vector fields of a switched system can be decomposed into two parts, namely, vector fields corresponding to a common equilibrium and vector fields corresponding to integrator dynamics. Based on such a decomposition, we study the relationship between conventional asymptotic stabilizability and practical asymptotic stabilizability of switched systems. Finally we focus on switched affine systems and derive methods for estimating the region of attraction. More sufficient conditions based on quadratic energy functions are proposed for switched affine systems.

It should be noted that practical asymptotic stabilization is a natural extension of asymptotic stabilization for switched systems when the origin is not an equilibrium for some or all of their subsystems. Such practical asymptotic stabilization is made possible by switchings among subsystems and thus is an important feature of switched systems. This feature makes switched system useful in many control problems for which conventional controller based on one dynamic equation is incapable of providing a solution. Our research on practical asymptotic stabilizability is thus the first step toward the study of many important problems such as asymptotic tracking, practical reachability (see, e.g., [10] for an introduction to such problems) and is expected to arouse more interest in this new research direction.

II. Switched Systems and Practical Stabilizability Notions

A. Switched Systems and Switching Laws

In this paper, we consider switched systems consisting of time-invariant subsystems

$$\dot{x} = f_i(x), \quad i \in I \stackrel{\triangle}{=} \{1, 2, \cdots, M\}. \tag{1}$$

In (1), every $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. The active subsystem at each time instant is orchestrated by a switching law, which will be formally defined below. Given any initial time t_0 and initial state $x(t_0)$, the switching law generates a switching sequence $\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_k, i_k), \cdots\big)$ ($t_0 < t_1 < \cdots < t_k < \cdots$, $i_k \in I$) which indicates that subsystem i_k is active in $[t_k, t_{k+1})$. For a switched system to be well-behaved, we only consider **nonZeno** sequences which switch at most a finite number of times in any finite time interval.

In the sequel, we will pay particular attention to switching sequences over the time interval $[0, \infty)$. Such sequences are usually generated by switching laws defined below.

Definition 1 (Switching Law over $[0, \infty)$*):* For switched system (1), a switching law $\mathcal{S}$ over $[0, \infty)$ is defined as a mapping $\mathcal{S} : \mathbb{R}^n \rightarrow \Sigma_{[0,\infty)}$ which specifies a nonZeno switching sequence $\sigma \in \Sigma_{[0,\infty)}$ for any initial state $x(0)$. Here $\Sigma_{[0,\infty)} \stackrel{\triangle}{=} \{\text{switching sequence } \sigma \text{ over } [0,\infty)\}$. □

Remark 1: $\mathcal{S}$ over $[0,\infty)$ (in the sequel, we simply call it switching law $\mathcal{S}$) is often determined by some rules or algorithms, which describe how to generate a switching sequence for a given $x(0)$, rather than mathematical formulae. □

Remark 2: Switching laws over any $[t_0,\infty)$ ($t_0 \in \mathbb{R}$) may be similarly defined. Since all subsystems in (1) are time-invariant, given any initial state, a time shift in the amount t_0 of any switching sequence over $[0,\infty)$ will result in the same amount of time shift in the state trajectory. Hence many properties that the switched system has under switching laws over $[0,\infty)$ will carry over to the cases in which the switching laws are over $[t_0,\infty)$, except for the time shift t_0. □

B. Practical Stabilizability Notions

Now we review some notions and results reported in [8], [9], [10]. We use $\|\cdot\|$ to denote the 2-norm, $B[x,r]$ to denote the closed ball $\{y \in \mathbb{R}^n : \|y-x\| \le r\}$ and $B(x,r)$ the open ball. Unlike the conventional stability concept, here we do not assume that $f_i(0) = 0$, $\forall i \in I$, i.e., the origin does not have to be an equilibrium for any or all of the subsystems. Without loss of generality, we only discuss the case of the origin and let the initial time be $t_0 = 0$.

Definition 2 (ϵ-Practical Stability): Assume that a switching law $\mathcal{S}$ is given for switched system (1). Given an $\epsilon > 0$, the system is said to be *ϵ-practically stable under $\mathcal{S}$* if there exists a $\delta = \delta(\epsilon) > 0$ such that $x(t) \in B[0,\epsilon]$, $\forall t \ge 0$, whenever $x(0) \in B[0,\delta]$. □

Definition 3 (Practical Stabilizability): Switched system (1) is said to be *practically stabilizable* if for every $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically stable under $\mathcal{S}$. □

Remark 3: In Definition 3, ϵ can be varied as opposed to the fixed ϵ in Definition 2. Such a definition provides us with the flexibility in design and trajectory tracking problems where the bound ϵ may vary depending on the specific task we are facing. □

In view of the results in [9], [10], we have the following lemma which provides a sufficient condition for the practical stabilizability of switched system (1).

Lemma 1: Switched system (1) is practically stabilizable if $0 \in \text{Int}(C)$. Here C is the convex hull of the set $\{f_i(0) : i \in I\}$, i.e., $C = \text{conv}\big(\{f_i(0) : i \in I\}\big) = \{\sum_{i=1}^{M} \lambda_i f_i(0) : \lambda_i \ge 0, i \in I \text{ and } \sum_{i=1}^{M} \lambda_i = 1\}$, and $\text{Int}(C)$ is the interior of C. □

Remark 4: In [9], we have actually obtained a δ as in Definition 2 for a given ϵ and constructed a valid switching law $\mathcal{S}$ to achieve ϵ-practical stability. □

C. Practical Asymptotic Stabilizability Notions

Practical stabilizability concerns the local behavior of the system trajectory within given bounds around the origin. In many cases, we are also interested in the behavior of the trajectory in a larger region around the origin. For example, we may want to know whether a switched system can exhibit "convergent behavior" similar to that of a conventional asymptotically stable system. Such behavior has been formally defined as practical asymptotic stabilizability in [11]. Below we review some notions related to practical asymptotic stabilizability. In the sequel, by a region D around the origin, we mean a open connected subset of $\mathbb{R}^n$ containing the origin along with some, none, or all of its boundary points.

Definition 4 (ϵ-Practical Attractivity): Assume that a region D around the origin is given. Also assume that a switching law $\mathcal{S}$ is given for switched system (1). Given an $\epsilon > 0$, the system is said to be *ϵ-practically attractive on D under $\mathcal{S}$* if for every $x(0) \in D$, there exists a finite $T = T\big(x(0)\big) \ge 0$ such that $x(T) \in B[0,\epsilon]$. □

Definition 5 (Practical Attractivity): Assume that a region D around the origin is given. Switched system (1) is said to be *practically attractive on D* if for every $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically attractive on D under $\mathcal{S}$. □

Combining the notions of practical attractivity and practical stabilizability, we can define the following useful concept.

Definition 6 (Practical Asymptotic Stabilizability): Assume that a region D around the origin is given. Switched system (1) is said to be *practically asymptotically stabilizable on D* if it is both practically attractive on D and practically stabilizable. Moreover, if $D = \mathbb{R}^n$, then it is said to be *globally practically asymptotically stabilizable*. □

Remark 5: For a practically asymptotically stabilizable system, given an $\epsilon > 0$, we can find a $\delta > 0$ as in Definition 2 and a switching law $\mathcal{S}_1$ that keeps every trajectory starting in $B[0,\delta]$ to be within $B[0,\epsilon]$. Furthermore, we can find a switching law $\mathcal{S}_0$ that can drive the state trajectory starting at every $x(0) \in D$ into $B[0,\delta]$ at some time instant $T\big(x(0)\big)$. Concatenating $\mathcal{S}_0$ and a time shifted version of $\mathcal{S}_1$ (shifted by $T\big(x(0)\big)$ for each $x(0)$), we can then obtain a switching law $\mathcal{S}$ that can bring the trajectory into and keep it within $B[0,\epsilon]$. This resembles the trajectory of a conventional asymptotically stable system. However, for a switched system, we can only keep the trajectory to be within certain ϵ-bound; yet we may not be able to drive the trajectory asymptotically to 0 as $t \rightarrow \infty$ (even though ϵ can be chosen to be very small). □

III. Sufficient Conditions for Practical Asymptotic Stabilizability

Methods using energy functions similar to the conventional Lyapunov function methods can be applied to determine the practical attractivity of system (1). In the sequel, we call a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ an *energy function* if it is positive definite, i.e., $V(0) = 0$ and $V(x) > 0$, $\forall x \ne 0$. An energy function V is said to be *radially unbounded* if $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ (see [3]).

Theorem 1: Assume that an energy function $V(x)$ is given for switched system (1). Also assume that a $\rho > 0$ is given and the set $\Omega_\rho = \{x \in \mathbb{R}^n : V(x) \le \rho\}$ is bounded.

Switched system (1) is practically asymptotically stable on Ω_ρ if the following two conditions are satisfied:

(a). For any $x \in \Omega_\rho - \{0\}$,

$$\min_{i\in I} \frac{\partial V}{\partial x} f_i(x) < 0. \quad (2)$$

(b). $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{f_i(0) : i \in I\}\big)$.

Proof: Due to Lemma 1, condition (b) guarantees that switched system (1) is practically stabilizable. Hence, given any $\epsilon > 0$, there exist a $\delta > 0$ and a switching law $\mathcal{S}_1$ such that $x(t) \in B[0,\epsilon], \forall t \geq 0$ whenever $x(0) \in B[0,\delta]$. Next we will construct a switching law $\mathcal{S}_0$ such that the system is δ-practically attractive on Ω_ρ under $\mathcal{S}_0$.

Let $\alpha = \min_{\|x\|=\delta} V(x)$, then $\alpha > 0$ since V is positive definite. Choose $\beta \in (0,\alpha)$ small enough such that $\beta < \min_{x\in\Omega_\rho \text{ and } \|x\|\geq\delta} V(x)$. Let $\Omega_\beta = \{x \in \mathbb{R}^n : V(x) \leq \beta\}$. Then $\Omega_\beta \subseteq B(0,\delta)$. Note that the local Lipschitz continuity of all $f_i(x)$'s leads to the continuity of all functions $h_i(x) \triangleq \frac{\partial V}{\partial x} f_i(x)$, $i \in I$ and the function $H(x) \triangleq \min_{i\in I} \frac{\partial V}{\partial x} f_i(x) = \min_{i\in I} h_i(x)$ on Ω_ρ. Since the set $\Omega_{\beta,\rho} \triangleq \Omega_\rho - \text{Int}(\Omega_\beta) = \{x \in \mathbb{R}^n : \beta \leq V(x) \leq \rho\}$ is compact, there exists a $\gamma > 0$ such that $\max_{x\in\Omega_{\beta,\rho}} H(x) = -\gamma$.

Because $h_i(x)$'s are uniformly continuous on the compact set $\Omega_{\beta,\rho}$, there exists a $r_1 > 0$ such that $|h_i(x_a) - h_i(x_b)| \leq \frac{\gamma}{2}$, $\forall i \in I$ whenever $x_a, x_b \in \Omega_{\beta,\rho}$ and $\|x_a - x_b\| \leq r_1$.

Now let us construct a valid switching law to achieve δ-practical attractivity. Let $L \triangleq \max_{i\in I}\{\max_{x\in\Omega_{\beta,\rho}} \|f_i(x)\|\}$. Define $T_d \triangleq \frac{r_1}{L}$. The switching law is as follows.

<u>Switching Law $\mathcal{S}_0$</u> (for system (1) with $x(0) \in \Omega_{\beta,\rho}$):

(1). Set $k = 0$.

(2). Repeat the following step until at some time instant $t \in [0,\infty)$, the state trajectory intersects Ω_β:

(2a). At $t_k = kT_d$, let subsystem $i_k = \arg\min_{i\in I} h_i\big(x(t_k)\big)$ be active in the time interval $[kT_d, (k+1)T_d)$. At time instant $(k+1)T_d$, set $k = k+1$.

This switching law will drive the state trajectory to intersect $\Omega_\beta \subseteq B(0,\delta)$ in finite time for any initial $x(0) \in \Omega_{\beta,\rho}$. We show this by contradiction as follows. Assume that the trajectory never intersects Ω_β. Then at any time instant kT_d, we must have $h_{i_k}\big(x(kT_d)\big) = H\big(x(kT_d)\big) \leq -\gamma$ due to step (2a). Moreover, from the definition of T_d, we conclude that $\|x(t) - x(kT_d)\| = \|\int_{kT_d}^t f_{i_k}\big(x(\tau)\big)\,d\tau\| \leq r_1$, $\forall t \in [kT_d, (k+1)T_d)$. This leads to $\|h_{i_k}\big(x(t)\big) - h_{i_k}\big(x(kT_d)\big)\| \leq \frac{\gamma}{2}$, $\forall t \in [kT_d, (k+1)T_d)$. Consequently

$$h_{i_k}\big(x(t)\big) \leq -\frac{\gamma}{2},\ \forall t \in [kT_d, (k+1)T_d) \quad (3)$$

(which also implies that $V\big(x(t)\big)$ decreases as t increases).

Due to (3), we have for any $t \in [KT_d, (K+1)T_d)$,

$$V\big(x(t)\big) = V\big(x(0)\big) + \sum_{k=0}^{K-1} \int_{kT_d}^{(k+1)T_d} h_{i_k}\big(x(\tau)\big)\,d\tau + \int_{KT_d}^{t} h_{i_K}\big(x(\tau)\big)\,d\tau \leq V\big(x(0)\big) - \frac{\gamma}{2}t. \quad (4)$$

In (4), the right-hand side will eventually become negative if t is large enough. This leads to a contradiction to the positive definiteness of $V(x)$. Hence the state trajectory must intersect Ω_β in finite time.

Finally, we can concatenate $\mathcal{S}_0$ and a time shifted version of $\mathcal{S}_1$ to obtain a switching law $\mathcal{S}$ which practically asymptotically stabilizes the system (1) on Ω_ρ (see Remark 5). □

In the case when $V(x)$ satisfies (2) for any $x \in \mathbb{R}^n - \{0\}$, we have the following corollary.

Corollary 1: Assume that a radially unbounded energy function $V(x)$ is given for switched system (1). Switched system (1) is globally practically asymptotically stable if the following two conditions are satisfied:

(a). $\min_{i\in I} \frac{\partial V}{\partial x} f_i(x) < 0,\ \forall x \in \mathbb{R}^n - \{0\}$.

(b). $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{f_i(0) : i \in I\}\big)$.

Proof: The radial unboundedness of $V(x)$ leads to the boundedness of Ω_ρ for any $\rho > 0$. Hence, by Theorem 1, under condition (a), the system is practically asymptotically stabilizable on Ω_ρ for any $\rho > 0$. □

Remark 6: A Lyapunov function in conventional Lyapunov stability theory for a switched system with a common equilibrium at the origin is an energy function with $\dot{V}(x)$ being negative definite (see [1], [2], [6], [7] for more on conventional stability theory of hybrid and switched systems). The energy function $V(x)$ in Theorem 1, although similar to a Lyapunov function, cannot be called a Lyapunov function because the origin is not necessarily a common equilibrium for all subsystems; hence the conventional stability theory of switched system cannot be directly applied here. However, the close relationship between energy functions and Lyapunov functions naturally leads us to consider Lyapunov functions for some switching dynamics with common equilibrium as candidate energy functions in the exploration of practical asymptotic stabilizability of switched system (1). □

To implement the idea in Remark 6, we first note that any subsystem vector field $f_i(x)$ in (1) can be written as $f_i(x) = f_i(x) - f_i(0) + f_i(0)$. Let us define a first auxiliary switched system

$$\dot{x} = g_i(x) \triangleq f_i(x) - f_i(0),\ i \in I, \quad (5)$$

and a second auxiliary switched system

$$\dot{x} = f_i(0),\ i \in I. \quad (6)$$

Note that the vector fields of the above two switched systems adds up to the vector fields of system (1), i.e., $g_i(x) + f_i(0) = f_i(x)$ for any $i \in I$. In this way, we decompose the vector fields of system (1) into two parts and obtain two corresponding auxiliary switched systems. Switched system (5) has a common equilibrium at the origin for all subsystems and switched system (6) is an integrator switched system.

Remark 7: Although the vector fields of the above two auxiliary switched systems add up to the vector fields of switched system (1), their corresponding state trajectories do not add up to that of system (1). Therefore, in a rigorous sense, (5) and (6) cannot be said to be decompositions of the original switched system (we may only say that they corresponds to the decomposition of the vector fields of the original switched system). However, since the Lyapunov function approach concerns only the vector fields, the two auxiliary systems are of importance in our subsequent studies of practical asymptotic stabilizability. □

Since switched system (5) has a common equilibrium at the origin, conventional Lyapunov stability theory for switched systems can be applied. For such a switched system, if there exists an energy function $V(x)$ whose derivatives along the solutions of all subsystems in (5) are negative definite, then $V(x)$ is called a *common Lyapunov function (CLF)* for (5); if the derivatives along the solutions of all subsystems are negative semidefinite, $V(x)$ is called a *weak CLF*. CLFs play important roles in stability analysis of switched systems in that the existence of CLFs guarantees the stability of switched systems under arbitrary switching laws. For more on CLFs, the reader is referred to [1], [2], [6], [7] and the references therein.

Theorem 2: Given switched system (1), assume that a weak CLF $V(x)$ exists for the corresponding auxiliary switched system (5). Also assume that a $\rho > 0$ is given and the set $\Omega_\rho = \{x \in \mathbb{R}^n : V(x) \leq \rho\}$ is bounded. Switched system (1) is practically asymptotically stable on Ω_ρ if $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{f_i(0) : i \in I\}\big)$. Moreover, if $V(x)$ is radially unbounded, then switched system (1) is globally practically asymptotically stable.

Proof: Since $V(x)$ is a weak CLF for switched system (5), then for any $x \in \Omega_\rho - \{0\}$, we have

$$\frac{\partial V}{\partial x} g_i(x) \leq 0, \ \forall i \in I. \tag{7}$$

On the other hand, due to the condition $0 \in \text{Int}(C)$, we can conclude that at every $x \in \Omega_\rho - \{0\}$ there exists an $i \in I$ such that $\frac{\partial V}{\partial x} f_i(0) < 0$. This conclusion is shown by contradiction as follows. Assume on the contrary that at some $x \in \Omega_\rho - \{0\}$, we have $\frac{\partial V}{\partial x} f_i(0) \geq 0$ for any $i \in I$. Then $\frac{\partial V}{\partial x} y \geq 0$ for any $y \in C$. This leads to the conclusion that there exists a separating hyperplane between C and $\{0\}$, which is contradictory to the condition $0 \in \text{Int}(C)$.

At any $x \in \Omega_\rho - \{0\}$, since there exists an $i \in I$ such that $\frac{\partial V}{\partial x} f_i(0) < 0$, we furthermore have

$$\min_{i \in I} \frac{\partial V}{\partial x} f_i(0) < 0. \tag{8}$$

In view of (7) and (8), we then conclude that for every $x \in \Omega_\rho - \{0\}$ and every $i \in I$,

$$\frac{\partial V}{\partial x} f_i(x) = \frac{\partial V}{\partial x} g_i(x) + \frac{\partial V}{\partial x} f_i(0) \leq \frac{\partial V}{\partial x} f_i(0), \tag{9}$$

which consequently leads to

$$\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) \leq \min_{i \in I} \frac{\partial V}{\partial x} f_i(0) < 0. \tag{10}$$

Theorem 1 and Corollary 1 can then be applied to establish the results. □

Remark 8: Note that the condition $\min_{i \in I} \frac{\partial V}{\partial x} g_i(x) < 0$, $\forall x \in \mathbb{R}^n - \{0\}$ in general does not guarantee the asymptotic stabilizability of the auxiliary switched system (5). For general nonlinear subsystem $g_i(x)$'s, the switching law which associates with each x the subsystem $i = \arg\min_{i \in I} \frac{\partial V}{\partial x} g_i(x)$ might cause the undesirable (and invalid) Zenoness or chattering phenomenon. By far most literature results based on the aforementioned condition are derived for switched linear systems where $g_i(x) = A_i x$ (e.g., asymptotic stabilization using state-dependent switching laws). One advantage of our results here is that when (2) holds, we can establish the practical asymptotic stabilizability of switched system (1) and nonZeno switching laws can be explicitly constructed for every ϵ to achieve ϵ-practical attractivity and ϵ-practical stabilizability. □

IV. Switched Affine Systems

When a switched system is practically asymptotically stabilizable, we are often interested in its largest region of attraction. However, finding the exact region of attraction analytically is difficult or impossible even for ordinary nonlinear systems (see [3]). In practice, we often content ourselves by finding a bounded subset of $\mathbb{R}^n$ containing the origin and regard it as an estimate of the region of attraction. The results in Section III can help us find such estimates. In fact, the set Ω_ρ in Theorem 1 is such an estimate. In this section, we focus on switched affine systems, and develop methods for finding estimates of regions of attraction.

Consider switched systems consisting of affine subsystems

$$\dot{x} = f_i(x) = A_i x + b_i, \ A_i \in \mathbb{R}^{n \times n}, b_i \in \mathbb{R}^n, \ i \in I. \tag{11}$$

The vector fields of system (11) can be decomposed into two parts which correspond to two auxiliary switched systems as in (5) and (6). The first one is a switched linear system with a common equilibrium

$$\dot{x} = A_i x, \ i \in I. \tag{12}$$

The second one is an integrator switched system

$$\dot{x} = b_i, \ i \in I. \tag{13}$$

For switched affine systems, we mainly consider quadratic energy function $V(x) = x^T P x$ where $P = P^T$ is a real positive definite matrix. If $PA_i + A_i^T P$ is negative definite for every $i \in I$, $V(x)$ is called a *common quadratic Lyapunov function (CQLF)* for switched system (12); if $PA_i + A_i^T P$ is negative semidefinite for every $i \in I$, $V(x)$ is called a *weak CQLF* for system (12).

Assume that a weak CQLF $V(x)$ exists for system (12). By Theorem 2, if $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{b_i : i \in I\}\big)$, then switched system (11) is globally practically asymptotically stabilizable (since $V(x)$ is radially unbounded). In the following, we assume that a weak CQLF does not exist. In such a case, given any $V(x) = x^T P x$, there exists some $i \in I$ such that $\lambda_{max,i} > 0$ where $\lambda_{max,i}$ is the maximum eigenvalue of $PA_i + A_i^T P$.

The derivative of $V(x)$ along the trajectory of the i-th subsystem is

$$\frac{\partial V}{\partial x} f_i(x) = x^T (PA_i + A_i^T P) x + 2x^T P b_i. \tag{14}$$

For any $x \neq 0$, the second term in (14) can be represented as $2x^T P b_i = \|x\| \cdot 2\big(\frac{x}{\|x\|}\big)^T P b_i = \|x\| \cdot 2y^T P b_i$ where $y \triangleq \frac{x}{\|x\|}$ (and $\|y\| = 1$). If we define $-\eta \triangleq$

$\max_{\|y\|=1} \min_{i\in I} 2y^T Pb_i$, then we have $\min_{i\in I} 2x^T Pb_i \leq -\eta\|x\|$. Note that $\eta > 0$ since $h(y) = \min_{i\in I} 2y^T Pb_i$ is continuous and negative on the compact set $\{y \in \mathbb{R}^n : \|y\| = 1\}$. Here the negativeness of $h(y)$ for any $\|y\| = 1$ can be shown by contradiction as follows. Assume on the contrary that at some y ($\|y\| = 1$), we have $2y^T Pb_i \geq 0$ for every $i \in I$. Then $2y^T Pz \geq 0$ for any $z \in C$. This implies that there exists a separating hyperplane between C and $\{0\}$, which is contradictory to $0 \in \text{Int}(C)$.

Now we return to (14). If we define $\lambda_{max} \triangleq \max_{i\in I} \lambda_{max,i}$, we then have

$$\min_{i\in I} \frac{\partial V}{\partial x} f_i(x) = \min_{i\in I}\{x^T(PA_i + A_i^T P)x + 2x^T Pb_i\}$$
$$\leq \min_{i\in I}\{\lambda_{max,i}\|x\|^2 + 2x^T Pb_i\} \leq \min_{i\in I}\{\lambda_{max}\|x\|^2 + 2x^T Pb_i\}$$
$$\leq \lambda_{max}\|x\|^2 + \min_{i\in I} 2x^T Pb_i \leq \lambda_{max}\|x\|^2 - \eta\|x\|. \quad (15)$$

In view of (15), $\min_{i\in I} \frac{\partial V}{\partial x} f_i(x) < 0$ holds true for every $0 < \|x\| < \frac{\eta}{\lambda_{max}}$. Thus by Theorem 1, any $\Omega_\rho = \{x \in \mathbb{R}^n : x^T Px \leq \rho\}$ is an estimate of the region of attraction if $\rho < \min_{\|x\|=\frac{\eta}{\lambda_{max}}} V(x)$.

Through the above discussions, we have in fact proved

Theorem 3: Assume that a switched affine system (11) is given and $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{b_i : i \in I\}\big)$. Also assume that a quadratic energy function $V(x) = x^T Px$ with $P = P^T > 0$ is given. Then the following conclusions hold.

(i). If $V(x)$ is a weak CQLF for the corresponding auxiliary switched linear system (12), then the switched affine system (11) is globally practically asymptotically stabilizable.

(ii). If $V(x)$ is not a weak CQLF for system (12), then the switched affine system (11) is practically asymptotically stabilizable on any $\Omega_\rho = \{x \in \mathbb{R}^n : x^T Px \leq \rho\}$ with $\rho < \min_{\|x\|=\frac{\eta}{\lambda_{max}}} V(x)$. Here $\eta \triangleq -\max_{\|y\|=1} \min_{i\in I} 2y^T Pb_i$ and $\lambda_{max} = \max_{i\in I} \lambda_{max,i}$, where $\lambda_{max,i}$ is the maximum eigenvalue of $PA_i + A_i^T P$. □

Remark 9: For switched affine systems (11) satisfying $0 \in \text{Int}(C)$, any quadratic function $V(x) = x^T Px$ with $P = P^T > 0$ can serve as an energy function and can be used to establish the practical asymptotic stabilizability of (11) on some region around the origin. This is true even when the corresponding auxiliary switched linear system (12) is not asymptotically stabilizable. Such a result is quite different from conventional Lyapunov stability results for system (12). For (12), it is possible that not every quadratic function can be a Lyapunov function or even no quadratic function can be a Lyapunov function.

In view of this, we find that the condition $0 \in \text{Int}(C)$ plays an important role in establishing the practical asymptotic stabilizability of switched affine system (11) in neighborhoods near the origin. In other words, the vector fields of the integrator switched system (13) determines the local behavior of system (11) around the origin. This explains why our previous results (see [8], [9], [10]) on practical stabilizability of switched systems have close relationship with the vector fields of the integrator switched system (6). □

Remark 10: Although the vector fields of the auxiliary integrator switched system (13) determine the local behavior of system (11) around the origin, the vector fields of both auxiliary switched systems (12) and (13) determine the region of attraction. This is evident from Theorem 3. In particular, the size of Ω_ρ is related to λ_{max}, which is relevant to the vector fields of system (12); while η is relevant to the vector fields of the integrator switched system (13). □

Remark 11: The choice of P will also affect λ_{max}, which in turn will affect the estimate of the region of attraction. In the calculations of the examples in this section, we simply choose $P = I$. It should be noted that for different P matrices, the estimates can be different. In fact, the union of all such estimates is a better estimate. □

Example 1: Consider a switched affine system (11) in $\mathbb{R}^2$ consisting of 3 subsystems $\dot{x} = A_i x + b_i$, $i = 1, 2, 3$, where $A_1 = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $A_2 = \begin{bmatrix} -0.1 & 0.5 \\ -0.5 & 0.1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $A_3 = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.2 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. There does not exist any weak CQLF for the corresponding auxiliary switched linear system since $\dot{x} = A_1 x$ and $\dot{x} = A_3 x$ are unstable (a necessary condition for the existence of a weak CLF is that all subsystems are stable). If we choose $V(x) = x^T x$, then computation gives us $\lambda_{max} = 1$ and $\eta = 0.8944$. Hence the switched system is practically asymptotically stabilizable on $B(0, 0.8944)$. Given $\epsilon = 0.1$, a switching law can be constructed to bring the trajectory into $B[0, \epsilon]$ and keep it within $B[0, \epsilon]$ (see the proof of Theorem 1). Fig. 1 shows $x_1(t)$ and $x_2(t)$ generated by such a switching law (with $x_1(0) = 0.6$, $x_2(0) = 0.5$). □

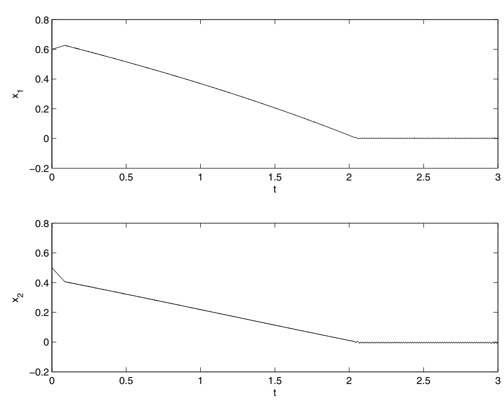

Fig. 1. $x_1(t)$ and $x_2(t)$ generated by the switching law (with initial condition $x_1(0) = 0.6$ and $x_2(0) = 0.5$).

In Theorem 3, when $V(x)$ is not a weak CQLF, practical asymptotic stabilizability can only be established on a bounded region Ω_ρ. This is sometimes quite restrictive. In many cases in which no weak CQLF exists for the auxiliary switched linear system (12), global practical asymptotic stabilizability can still be established. In the following, we present such a case.

Assume that a quadratic energy function $V(x) = x^T Px$

is given. Also assume that there exists an $\epsilon_0 > 0$ such that

$$\min_{i \in I} x^T (PA_i + A_i^T P)x \leq -\epsilon_0 \|x\|^2, \ \forall x \neq 0. \tag{16}$$

The condition in (16) is closely related to quadratic stabilizability of switched linear systems (for more details on quadratic stabilization, see [7]). Note that such an ϵ_0 may exist even if A_i is not stable since the minimum is taken.

In this case, if we define

$$\zeta \stackrel{\triangle}{=} \max_{i \in I} 2\|Pb_i\|, \tag{17}$$

then we have

$$2x^T Pb_i \leq \|x\| \cdot 2\|Pb_i\| \leq \zeta \|x\|, \ \forall i \in I. \tag{18}$$

In view of (16) and (18), we have

$$\begin{aligned} &\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) = \min_{i \in I} \{x^T (PA_i + A_i^T P)x + 2x^T Pb_i\} \\ &\leq \ \min_{i \in I} \{x^T (PA_i + A_i^T P)x + \zeta \|x\|\} \\ &\leq \min_{i \in I} \{x^T (PA_i + A_i^T P)x\} + \zeta \|x\| \leq -\epsilon_0 \|x\|^2 + \zeta \|x\|. \end{aligned} \tag{19}$$

From (19), if $\|x\| > \frac{\zeta}{\epsilon_0}$, then $\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) < 0$. Combining this with the result (ii) in Theorem 3, we obtain

Theorem 4: Assume that a switched affine system (11) is given and assume $0 \in \text{Int}(C)$ where $C = \text{conv}\big(\{b_i : i \in I\}\big)$. Also assume that a quadratic energy function $V(x) = x^T Px$ with $P = P^T > 0$, which is not a weak CQLF for the corresponding auxiliary switched linear system (12), is given such that $\min_{i \in I} x^T (PA_i + A_i^T P)x \leq -\epsilon_0 \|x\|^2, \ \forall x \neq 0$. Then the switched affine system (11) is globally practically asymptotically stabilizable if $\frac{\zeta}{\epsilon_0} < \frac{\eta}{\lambda_{max}}$. Here η, λ_{max}, and ζ are defined in Theorem 3, and (17), respectively.

Proof: For every $x \neq 0$, since $\frac{\zeta}{\epsilon_0} < \frac{\eta}{\lambda_{max}}$, at least one of (15) and (19) can be applied to show $\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) < 0$ (apply (15) when $\|x\| < \frac{\eta}{\lambda_{max}}$ and (19) when $\|x\| > \frac{\zeta}{\epsilon_0}$). □

Remark 12: If (16) is satisfied, then by (19), $\min_{i \in I} \frac{\partial V}{\partial x} f_i(x) < 0$ can be established outside some ball around the origin. In other words, the system trajectory can be driven toward (i.e., attracted toward) the origin outside that ball by appropriate switchings. This indicates that the behavior of the system trajectory far from the origin is mainly determined by the vector fields of the corresponding auxiliary switched linear system (12). Theorem 4 simply says that if the "attracting regions" determined by (15) and (19) intersects, then the whole $\mathbb{R}^n$ is covered by attracting region. □

Example 2: Consider a switched affine system (11) in $\mathbb{R}^2$ consisting of 3 subsystems $\dot{x} = A_i x + b_i$, $i = 1, 2, 3$, where $A_1 = \begin{bmatrix} 0.1 & 0.2 \\ -0.01 & 0.1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $A_2 = \begin{bmatrix} 0.1 & 0.01 \\ -0.2 & 0.1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $A_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.
There does not exist any weak CQLF for the corresponding auxiliary switched linear system since $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ are unstable. However, we can still prove that the system is globally practically stabilizable by Theorem 4. For this example, we choose $V(x) = x^T x$. Computation gives us $\lambda_{max} = 0.3900$, $\epsilon_0 = 2$, $\eta = 0.8944$, $\zeta = 2.8284$, and therefore $\frac{\zeta}{\epsilon_0} = 1.4142 < 2.2933 = \frac{\eta}{\lambda_{max}}$. Given $\epsilon = 0.1$, a switching law can be constructed to bring the trajectory into $B[0, \epsilon]$ and keep it within $B[0, \epsilon]$ (see the proof of Theorem 1). Fig. 2 shows $x_1(t)$ and $x_2(t)$ generated by such a switching law (with $x_1(0) = 2$, $x_2(0) = 2$). □

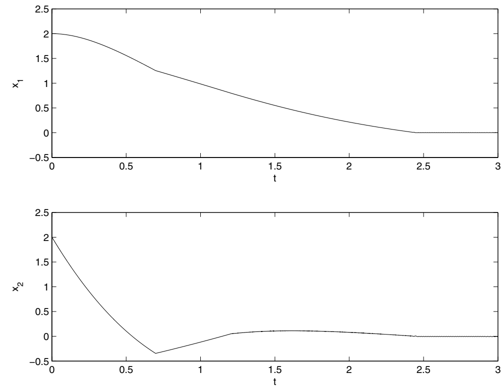

Fig. 2. $x_1(t)$ and $x_2(t)$ generated by the switching law (with initial condition $x_1(0) = 2$ and $x_2(0) = 2$).

V. Conclusion

This paper reports some new results on practical asymptotic stabilizability of switched systems consisting of time-invariant subsystems. Sufficient conditions based on energy functions have been presented. Decomposition of the system vector fields has been proposed. Based on such decompositions, more sufficient conditions have been proposed. In particular, we present methods for estimating the region of attraction for switched affine systems. Future research includes studies of the practical asymptotic stabilizability of switched systems with time-varying subsystems and applications in asymptotic tracking problems.

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