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Vertically transverse functions as an extension of the transverse function control approach for second-order systems

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Abstract— **In recent joint work, P. Morin and C. Samson have advocated the use of transverse functions on tori to design practical stabilizers (even asymptotic stabilizers whenever possible) and tracking controllers for driftless, control-affine systems. By design, those stabilizers and controllers retain their performance even in the presence of known disturbance drift terms and, moreover, they exhibit additional robustness features that derive from their smoothness in terms of the state. As yet, however, no systematic procedure has been devised to address the same problem in the case of systems whose *drift* vector fields are *required* to ensure local accessibility, including, for instance, second-order and simple mechanical systems. In this paper we show that one way to extend the usefulness of transverse functions to the realm of second-order systems is by taking their associated tangent mappings, a process that results in "vertically transverse functions." We believe that, although work remains to be done in this direction, this extension provides an initial step towards a more thorough theory that addresses both practical stabilization and tracking for such systems. Included is an example that illustrates how these ideas may be put to use for control purposes.**

I. INTRODUCTION

The transverse function formalism of [8], [9], [10] provides a unified treatment of practical stabilization and trajectory tracking for control-affine systems whose *control* vector fields ensure local accessibility. The transverse function approach may be tailored to control systems of the form

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u^i X_i(x), \tag{1}$$

with $X_0, X_1, \ldots, X_m$ smooth vector fields on an n-dimensional, smooth, connected manifold M, such that the Lie algebra $\mathrm{Lie}(\boldsymbol{X})$ generated by $\boldsymbol{X} = \{X_1, \ldots, X_m\}$ spans T_pM at a given point $p \in M$ (i.e., $\boldsymbol{X}$ satisfies the "Lie algebra rank condition at p"). Included in this class are controllable *driftless* systems subject to (possibly null) additive disturbances represented by the drift term X_0, such as the so-called 3-state, 2-input Chained Form (CF) $\dot{x} = u^1(\frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}) + u^2 \frac{\partial}{\partial x^2}$.

Feedback laws derived from the transverse function approach allow one to circumvent some difficulties linked to the control of "critical" systems, i.e., systems that do not meet (generalizations of) Brockett's necessary conditions for asymptotic stabilization. For instance, they address stabilization of equilibria and more general trajectories, the trade-off being the replacement of convergence to the desired value by convergence to a given *neighborhood* of that value (practical stability/tracking). This agrees with results in [5], which point out that constructing "universal" controllers that stabilize arbitrary system trajectories seems hopeless for some classes of systems, among which the CF. The approach also produces feedback laws that ensure practical stabilization of "non-feasible" trajectories and, moreover, those feedback laws are typically *smooth*, which prevents them from exhibiting some nonrobustness issues, for instance the ones alluded to in [6]. More recently, the transverse function approach was enhanced in [10] to produce controllers that ensure *asymptotic* stabilization to a point whenever this is allowed by the drift vector field.

In this paper we take initial steps towards extending this formalism to second-order systems, in particular those defined on (tangent) Lie groups, under the hypothesis that their *drift* vector fields play a fundamental role in guaranteeing local accessibility. In particular, we consider second-order systems of the form (1) under the assumption that, whereas the set $\widetilde{\boldsymbol{X}} = \{X_0, X_1, \ldots, X_m\}$ satisfies the rank condition at $p \in M$, $\boldsymbol{X}$ itself *no longer does*. Examples include underactuated mechanical systems evolving on Lie groups, such as underactuated manipulators, blimp-like systems and underwater vehicles. The principle underlying our results is that the tangent functor, as applied to manifolds and mappings involved in the constructions by Morin and Samson, yields what may be termed "vertically transverse functions," which extend the notion of transverse functions on tori introduced in [8]. Our presentation of the theoretical constructs is entirely coordinate-free; nevertheless, an example with computations in coordinates is included in Section VI for concretion. Due to space limitations, the proofs and several details are omitted.

II. PRELIMINARY NOTIONS

A. Basic concepts

Given a smooth manifold Q, and its first and second tangent bundles TQ and TTQ, we let $\pi_Q : TQ \longrightarrow Q$ and $\pi_{TQ} : TTQ \longrightarrow TQ$ denote their respective bundle projections. The tangent space to Q at a point $q \in Q$ is denoted T_qQ. Given smooth manifolds Q, P and a mapping $f : Q \longrightarrow P$, we write $T_qf : T_qQ \longrightarrow T_{f(q)}P$ for the tangent mapping of f at q and Tf for the respective bundle map covering f. If q is clear from the context, we sometimes use the notation $Tf(v)$ instead of $T_qf(v)$. The sets of *smooth*

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vector fields on Q and on TQ will be denoted by $\Gamma(TQ)$ and $\Gamma(TTQ)$, respectively. For simplicity we shall frequently write X_q instead of $X(q)$ for the value of a vector field X at a point q.

B. Second-order and vertical constructs

More details on the concepts recalled in this section can be found in e.g. [1], [7]. A vector field $X \in \Gamma(TTQ)$ is said to be a ***second-order vector field*** (one also says that "X defines a second-order equation on Q" or simply that "X is second order") if $T\pi_Q \circ X = \mathrm{id}_{TQ}$. This definition extends naturally to vector fields along a curve in TQ, namely, if $\gamma : (t_0, t_1) \longrightarrow TQ$ is a curve and X is given by $X_{\gamma(t)} = T_t\gamma\left(\frac{\partial}{\partial r}\big|_t\right)$, then X is said to be ***second-order along*** γ if for every $t \in (t_0, t_1)$, $T\pi_Q(X_{\gamma(t)}) = \gamma(t)$. Associated with any such curve γ is the corresponding ***base curve*** $\pi_Q \circ \gamma : (t_0, t_1) \longrightarrow Q$. Given $v \in TQ$, the ***vertical space over*** v is $\ker(T_v\pi_Q)$, a subspace of T_vTQ which we denote by $(T_vTQ)^{\text{vert}}$. The disjoint union of vertical spaces over points in TQ inherits a natural structure that makes it a vector subbundle of TTQ, called the vertical subbundle TTQ^{vert}. A section $X \in \Gamma(TTQ^{\text{vert}})$ of this subbundle is called a ***vertical vector field***. A subbundle being a constant-dimensional distribution, TTQ^{vert} is also referred to as the *vertical distribution on* TQ. Given tangent vectors $v, w \in TQ$ such that $\pi_Q(v) = \pi_Q(w)$, one defines the ***vertical lift of*** w ***by*** v as the vector in T_vTQ given by $\mathrm{lift}(v, w) = T_0\gamma_{v,w}\left(\partial/\partial r|_0\right)$, where $\gamma_{v,w} : \mathbb{R} \longrightarrow TQ$ is the curve determined by $\gamma_{v,w}(t) = v + tw$. Given a vector field $X \in \Gamma(TQ)$, the ***vertical lift of*** X is the vector field $X^{\text{lift}} \in \Gamma(TTQ)$ defined by $X^{\text{lift}}_v = \mathrm{lift}(v, X_{\pi_Q(v)})$. System (1), under the assumption that $X_0, X_1, \ldots, X_m \in \Gamma(TTQ)$, is said to be a ***second-order (control-affine) system on*** TQ if $X_0 + \sum_{i=1}^m u^i X_i$ is a second-order vector field for every $u = (u^1, \ldots, u^m) \in \mathbb{R}^m$. One easily checks that if (1) is second order, then X_0 is itself second order whereas the vector fields $X_1, \ldots, X_m$ are vertical.

C. Tangent Lie groups

Given an n-dimensional Lie group G, we consider its associated *tangent group* TG (cf. [7, Chap. 9]) with Lie group structure determined by the multiplication $\mu : TG \times TG \longrightarrow TG$ given by

$$\mu(v, w) = T_{\pi_G(w)}\widehat{L}_{\pi_G(v)}(w) + T_{\pi_G(v)}\widehat{R}_{\pi_G(w)}(v), \qquad (2)$$

where $\widehat{L}_g : h \mapsto gh$ and $\widehat{R}_g : h \mapsto hg$ denote left and right translations on G, respectively. Endowed with this structure, TG admits 0_e (the zero vector in T_eG) as identity element. In the sequel, when both a Lie group G and its tangent group TG are involved in a discussion, we systematically use $\widehat{R}, \widehat{L}$ to denote translations on G, and R, L to represent translations on TG.

III. Transverse functions for driftless systems

In this section we recall the results of [8] on the existence and construction of transverse functions for driftless systems. Consider a set of vector fields $\boldsymbol{X} = \{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ and a point $p \in Q$ such that $\{Y_p : Y \in \mathrm{Lie}(\boldsymbol{X})\} = T_pQ$. It follows from [8, Thm. 1] that, given a neighborhood U of p, there exist an integer $\kappa \geq n - m$ and a *transverse function* $f : \mathbb{T}^\kappa \longrightarrow Q$ that satisfies $f(\mathbb{T}^\kappa) \subset U$ and, for every $\theta \in \mathbb{T}^\kappa$,

$$T_{f(\theta)}Q = T_\theta f(T_\theta\mathbb{T}^\kappa) + \mathrm{span}_{\mathbb{R}}\{X_{1,f(\theta)}, \ldots, X_{m,f(\theta)}\}, \qquad (3)$$

where $\mathbb{T}^\kappa$ denotes the κ-torus $(\mathbb{R}/2\pi\mathbb{Z})^\kappa$. In the sequel we shall refer to any such mapping as ***a Morin–Samson function for X (near*** p***)***. Note that, while in general the sum in (3) is not direct, i.e., κ need not equal $n - m$, in some cases f can be chosen so that it is, for instance when $Q = G$ is an n-dimensional Lie group and the vector fields $X_1, \ldots, X_m$ are left-invariant. In the latter case, the explicit construction in [9] of a transverse function can be easily detailed as follows. Let $\xi_1, \ldots, \xi_m$ be elements of $\mathfrak{g}$, the Lie algebra of G, such that $\mathrm{Lie}(\{\xi_1, \ldots, \xi_m\}) = \mathfrak{g}$ and assume that $X_{i,g} = T_e\widehat{L}_g(\xi_i)$ for $i = 1, \ldots, m$ and $g \in G$. Define inductively a family $(G_k)_{k\in\mathbb{N}}$ of subspaces of $\mathfrak{g}$ by setting $G_0 = \mathrm{span}_{\mathbb{R}}\{\xi_1, \ldots, \xi_m\}$ and $G_k = G_{k-1} + [G_0, G_{k-1}]$ for $k \geq 1$. Then consider mappings $\lambda, \rho : \{m+1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and an ordered basis $\{\zeta_1, \ldots, \zeta_n\}$ of $\mathfrak{g}$ such that **(1)** $G_k = \mathrm{span}_{\mathbb{R}}\{\zeta_1, \ldots, \zeta_{\dim(G_k)}\}$ for $k = 1, \ldots, \min\{k : G_k = \mathfrak{g}\}$; and **(2)** Whenever $k \geq 2$ and $\dim(G_{k-1}) \leq i \leq \dim(G_k)$, one has $\zeta_i = [\zeta_{\lambda(i)}, \zeta_{\rho(i)}]$, with $\zeta_{\lambda(i)} \in G_a$, $\zeta_{\rho(i)} \in G_b$ and $a + b = k$. The set $\{\zeta_1, \ldots, \zeta_n\}$, together with the mappings λ and ρ, constitute a ***graded basis*** of $\mathfrak{g}$. With such basis one associates an n-tuple $(r_1, \ldots, r_n)$, referred to as a ***weight vector***, by requiring that $r_i = k$ iff $\zeta_i \in G_k \setminus G_{k-1}$. Given a graded basis, the construction of the transverse function proceeds by selecting strictly positive reals $\varepsilon_{m+1}, \ldots, \varepsilon_n$ and by defining mappings $f_i : \mathbb{T} \longrightarrow G$ $(i = m+1, \ldots, n)$ as follows:

$$f_i(\theta) = \exp\left(\varepsilon_i^{r_{\lambda(i)}}\sin(\theta)\zeta_{\lambda(i)} + \varepsilon_i^{r_{\rho(i)}}\cos(\theta)\zeta_{\rho(i)}\right).$$

With these mappings at hand, a Morin–Samson function $f : \mathbb{T}^{n-m} \longrightarrow G$ is then obtained by setting

$$f(\theta_{m+1}, \ldots, \theta_n) = f_n(\theta_n)f_{n-1}(\theta_{n-1})\cdots f_{m+1}(\theta_{m+1}).$$

IV. "Vertically transverse functions" for second-order systems

In this section we show how tangent mappings of Morin–Samson functions for driftless systems define "vertically-transverse functions," which may be regarded as second-order generalizations of transverse functions for second-order systems. Let Q be a manifold (referred to as the *configuration manifold*, by analogy with the case of mechanical systems) and let $p \in Q$. Starting with a set of vector fields $\{X_1, \ldots, X_m\} \subset \Gamma(TQ)$ that satisfies the Lie algebra rank condition at p, we define a "lifted" system on TQ by selecting a second-order vector field $Z \in \Gamma(TTQ)$ and by considering

$$\dot{v} = Z_v + \sum_{i=1}^m u^i X^{\text{lift}}_{i,v}. \qquad (4)$$

The approach we follow proceeds by assuming that the *target system* (the system under control) is of the form (4). We remark that, even though this assumed structure for the target system somewhat restricts the applicability of this approach, it encompasses a large class of second-order and mechanical systems, in particular the class of simple mechanical systems (fully actuated or underactuated) possibly subject to constraints (see e.g. [4]). The ultimate goal will be to provide control laws for the target system (4) building upon the properties of any Morin–Samson function f for $\{X_1,\ldots,X_m\}$, a function whose existence is guaranteed in view of the stated assumptions. The first of our main results in this vein states that the tangent map Tf satisfies a condition that somehow extends (3), namely, that along the image of Tf, the image of the vertical subbundle $(TT\mathbb{T}^\kappa)^{\text{vert}}$ by TTf, together with the distribution spanned by the lifted control vector fields $\{X_1^{\text{lift}},\ldots,X_m^{\text{lift}}\}$, generate the vertical subbundle of TTQ over $Tf(T\mathbb{T}^\kappa)$. This is made precise in the following proposition.

Proposition 1 *Let $\boldsymbol{X}=\{X_1,\ldots,X_m\}\subset\Gamma(TQ)$ satisfy the Lie algebra rank condition at a point $p\in Q$, and let $f:\mathbb{T}^\kappa\longrightarrow Q$ be a Morin–Samson function for $\boldsymbol{X}$ near p. Then, for every $\omega\in T\mathbb{T}^\kappa$,*

$$\begin{aligned} T_{Tf(\omega)}TQ^{\text{vert}} &= T_\omega Tf((T_\omega T\mathbb{T}^\kappa)^{\text{vert}}) + \\ &\quad \operatorname{span}_{\mathbb{R}}\{X^{\text{lift}}_{1,Tf(\omega)},\ldots,X^{\text{lift}}_{m,Tf(\omega)}\}. \end{aligned} \tag{5}$$

Moreover, if f is such that the sum in (3) is direct, then the sum in (5) is direct as well.

We are following [8] in calling Tf "vertically *transverse*," although, as pointed out therein, this rather liberal use of the adjective "transverse" is merely motivated by the formal similarity between conditions (3) and (5) and the condition required in the definition of transverse mapping in differential topology (cf e.g. [2, Chap. 3]).

V. Applications of vertically transverse functions to control

In this section we take a step forward in the direction of using our generalization for control purposes, albeit we hasten to mention that this subject is open and research is still in progress. As a first example of how "vertically transverse" functions may be applied for control, suppose that the configuration manifold G is a Lie group and that the vector fields $X_i\in\Gamma(TG)$ $(i=1,\ldots,m)$ are obtained by left-translating vectors $\xi_i\in\mathfrak{g}$ that satisfy $\operatorname{Lie}(\{\xi_1,\ldots,\xi_m\})=\mathfrak{g}$. In this case, using the procedure recalled in Section III, for any open set $U\subset G$ one can define a Morin–Samson function $f:\mathbb{T}^{n-m}\longrightarrow U$ and, from Proposition 1, we see that Tf satisfies, for every $\omega\in T\mathbb{T}^{n-m}$,

$$\begin{aligned} T_{Tf(\omega)}TG^{\text{vert}} &= T_\omega Tf((T_\omega T\mathbb{T}^{n-m})^{\text{vert}}) \oplus \\ &\quad \operatorname{span}_{\mathbb{R}}\{X^{\text{lift}}_{1,Tf(\omega)},\ldots,X^{\text{lift}}_{m,Tf(\omega)}\}. \end{aligned} \tag{6}$$

Mimicking the procedure described in [9], we extend system (4) dynamically by selecting a global frame for $(TT\mathbb{T}^{n-m})^{\text{vert}}$, that is, a set $\{\Omega_1,\ldots,\Omega_{n-m}\}\subset\Gamma(T\mathbb{T}^{n-m})$ such that $\operatorname{span}_{\mathbb{R}}\{\Omega_{1,\omega},\ldots,\Omega_{n-m,\omega}\}=(T_\omega T\mathbb{T}^{n-m})^{\text{vert}}$ for all $\omega\in T\mathbb{T}^{n-m}$. The existence of a global frame is guaranteed by the triviality of $TT\mathbb{T}^{n-m}$ as a vector bundle, which is easily established. Then pick a second-order vector field $\Delta\in\Gamma(TT\mathbb{T}^{n-m})$ and define the *auxiliary system*

$$\dot\omega=\Delta_\omega+\sum_{i=1}^{n-m}w^i\Omega_{i,\omega}. \tag{7}$$

If $\omega:(t_0,t_1)\longrightarrow T\mathbb{T}^{n-m}$ and $w:(t_0,t_1)\longrightarrow\mathbb{R}^{n-m}$ are functions of classes C^1 and C^0, respectively, which satisfy the differential equation (7), we shall refer to the couple (ω,w) as an *auxiliary trajectory*. At this point we can define an *error signal* whose intent, intuitively speaking, is "to quantify the deviation of the state v of (4) from the image by Tf of the state ω of (7)." The definition makes use of the Lie group structure on TG determined by μ, thus we set $z=\mu(v,(Tf(\omega))^{-1})$ which, for the sake of simplicity, we write as[1]

$$z=v\cdot Tf(\omega)^{-1}.$$

The natural question arising at this point is whether there exists a general expression for the error dynamics and, if so, what structure it may have. To respond to the question we shall proceed in several simple steps. If (ω,w) is any arbitrary auxiliary trajectory and B is defined along the curve $Tf\circ\omega:(t_0,t_1)\longrightarrow TG$ by

$$B_{Tf\circ\omega(t)}=(Tf\circ\omega)'(t)=T_t(Tf\circ\omega)\left(\left.\frac{\partial}{\partial r}\right|_t\right),$$

then B satisfies a second-order equation. The following result is the key to writing an explicit expression for the error dynamics.

Proposition 2 *Let TG be a tangent Lie group, $A\in\Gamma(TTG)$ a complete, second-order vector field (not necessarily left-invariant) and let B be a second-order vector field defined along a smooth curve $b:(t_0,t_1)\longrightarrow TG$ by $\dot b(t)=B_{b(t)}$. Then **(i)** if $a:(t_0,t_1)\longrightarrow TG$ is an integral curve of A, the curve $c=ab^{-1}$ satisfies, for $t\in(t_0,t_1)$,*

$$\dot c(t)=T_{a(t)}R_{b^{-1}(t)}(A_{a(t)}-T_{b(t)}L_{c(t)}(B_{b(t)})); \tag{8}$$

*and **(ii)** (8) represents a (non-autonomous) second-order differential equation on TG.*

In order to apply Proposition 2, we define curves $a=v$ and $b=Tf\circ\omega$, as well as the corresponding vector fields

$$\begin{aligned} A_v &= Z_v+\sum_{i=1}^m u^iX^{\text{lift}}_{i,v} \\ B_{Tf(\omega)} &= T_\omega Tf\left(\Delta_\omega+\sum_{i=1}^{n-m}w^i\Omega_{i,\omega}\right), \end{aligned}$$

[1] We shall often write $Tf(\omega)$ for $Tf\circ\omega$ and $Tf(\omega)^{-1}$ for $(Tf(\omega))^{-1}$ to simplify the exposition.

which then yield the *error dynamics*

$$\dot{z} = T_v R_{Tf(\omega)^{-1}} \left(Z_v + \sum_{i=1}^{m} u^i X^{\text{lift}}_{i,v} - T_{Tf(\omega)} L_z \circ T_\omega Tf \left(\Delta_\omega + \sum_{i=1}^{n-m} w^i \Omega_{i,\omega} \right) \right).$$

After some simplifications one gets

$$\begin{aligned} \dot{z} &= T_{z\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} (Z_{z\cdot Tf(\omega)} \\ &\quad - T_{Tf(\omega)} L_z \circ T_\omega Tf(\Delta_\omega)) + \\ &\quad T_{z\cdot Tf(\omega)} R_{Tf(\omega)^{-1}} \circ T_{Tf(\omega)} L_z \left(\sum_{i=1}^{m} u^i X^{\text{lift}}_{i,Tf(\omega)} \right. \\ &\quad \left. - \sum_{i=1}^{n-m} w^i T_\omega Tf(\Omega_{i,\omega}) \right). \end{aligned} \tag{9}$$

Equation (9), a second-order equation in view of Proposition 2-**(ii)**, is the form of the error dynamics which we shall use in the sequel. Let us remark that one can use alternative definitions of the error, e.g. $z = Tf(\omega) \cdot v^{-1}$, for which the approach leads to analogous results, *mutatis mutandis*.

Now, *How can one profit from vertical transversality for control purposes?* The answer to this question generalizes the way one uses "transversality" in the approach of Morin and Samson. Indeed, for second-order systems the control inputs can *only* shape the second-order time derivatives of the base trajectories, which amounts to assigning them values in the vertical subbundle. Thus the fact that, as stated in (5), the image of TTf complements the control distribution to span the whole vertical subbundle provides full control over the error system, and this can in turn be used to derive feedback laws that impose any desired error dynamics. The following result makes this statement precise.

Theorem 1 *Given a second-order vector field $S \in \Gamma(TTG)$, there exists a smooth feedback law $\alpha = (\alpha^1, \ldots, \alpha^n) : TG \times T\mathbb{T}^{n-m} \longrightarrow \mathbb{R}^n$ such that the error dynamics (9) with control inputs $u^i(z,\omega) = \alpha^i(z,\omega)$ $(i = 1,\ldots,m)$ and $w^j(z,\omega) = \alpha^{j+m}(z,\omega)$ $(j = 1,\ldots,n-m)$ writes as $\dot{z} = S_z$.*

Notice that, while Theorem 1 ensures that any dynamics can be imposed on the error signal, one cannot say much about the stability properties of the target system without further investigation. In particular, if in the theorem one selects the desired vector field S to have 0_e as an Exponentially Stable (ES) equilibrium, then the problem becomes one of *output* regulation to 0_e, where the output function is the one that gives "the error," $h : (v,\omega) \mapsto v \cdot Tf(\omega)^{-1}$. In such case, the ultimate behavior of the target system will be determined by the resulting zero-dynamics, whose stability characteristics require a detailed assessment. This is where the contribution of this paper reaches its boundary, as we are currently searching in that direction. Nevertheless, the specific example presented below suggests that under appropriate conditions imposed on Z—the drift term of the target system—one can expect the base trajectory $(\pi_G \circ v)(t)$ to converge to U, the open set that contains the image of f, whereas $v(t)$ itself converges towards a neighborhood V of the zero-section in TTG—the caveat at this point being that V cannot be fixed in advance, for it depends on the initial conditions.

VI. Example

Here we apply the above constructions to a particular example of mechanical system, namely the (idealized) underactuated, horizontal PPR (Prismatic-Prismatic-Rotational) manipulator, the rotational joint of which is passive. It was shown in [3] that the PPR's dynamic model is locally feedback-equivalent to the "Extended Chained Form" (ECF)

$$\begin{cases} \ddot{q}_1 = u_1 \\ \ddot{q}_2 = u_2 \\ \ddot{q}_3 = u_1 q_2, \end{cases} \tag{10}$$

which we shall use in this example. Clearly, the ECF bears a strong resemblance with the first-order chained form—which involves $\dot{q}_i$s instead of $\ddot{q}_i$s—and, in a sense, this "resemblance" is the key to our approach. Consider the Lie group G with underlying manifold structure $\mathbb{R}^3$ and group composition law given by

$$\widehat{\mu}(x,y) = (\ x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1\ ).$$

We take $q = \mathrm{id}_G$ and consider natural (global) coordinates $(TG, (q,\dot{q}))$ on the tangent group TG. Using these coordinates, the group operation on TG is easily computed to be

$$\begin{aligned} \mu(x,y) = (\ & x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3 + x_2 y_1, \\ & x_4 + y_4,\ x_5 + y_5,\ x_6 + y_6 + x_2 y_4 + x_5 y_1\ ). \end{aligned}$$

Now, letting the vector fields $X_1, X_2 \in \Gamma(TG)$ be given by

$$X_{1,q} = \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_3}, \quad X_{2,q} = \frac{\partial}{\partial q_2},$$

we see that (10) defines the target system as a second-order system on TG of the form (4), provided we consider, for $v = (q,\dot{q})$,

$$Z_v = \sum_{i=1}^{3} \dot{q}_i \frac{\partial}{\partial q_i}, \quad X^{\text{lift}}_{1,v} = \frac{\partial}{\partial \dot{q}_1} + q_2 \frac{\partial}{\partial \dot{q}_3}, \quad X^{\text{lift}}_{2,v} = \frac{\partial}{\partial \dot{q}_2}.$$

One easily verifies that both X_i and X^{lift}_i $(i = 1,2)$ are left-invariant under the respective left translations in G and TG. It is also straightforward to check that $[X_1, X_2] = -\frac{\partial}{\partial q_3}$, so that $\mathrm{Lie}(\{X_{1,q}, X_{2,q}\})$ spans $T_q G$ at every $q \in G$. Consequently one can apply the theory of [9], recalled in Section III, to construct a Morin–Samson function $f : \mathbb{T} \longrightarrow G$ for the CF $\dot{q} = u_1 X_{1,q} + u_2 X_{2,q}$ near any point in G, e.g. $q = 0$. Taking (U,θ) to be an "angular" coordinate system on $\mathbb{T} \simeq S^1 \subset \mathbb{R}^2$ (for instance $U = S^1 \setminus \{(0,1)\}$ and $\theta(p) = 2\arctan(\frac{p_1}{1-p_2})$), we obtain

$$f(\theta) = \left(\ \varepsilon \sin(\theta),\ \varepsilon \cos(\theta),\ \frac{1}{4}\varepsilon^2 \sin(2\theta)\ \right),$$

with $\varepsilon > 0$ arbitrary. In view of the periodicity of this representative of f, we extend it by continuity so that f is globally defined. (In the sequel we often write $\sin = \mathrm{s}$ and $\cos = \mathrm{c}$ to save space.) Now, the transversality condition (3) amounts to the determinant of the matrix with columns $X_{1,f(\theta)}$, $X_{2,f(\theta)}$ and $f'(\theta)$ being constant and equal to $-\frac{1}{2}\varepsilon^2 (\neq 0)$. Using natural coordinates $(\theta, \dot\theta)$ for $T\mathbb{T}$, the value of the associated tangent mapping Tf at $\omega = (\theta, \dot\theta) \in T_\theta\mathbb{T}$ is then:

$$T_\theta f(\omega) = \Big(\varepsilon \mathrm{s}(\theta),\ \varepsilon \mathrm{c}(\theta),\ \frac{1}{4}\varepsilon^2 \mathrm{s}(2\theta), \varepsilon \mathrm{c}(\theta)\dot\theta,\ -\varepsilon \mathrm{s}(\theta)\dot\theta,\ \frac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\dot\theta \Big).$$

The vertical transversality condition (5) is easy to establish although we omit some computations for expediency. Considering natural coordinates $(\theta, \dot\theta, \alpha_L, \alpha_H)$ for $TT\mathbb{T}$, one first evaluates *the tangent* of Tf at a vertical vector $\alpha \in \ker(T_\omega \pi_{\mathbb{T}}) \subset T_\omega T\mathbb{T}$. Since $T_\omega \pi_{\mathbb{T}}$ maps $(\theta, \dot\theta, \alpha_L, \alpha_H)$ to (θ, α_L), α is in the kernel of $T_\omega \pi_{\mathbb{T}}$ iff it has the form $\alpha = (\theta, \dot\theta, 0, \alpha_H)$, so for simplicity we take $\tilde\alpha = (\theta, \dot\theta, 0, 1)$. Carrying out the operations one obtains

$$T_\omega T f(\tilde\alpha) = \Big(0,\ 0,\ 0,\ \varepsilon \mathrm{c}(\theta), -\varepsilon \mathrm{s}(\theta),\ \frac{1}{2}\varepsilon^2 \mathrm{c}(2\theta) \Big).$$

Now let us check that X_1^{lift}, X_2^{lift} and $T_\omega T f(\tilde\alpha)$ span the vertical subspace $(T_{Tf(\omega)}TG)^{\mathrm{vert}}$. In this case, any vector in the latter is of the form $\sum_{i=1}^3 \alpha_i \frac{\partial}{\partial \dot q_i}$, that is, its first three components are zero. Hence the verification reduces to computing the determinant of the submatrix consisting of the lower three rows of the matrix with columns $X_{1,Tf(\omega)}^{\mathrm{lift}}$, $X_{2,Tf(\omega)}^{\mathrm{lift}}$ and $T_\omega T f(\tilde\alpha)$. But this is exactly the matrix $(X_{1,f(\theta)}, X_{2,f(\theta)}, f'(\theta))$ considered above, with determinant equal to $-\frac{1}{2}\varepsilon^2$, so Tf indeed satisfies (6).

We define an auxiliary system (7), in this case given by the second-order system on $T\mathbb{T}$

$$\ddot\theta = w, \tag{11}$$

and the corresponding error $z = \mu(v, Tf(\omega)^{-1})$

$$\begin{aligned} z = \Big(& q_1 - \varepsilon \mathrm{s}(\theta),\ q_2 - \varepsilon \mathrm{c}(\theta), \\ & q_3 + \frac{1}{4}\varepsilon^2 \mathrm{s}(2\theta) - q_2 \varepsilon \mathrm{s}(\theta),\ v_1 - \varepsilon \mathrm{c}(\theta)\dot\theta, \\ & v_2 + \varepsilon \mathrm{s}(\theta)\dot\theta,\ v_3 - v_2 \varepsilon \mathrm{s}(\theta) - q_2 \varepsilon \mathrm{c}(\theta)\dot\theta \\ & + \frac{1}{2}\varepsilon^2 \mathrm{c}(2\theta)\dot\theta \Big). \end{aligned}$$

Differentiating this expression we get the error dynamics

$$\dot z = F(z, \omega) + \sum_{i=1}^3 u_i G_i(z, \omega), \tag{12}$$

with $u_3 = w$ and the components of F and the G_is given by

$$\begin{aligned} F(z,\omega) &= \Big(z_4,\ z_5,\ z_6,\ \dot\theta^2 \varepsilon \mathrm{s}(\theta),\ \varepsilon \mathrm{c}(\theta)\dot\theta^2, \\ &\qquad \frac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot\theta^2 + \varepsilon \mathrm{s}(\theta) z_2 \dot\theta^2 \\ &\qquad -2\varepsilon \mathrm{c}(\theta) z_5 \dot\theta \Big) \\ G_1(z,\omega) &= (\ 0,\ 0,\ 0,\ 1,\ 0,\ z_2 + \varepsilon \mathrm{c}(\theta)\ ) \\ G_2(z,\omega) &= (\ 0,\ 0,\ 0,\ 0,\ 1,\ -\varepsilon \mathrm{s}(\theta)\ ) \\ G_3(z,\omega) &= \Big(0,\ 0,\ 0,\ -\varepsilon \mathrm{c}(\theta),\ \varepsilon \mathrm{s}(\theta), \\ &\qquad -\frac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z_2 \Big). \end{aligned}$$

Each of the G_is, as well as F, may be seen as a family of vector fields on TG indexed by $\omega = (\theta, \dot\theta) \in T_\theta\mathbb{T}$. Moreover, it is clear that $F(\cdot, \omega)$ is second order whereas $G_i(\cdot, \omega)$ is vertical ($i = 1, 2, 3$), thus the error dynamics (12) is second order for every $\omega \in T\mathbb{T}$.

In order to construct a control law as outlined in Section V, and Theorem 1 in particular, we select for the desired dynamics a second-order vector field $S \in \Gamma(TTG)$ which has 0 as ES equilibrium point, for instance

$$\begin{aligned} S_z = (& z_4,\ z_5,\ z_6, \\ & -k_1 z_1 - k_2 z_4,\ -k_1 z_2 - k_2 z_5,\ -k_1 z_3 - k_2 z_6), \end{aligned}$$

where the control gains k_1, k_2 are strictly positive. From this point on, the control design translates into the search for a function $u : TG \times T\mathbb{T} \longrightarrow \mathbb{R}^3$ such that

$$F(z,\omega) + \sum_{i=1}^3 u_i(z,\omega) G_i(z,\omega) = S_z$$

for all $(z, \omega) \in TG \times T\mathbb{T}$. From the structure of the error dynamics (12) this boils down to solving for u in the following matrix equation

$$\begin{pmatrix} 1 & 0 & -\varepsilon \mathrm{c}(\theta) \\ 0 & 1 & \varepsilon \mathrm{s}(\theta) \\ z_2 + \varepsilon \mathrm{c}(\theta) & -\varepsilon \mathrm{s}(\theta) & -\frac{1}{2}\varepsilon^2 - \varepsilon \mathrm{c}(\theta) z_2 \end{pmatrix} u = \begin{pmatrix} -\dot\theta^2 \varepsilon \mathrm{s}(\theta) - k_1 z_1 - k_2 z_4 \\ -\dot\theta^2 \varepsilon \mathrm{c}(\theta) - k_1 z_2 - k_2 z_5 \\ -\frac{1}{2}\varepsilon^2 \mathrm{s}(2\theta)\dot\theta^2 - \varepsilon \mathrm{s}(\theta) z_2 \dot\theta^2 + 2\varepsilon \mathrm{c}(\theta) z_5 \dot\theta - k_1 z_3 - k_2 z_6 \end{pmatrix}.$$

This equation is solvable since the invertibility of the coefficient of u is equivalent to the invertibility of the matrix which ensures the vertical transversality of Tf; its determinant, in particular, is equal to $\frac{1}{2}\varepsilon^2$. For economy of space we do not give its explicit expression, but it is straightforward to check that the solution is smooth. In order to illustrate the time evolution for the error and target systems, we include a simple numerical simulation with $v_0 = (1, -2, 2, 0, 0, 0)$, $\omega_0 = (0, 0)$, $\varepsilon = 0.5$ and control gains $k_1 = 0.0707$, $k_2 = 0.3827$. By inspection of Figure 1, the error tends to zero whereas the logarithm of its norm decays sublinearly, so that $z(t) \to 0$ exponentially. As for the target system we

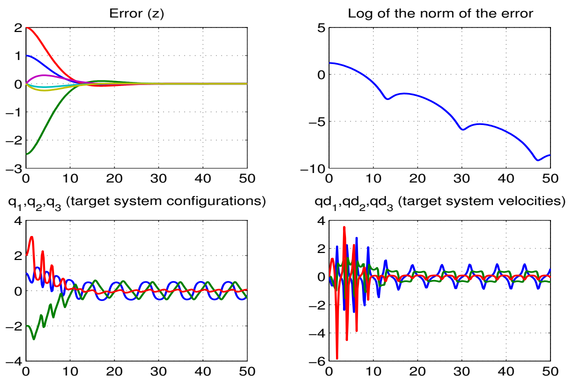

Fig. 1. Time histories of the error and target systems ($qd_i = \dot{q}_i$).

also observe that, after a short transient, the configurations $q(t)$ and the velocities $\dot{q}(t)$ seem to converge to a periodic motion and, moreover, the base curve $q(t) = (\pi_G \circ v)(t)$ ultimately converges to a bounded set, the extent of which can be made arbitrarily small by decreasing ε. Note, however, that two side-effects of taking smaller values of ε are an increase in the peak excursions of the control signals (as in the transverse function approach) and an increase in the frequency of the steady-state oscillations. Concerning the velocities $\dot{q}(t)$, they also converge to a bounded set, but the extent of that set depends on the initial conditions (v_0, ω_0), hence *one cannot specify it in advance*. in this sense our current result is weaker than *practical* stabilization. The long term behavior of the target system is governed by the zero-dynamics of the compound system. Thus, it suffices to analyze the ultimate behavior of the auxiliary system (11) with input equal to $u_3(0, \omega(t))$. After simple computations we get the *zero-dynamics*

$$\ddot{\theta} = -\sin(2\theta)\dot{\theta}^2. \tag{13}$$

It is interesting to note that this system admits an interpretation as a simple mechanical system on $\mathbb{T}$ with connection $\nabla : \Gamma(T\mathbb{T}) \longrightarrow \Gamma(T^*\mathbb{T} \otimes T\mathbb{T})$ determined from the (unique) Christoffel symbol $\Gamma(\theta) = \sin(2\theta)$ by the rule $\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = \Gamma(q)\frac{\partial}{\partial\theta}$. (We are adopting here the geometric approach to modeling of Lagrangian systems; the reader is referred to e.g. [4] and references therein for more details on such approach.) Now, ∇ is clearly torsionless, so one may expect it to represent the Levi-Cività connection associated with a particular metric tensor field g on $\mathbb{T}$; to obtain g we simply reverse the usual procedure that yields the Levi-Cività connection from the data of g. Typically that would involve PDEs, but in this case the equations reduce to a simple ODE with (family of) solutions given by $g_\theta = Ae^{-\cos(2\theta)}$, $A > 0$. One immediately checks that, by defining the Lagrangian function $L : T\mathbb{T} \longrightarrow \mathbb{R}$ as

$$L(\omega) = \frac{1}{2} g_\theta(\dot{\theta}, \dot{\theta}) = \frac{1}{2} Ae^{-\cos(2\theta)}\dot{\theta}^2,$$

the associated Euler-Lagrange equation $\nabla_{\dot{\theta}}\dot{\theta} = 0$ precisely coincides with the zero-dynamics (13), irrespective of the value of A. Therefore, the zero-dynamics is, by itself, a *virtual*, unforced simple mechanical system with zero potential! Since in this case $(L \circ \omega)' = 0$, the (kinetic) energy is a conserved quantity and, given that it is bounded with respect to θ and depends quadratically on $\dot{\theta}$, it follows that $\dot{\theta}(t)$ remains bounded for all $t \in [t_0, \infty)$. Consequently, both $Tf(\omega(t))$ and $v(t)$ converge to a bounded neighborhood of the zero section in TG. In this case it is also clear that such neighborhood depends on the initial conditions. Intuitively, in fact, one may think of the auxiliary system as storing (kinetic) energy for as long as the error is nonzero. If $z(t)$ reached 0 in finite-time $t = T$, remaining at 0 thereafter, then the cumulated energy E_T would be constant for $t \geq T$, so the peak excursions of $\dot{\theta}(t)$ would depend on it. (Of course, by smoothness of the error dynamics, $z(t)$ cannot reach 0 unless $z = 0$, but this argument is helpful to acquire an intuitive picture of the situation.)

VII. Conclusions and Future Work

In this paper we single out *vertical transversality*, a property that proves instrumental in extending the notion of transverse functions on tori, introduced in [8], to the domain of second-order systems. We first show how transverse functions naturally give rise, upon differentiation, to vertically transverse ones, then outline the way these can be exploited for control purposes. We believe these results may be further developed into a more thorough theory that addresses practical stabilization and tracking for second-order systems. Research work remains to be done, however, and our current efforts aim at two issues: the characterization of the long-term behavior of the zero-dynamics and the modification of the current construction so that it forces the state of the target system to converge to the zero-section in TQ, at least when this is possible, so that the long-term behavior is not oscillatory.

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