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Adaptive Neural Tracking for a Class of SISO Uncertain and Stochastic Nonlinear Systems

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***Abstract*—Adaptive neural control schemes based on the backstepping technique are developed to solve the tracking control problem of a combined stochastic and uncertain nonlinear system. As shown by an extensive stability analysis the proposed control scheme ensures that all the error variables are bounded in probability while the mean square tracking error becomes semiglobally uniformly ultimately bounded in an arbitrarily small area around the origin. The effectiveness of the design approach is illustrated by simulation results.**

I. INTRODUCTION

THE tracking control design for systems with nonlinear deterministic uncertainties has been of special attention by many researchers [1-4]. On the other hand, numerous backstepping-based control laws for stochastic strict-feedback systems that are affected only by a Wiener process have been developed to guarantee stability, known as stability in probability [5-8]. However, the combined problem of the control of stochastic nonlinear systems with simultaneous nonlinear uncertainties remains open. To the authors' knowledge, only in [9] this problem has been addressed for discrete-time nonlinear systems.

In this paper we confront in a new manner the tracking control problem for continuous-time uncertain nonlinear systems that are additionally *disturbed by "noise"*. Our method starts from the backstepping technique and the transformation of the system variables into error variables. The Lyapunov functions used in all steps of the backstepping technique are quadratic except from the last step wherein a quartic form of the Lyapunov function is used. This makes possible a significantly simplified adaptive neural network (NN) implementation and the approximation of all the unknown functions by suitable radial NN designs. Extensive stability analysis proves that all the error variables are bounded in probability while the mean square (average) tracking error is semiglobally uniformly ultimately bounded (SGUUB). As verified by simulation results all the performance requirements posed are satisfied. Particularly, the output tracks the reference signal in a very satisfactory way while suitable tuning of the design parameters leads to a very good response of the adaptation mechanism and the NN controller.

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II. PRELIMINARIES

A. General Assumptions

Consider the stochastic and uncertain nonlinear system of the form

$$y^{(n)} = a\left(y, y^{(1)}, \dots, y^{(n-1)}\right) + b\left(y, y^{(1)}, \dots, y^{(n-1)}\right)u + \underbrace{+\, g\left(y, y^{(1)}, \dots, y^{(n-1)}\right)^T n_w}_{noise\ term} \quad (1)$$

where u is the controlled input and n_w white noise. Eq. (1) can be written in a state space form as a single-input single-output (SISO) system with stochastic differential equations

$$\left.\begin{aligned} dx_1 &= x_2 dt \\ &\vdots \\ dx_{n-1} &= x_n dt \\ dx_n &= a(x)dt + b(x)udt + g(x)^T dw \\ y &= x_1 \end{aligned}\right\} \quad (2)$$

where w is an independent r-dimensional Wiener process and $dw = n_w dt$, defined on the probability space $(\Omega, \mathbb{F}, P)$ and $a(x), b(x)$ are unknown nonlinear functions.

However, for systems of the form

$$dx = f(x)dt + h(x)dw \quad (3)$$

where $f(\cdot)$ and $h(\cdot)$ are locally Lipschitz in x, and w is an independent r-dimensional Wiener process, defined on the probability space $(\Omega, \mathbb{F}, P)$, we define the operator L known as infinitesimal generator for twice continuously differentiable function, $V(x)$ as follows:

$$LV(x) = \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} tr\left\{\frac{\partial^2 V(x)}{\partial x^2} h(x) h^T(x)\right\} \quad (4)$$

where tr is the matrix trace. For the nonlinear stochastic system we need the following definition.

Definition 1: (see [10]) The solution process $\{x(t), t \geq 0\}$ of stochastic system (7) is said to be bounded in probability, if $\lim_{c\to\infty} \sup_{0\leq t<\infty} P\{\|x(t)\| > c\} = 0$ where $P\{A\}$ denotes the probability of event A.

Corresponding to the definition above we recall the following theorem [8].

Theorem 1: Consider the stochastic nonlinear system (3). If there exists a positive definite, radially unbounded, twice continuously differentiable Lyapunov function $V:\mathbb{R}^n \to \mathbb{R}$, and constants $c_1 > 0$, $c_2 > 0$ such that

$LV(x) \le -c_1 V(x) + c_2$

then i) the system has a unique solution almost surely and ii) the system is bounded in probability.

Let now the reference signals $y_d(t)$, $y_d^{(1)}(t), \ldots$, $y_d^{(n)}(t)$ that are smooth and bounded. In the following a controller is proposed that achieves tracking of the output of system (2) to the reference signal guaranteeing that the system is bounded in probability. Before proceeding with our approach we start by using the backstepping technique.

B. System Transformation by Backstepping Design

By iteratively viewing x_i as virtual control we select the error variables

$$z_i = x_i - a_{i-1}\left(z_{[i-1]}, y_d^{[i-1]}\right), \quad i = 1, 2, \ldots, n \tag{5}$$

where $a_{i-1}\left(z_{[i-1]}, y_d^{[i-1]}\right)$ a function to be defined, $y_d^{[i]} = \begin{bmatrix} y_d & y_d^{(1)} & \ldots & y_d^{(i)} \end{bmatrix}^T$, $z_{[i]} = \begin{bmatrix} z_1 & z_2 & \cdots & z_i \end{bmatrix}^T$.

Let the candidate Lyapunov function

$$V_i = \sum_{k=1}^{i} \frac{z_k^2}{2}, \quad i = 1, 2, \ldots, n-1 \tag{6}$$

We choose

$$\alpha_k\left(z_{[k]}, y_d^{[k]}\right) = d_{k,1} z_1 + \ldots + d_{k,k} z_k + y_d^{(k)}$$

with the following selection of the coefficients $d_{k,i}$, $i = 1, \ldots, k$, $k = 1, \ldots, n-1$

$$\begin{aligned} d_{k,i} = &-c_k \delta_{k,i} - \delta_{k,i+1} + d_{k-1,i-1} - \\ &- c_{k-1} d_{k-1,i} - d_{k-1,i+1}, \quad i = 1, 2, \ldots, k \end{aligned}$$

where $d_{k,i} = 0$ for $i > k$ or if $i \cdot k = 0$. The function $\delta_{m,n}$ is the Kronecker delta function given by

$$\delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \ne n \end{cases}$$

Then, by using the induction method, we prove that the following relations must hold true

$$\begin{aligned} dz_i &= \left[z_{i+1} - c_i z_i - z_{i-1}\right] dt \quad i = 1, \ldots, k \\ dV_k &= \left[-c_1 z_1^2 - \ldots - c_k z_k^2\right] dt + z_k z_{k+1} dt \\ &\qquad k = 1, \ldots, n-1. \end{aligned} \tag{7}$$

wherein we define $z_0 = 0$ and $a_0(y_d) = y_d$. For k = 1, equations (7) are obviously true. For k = 2 it can be easily proved for

$$\alpha_2\left(z_{[2]}, y_d^{[2]}\right) = -\left(c_1 + c_2\right) z_2 + \left(c_1^2 - 1\right) z_1 + y_d^{(2)}$$

that (7) hold true. Suppose they are true for k = j, i.e. it holds true that

$$\begin{aligned} dz_i &= \left[z_{i+1} - c_i z_i - z_{i-1}\right] dt \quad i = 1, \ldots, j \\ dV_j &= \left[-c_1 z_1^2 - \ldots - c_j z_j^2\right] dt + z_j z_{j+1} dt \\ \alpha_j\left(z_{[j]}, y_d^{[j]}\right) &= d_{j,1} z_1 + \ldots + d_{j,j} z_j + y_d^{(j)} \end{aligned} \tag{8}$$

We will prove it for $k = j+1$. So, for the candidate Lyapunov function

$$V_{j+1} = V_j + \frac{z_{j+1}^2}{2} \tag{9}$$

and since

$$z_{j+1} = x_{j+1} - \alpha_j \tag{10}$$

we have from (8) that

$$\begin{aligned} dz_{j+1} = &\left(z_{j+2} + \alpha_{j+1}\right) dt - \\ &- \left[\sum_{m=1}^{j} d_{j,m}\left(z_{m+1} - c_m z_m - z_{m-1}\right) + y_d^{(j+1)}\right] dt \end{aligned} \tag{11}$$

and

$$\begin{aligned} dV_{j+1} = &\left[-c_1 z_1^2 - \ldots - c_j z_j^2\right] dt + \\ &+ z_{j+1} z_{j+2} dt + z_{j+1}\Big[z_j + \alpha_{j+1} - \\ &- \sum_{m=1}^{j} d_{j,m}\left(z_{m+1} - c_m z_m - z_{m-1}\right) - y_d^{(j+1)}\Big] dt \end{aligned} \tag{12}$$

Selecting now

$$\alpha_{j+1}\left(z_{[j+1]}, y_d^{(j+1)}\right) = d_{j+1,1} z_1 + \ldots + d_{j+1,j+1} z_{j+1} + y_d^{(j+1)}$$

with

$$\begin{aligned} d_{j+1,i} = &-c_{j+1}\delta_{j+1,i} - \delta_{j+1,i+1} + d_{j,i-1} - \\ &- c_j d_{j,i} - d_{j,i+1}, \quad i = 1, 2, \ldots, j+1 \end{aligned}$$

we have that

$$\begin{aligned} \alpha_{j+1}\left(z_{[j+1]}, y_d^{(j+1)}\right) &= -c_{j+1} z_{j+1} - z_j + \\ &+ \sum_{m=1}^{j} d_{jm}\left(z_{m+1} - c_m z_m - z_{m-1}\right) + y_d^{(j+1)} \\ &= d_{j+1,1} z_1 + \ldots + d_{j+1,j+1} z_{j+1} + y_d^{(j+1)} \end{aligned} \tag{11}$$

and (11) and (12) yield

$$\begin{aligned} dz_{j+1} &= \left[z_{j+2} - c_{j+1} z_{j+1} - z_j\right] dt \\ dV_{j+1} &= \left[-c_1 z_1^2 - \ldots - c_{j+1} z_{j+1}^2\right] dt + z_{j+1} z_{j+2} dt \end{aligned}$$

and the proof is completed.

The relations above hold up to $k = n-1$. Now, at the last step, one can define the candidate Lyapunov function as the sum of V_{n-1} plus a quartic term instead of a quadratic term with time-varying gain [5]

$$V_n = V_{n-1} + \frac{z_n^4}{4b(x)} \tag{13}$$

where

$$\begin{aligned} dz_n = &\, a(x) dt + b(x) u dt - \\ &- \sum_{i=1}^{n-1} d_{n-1,i}\left(z_{i+1} - c_i z_i - z_{i-1}\right) dt - y_d^{(n)} dt + g^T(x) dw \end{aligned}$$

Now the following assumptions are considered.

Assumption 1: The function $b(x)$ is independent from x_n and there exist a known positive scalar $\underline{b}$ such that $0 < \underline{b} \le b(x) \quad \forall x \in \mathbb{R}^n$.

In fact, many practical systems, such as pendulum plants and single-link robot arms with flexible joints and others posses this property [11].

Since the function $g(x)$ is transformed to $\overline{g}\left(z, y_d^{[n-1]}\right)$ and its Taylor expansion with respect to *z* has the form

$$\overline{g}\left(z, y_d^{[n-1]}\right) = g_0\left(y_d^{[n-1]}\right) + \sum_{i=1}^{n} z_i g_{i1}\left(y_d^{[n-1]}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j g_{ij2}\left(y_d^{[n-1]}\right) + \ldots = g_0\left(y_d^{[n-1]}\right) + \sum_{i=1}^{n} z_i g_{i1}\left(y_d^{[n-1]}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j f_{ij2}\left(z, y_d^{[n-1]}\right)$$

it can be easily shown that there exist a nonnegative function $\psi_0\left(y_d^{[n-1]}\right)$ and a smooth nonnegative function $\psi_1\left(z, y_d^{[n-1]}\right)$, such that

$$\left\|g(x)\right\|^2 = \left\|\overline{g}\left(z, y_d^{[n-1]}\right)\right\|^2 \le \le \psi_0\left(y_d^{[n-1]}\right) + \psi_1\left(z, y_d^{[n-1]}\right)\left\|z\right\|^2 \quad \forall z \in \mathbb{R}^n \tag{14}$$

Therefore, one can express the following assumption.

Assumption 2: Inequality (14) holds true for some known nonnegative function $\psi_0\left(y_d^{[n-1]}\right)$ and some known smooth nonnegative function $\psi_1\left(z, y_d^{[n-1]}\right)$.

III. Controller Design

From (7) and (13) we have that

$$dV_n = \left[-c_1 z_1^2 - \ldots - \frac{c_n}{2b(x)} z_n^4\right]dt + \frac{z_n^4}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1}dt + \frac{z_n}{b(x)}\Big[b(x)z_{n-1} + \frac{c_n}{2}z_n^3 + z_n^2\Big(a(x) + b(x)u - \sum_{i=1}^{n-1} d_{n-1,i}\left(z_{i+1} - c_i z_i - z_{i-1}\right) - y_d^{(n)}\Big)\Big]dt + \frac{3}{2b(x)}z_n^2\overline{g}^T\overline{g}dt + \frac{z_n^3}{b(x)}\overline{g}^T dw \tag{15}$$

Obviously from (14) we can write

$$\frac{3}{2b(x)}z_n^2\overline{g}^T\overline{g} \le \frac{3}{2b(x)}z_n^2\psi_0 + \frac{3}{2b(x)}z_n^2\psi_1\left\|z\right\|^2 \tag{16}$$

However the two terms in the right-hand part of (16) are bounded as follows

i) $$\frac{3}{2b(x)}z_n^2\psi_0 \le \frac{1}{2\xi} + \frac{9}{8b^2(x)}\xi\psi_0^2 z_n^4 \tag{17}$$

for some positive scalar ξ and

ii) $$\frac{3}{2b(x)}z_n^2\psi_1\left\|z\right\|^2 = \sum_{i=1}^{n}\frac{3}{2b(x)}z_i^2 z_n^2\psi_1 = \sum_{i=1}^{n-1}\frac{3}{2b(x)}z_i^2 z_n^2\psi_1 + \frac{3}{2b(x)}\psi_1 z_n^4 \le \le \sum_{i=1}^{n-2}\varepsilon_i c_i z_i^2 + \sum_{i=1}^{n-2}\frac{9\psi_1^2}{16\varepsilon_i c_i b^2(x)}z_i^2 z_n^4 + \frac{\varepsilon_{n-1}c_{n-1}}{2}z_{n-1}^2 + \frac{9\psi_1^2}{8\varepsilon_{n-1}c_{n-1}b^2(x)}z_{n-1}^2 z_n^4 + \frac{3}{2b(x)}\psi_1 z_n^4 = \sum_{i=1}^{n-2}\varepsilon_i c_i z_i^2 + \frac{\varepsilon_{n-1}c_{n-1}}{2}z_{n-1}^2 + \frac{3}{2b(x)}\psi_1\left[1 + \frac{3\psi_1}{4b(x)}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]z_n^4 \tag{18}$$

Since the following inequality holds

$$z_{n-1}z_n \le \frac{\varepsilon_{n-1}c_{n-1}}{2}z_{n-1}^2 + \frac{1}{2\varepsilon_{n-1}c_{n-1}}z_n^2 \le \le \frac{1}{2\xi} + \frac{\varepsilon_{n-1}c_{n-1}}{2}z_{n-1}^2 + \frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2}z_n^4 \tag{19}$$

inequality (15) results in

$$dV_n \le \frac{z_n^3}{b(x)}\overline{g}^T dw + \frac{z_n^4}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1}dt + \frac{1}{\xi}dt - -\Big[\left(1-\varepsilon_1\right)c_1 z_1^2 + \ldots + \left(1-\varepsilon_{n-1}\right)c_{n-1}z_{n-1}^2 + \frac{\left(1-\varepsilon_n\right)c_n}{2b(x)}z_n^4\Big]dt + z_n^3\Bigg\{\Bigg\{\frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2} + \left(1-\varepsilon_n\right)\frac{c_n}{2b(x)} + \frac{9\xi\psi_0^2}{8b^2(x)} + \frac{3}{2b(x)}\psi_1\left[1 + \frac{3\psi_1}{4b(x)}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]\Bigg\}z_n + \frac{1}{b(x)}\Big(a(x) + b(x)u - \sum_{i=1}^{n-1} d_{n-1,i}\left(z_{i+1} - c_i z_i - z_{i-1}\right) - y_d^{(n)}\Big)\Bigg\}dt \tag{20}$$

Then, inequality (20) takes the form

$$dV_n \le \frac{1}{\xi}dt - 2\underline{c}V_n dt + \frac{z_n^3}{b(x)}\overline{g}^T dw +$$
$$+z_n^3\left\{u + \frac{a(x)+v_1}{b(x)} + \frac{z_n}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1} + \right. \tag{21}$$
$$+z_n\left\{\frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2} + \frac{9\xi\psi_0^2}{8\underline{b}^2} + (1-\varepsilon_n)\frac{c_n}{2\underline{b}} + \right.$$
$$\left.\left. + \frac{3}{2\underline{b}}\psi_1\left[1 + \frac{3\psi_1}{4\underline{b}}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]\right\}\right\}dt$$

where v_1 is a known function given by

$$v_1 = -\sum_{i=1}^{n-1} d_{n-1,i}(z_{i+1} - c_i z_i - z_{i-1}) - y_d^{(n)} \tag{22}$$

and $\underline{c} = \min_{1\le i\le n}\{(1-\varepsilon_i)c_i\} > 0$.

At this point, we are ready to prove Theorem 2 which provides a desired control that ensures boundedness in probability for system (2); furthermore, it is shown that the mean square error enters in finite-time in an arbitrarily small region of the origin i.e. it is semiglobally uniformly ultimately bounded (SGUUB).

Theorem 2: For the stochastic uncertain nonlinear system (2) satisfying Assumptions 1-2, if the control law

$$u = -\frac{\xi}{4}z_n^3 - \hat{W}^T\Phi(x, v_1, z_n) - \left\{(1-\varepsilon_n)\frac{c_n}{2\underline{b}} + \frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2} + \right.$$
$$\left. + \frac{9\xi\psi_0^2}{8\underline{b}^2} + \frac{3}{2\underline{b}}\psi_1\left[1 + \frac{3\psi_1}{4\underline{b}}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]\right\}z_n \tag{23}$$

and the update law

$$\dot{\hat{W}} = \Gamma\Phi(x, v_1, z_n)z_n^3 - \sigma\Gamma(\hat{W} - W^0)$$

are selected, where v_1 is a known signal given by (22), then the tracking error is bounded in probability and the mean square tracking error enters inside the region

$$\Omega_1 := \left\{y(t)\in\mathbb{R} \mid E\left[|y(t) - y_d(t)|^2\right] \le \right.$$
$$\left. \le \frac{2}{\min\{\underline{c}, \sigma\lambda_{\min}(\Gamma)\}}\left[\frac{1+\epsilon_{M1}^2}{\xi} + \frac{\sigma}{2}\left\|W^* - W^0\right\|^2\right] \forall t \ge T_1\right\}$$

wherein it remains for all time thereafter.

Proof: Let the control input

$$u_1^* = -k_1(t)z_n - \frac{a(x)+v_1}{b(x)} - \frac{z_n}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1} \tag{24}$$

where

$$k_1(t) = k_2(t) + (1-\varepsilon_n)\frac{c_n}{2\underline{b}} + \frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2} + \frac{9\xi\psi_0^2}{8\underline{b}^2} +$$
$$+ \frac{3}{2\underline{b}}\psi_1\left[1 + \frac{3\psi_1}{4\underline{b}}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]$$

and the gain $k_2(t) \ge 0$, for all t. For this control input, inequality (21) sequentially yields

$$dV_n \le -2\underline{c}V_n dt + \frac{1}{\xi}dt - k_1(t)z_n^4 dt + \frac{z_n^3}{b(x)}\overline{g}^T dw$$
$$\le -2\underline{c}V_n dt + \frac{1}{\xi}dt + \frac{z_n^3}{b(x)}\overline{g}^T dw \tag{25}$$

Thus from (25) it holds true that

$$LV_n \le -2\underline{c}V_n + \frac{1}{\xi} \tag{26}$$

This control input can be written as

$$u_1^* = -k_1(t)z_n - u_{11}^*(\sigma_1), \tag{27}$$

where

$$\sigma_1 = \left[x^T, v_1, z_n\right]^T,$$
$$u_{11}^*(\sigma_1) = \frac{a(x)+v_1}{b(x)} + \frac{z_n}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1}. \tag{28}$$

Since $a(x)$ and $b(x)$ are unknown functions, the desired controller u_1^* cannot be implemented directly. An approximation of u_{11}^* based on neural networks is proposed as follows.

Let the compact set

$$\Omega_{\sigma_1} = \left[(x, v_1, z_n) \mid x\in\Omega_x, y_d^{[n]}\in\Omega_d\right]$$

It is well-known that for any compact set any smooth nonlinear function can be approximated by NNs with arbitrarily small accuracy [2]. Hence, as the function $u_{11}^*(\sigma_1)$ is smooth, for a given positive number ϵ_{M1} there exists a NN approximation such that

$$u_{11}^*(\sigma_1) = W^{*T}\Phi(\sigma_1) + \epsilon_1 \tag{29}$$

with bounded function approximation error ϵ_1 satisfying $|\epsilon_1| \le \epsilon_{M1}$ and the ideal weight W^* giving the optimal approximation

$$W^* := \arg\min_{W\in\mathbb{R}^l}\left\{\sup_{\sigma_1\in\Omega_{\sigma_1}}\left|u_{11}^*(\sigma_1) - W^T\Phi(\sigma_1)\right|\right\}$$

The magnitude of ϵ_{M1} depends on the choice of the basis functions and the number of nodes. Several NN basis functions can be used for $\Phi(\sigma_1)$, such as radial basis function (RBF), higher-order NN, or fuzzy systems [2]. An RBF implementation is used. The subspaces Ω_x, Ω_d are defined as the compact sets through which the state and reference trajectories may travel. Let $\hat{W}$ be an estimate of the ideal NN weight W^*. Motivated by the desired control structure (24), the following adaptive controller is presented:

$$u = -k_2(t)z_n - \hat{W}^T\Phi(\sigma_1) - \left\{(1-\varepsilon_n)\frac{c_n}{2\underline{b}} + \frac{\xi}{8\varepsilon_{n-1}^2 c_{n-1}^2} + \right.$$
$$\left. + \frac{9\xi\psi_0^2}{8\underline{b}^2} + \frac{3}{2\underline{b}}\psi_1\left[1 + \frac{3\psi_1}{4\underline{b}}\left(\frac{z_{n-1}^2}{\varepsilon_{n-1}c_{n-1}} + \sum_{i=1}^{n-2}\frac{z_i^2}{2\varepsilon_i c_i}\right)\right]\right\}z_n \tag{30}$$

If one selects the candidate Lyapunov function as

$$\bar{V}_n = V_n + \frac{1}{2}\tilde{W}^T \Gamma^{-1} \tilde{W}$$

then its differential takes the form

$$d\bar{V}_n \le \frac{1}{\xi}dt - 2\underline{c}V_n dt + z_n^3\Big[-k_2(t) z_n - -\hat{W}^T\Phi(\sigma_1) + \frac{a(x)+v_1}{b(x)} + \frac{z_n}{4b^2(x)}\sum_{i=1}^{n-1}\frac{\partial b}{\partial x_i}x_{i+1}\Big]dt + \frac{z_n^3}{b(x)}\bar{g}^T dw + \tilde{W}^T\Gamma^{-1}\dot{\hat{W}}dt$$

Let the adaptive law

$$\dot{\hat{W}} = \Gamma\Phi(\sigma_1) z_n^3 - \sigma\Gamma\left(\hat{W} - W^0\right) \tag{31}$$

with gain matrix $\Gamma > 0$. This weight update law uses a σ-modification term [12] which prevent parameter drift of the network weights.

Taking into account, the previous NN approximation (29) and the control input (30), the differential of $\bar{V}_n$ takes the following form

$$\begin{aligned} d\bar{V}_n \le{}& \frac{1}{\xi}dt - 2\underline{c}V_n dt + \frac{z_n^3}{b(x)}\bar{g}^T dw + z_n^3\Big[-k_2(t) z_n - \\ & -\hat{W}^T\Phi(\sigma_1) + W^{*T}\Phi(\sigma_1) + \in_1\Big]dt + \\ & + \Big[\tilde{W}^T\Phi(\sigma_1) z_n^3 - \sigma\tilde{W}^T\left(\hat{W} - W^0\right)\Big]dt \end{aligned} \tag{32}$$

Since

$$-\tilde{W}^T\left(\hat{W} - W^0\right) = -\frac{1}{2}\left\|\tilde{W}\right\|^2 - \frac{1}{2}\left\|\hat{W} - W^0\right\|^2 + + \frac{1}{2}\left\|W^* - W^0\right\|^2 \le -\frac{1}{2}\left\|\tilde{W}\right\|^2 + \frac{1}{2}\left\|W^* - W^0\right\|^2$$

and

$$-\frac{\sigma}{2}\left\|\tilde{W}\right\|^2 \le -\frac{\sigma\lambda_{\min}(\Gamma)}{2}\tilde{W}^T\Gamma^{-1}\tilde{W}$$

(32) yields

$$d\bar{V}_n \le \lambda_1 dt - 2\mu_1\bar{V}_n dt + \frac{z_n^3}{b(x)}\bar{g}^T dw + \frac{\in_1^2 z_n^2}{4k_2(t)}dt$$

where $\lambda_1 := \frac{1}{\xi} + \frac{\sigma}{2}\left\|W^* - W^0\right\|^2$ and

$\mu_1 := \min\left\{\underline{c}, \sigma\lambda_{\min}(\Gamma)\right\}$.

If the following simple gain selection is made

$$k_2(t) = \frac{\xi}{4}z_n^2 \tag{33}$$

then

$$d\bar{V}_n \le -2\mu_1\bar{V}_n dt + \lambda_2 dt + \frac{z_n^3}{b(x)}\bar{g}^T dw \tag{34}$$

where $\lambda_2 := \lambda_1 + \frac{\in_{M1}^2}{\xi}$. From (34) we have that

$$L\bar{V}_n \le -2\mu_1\bar{V}_n + \lambda_2$$

Hence from Theorem 1 the system is bounded in probability and the mean value of the Lyapunov function satisfies

$$\frac{d}{dt}\left[E\left(\bar{V}_n\right)\right] \le -2\mu_1 E\left[\bar{V}_n(t)\right] + \lambda_2$$

Solving the above inequality by using Lemma B.5 of [13] we have

$$\begin{aligned} E\left[\bar{V}_n(t)\right] &\le e^{-2\mu_1 t}V_n(0) + \lambda_2\int_0^t e^{-2\mu_1(t-\tau)}d\tau \\ &\le e^{-2\mu_1 t}V_n(0) + \frac{\lambda_2}{2\mu_1} \quad \forall t \ge 0 \end{aligned}$$

So there exists a time T_1

$$T_1 = \max\left\{0, \frac{1}{2\mu_1}\ln\left[\frac{2\mu_1 V_n(0)}{\lambda_2}\right]\right\}$$

such that

$$E\left[\left(y(t) - y_d(t)\right)^2\right] \le 2E\left[\bar{V}_n(t)\right] \le \frac{2\lambda_2}{\mu_1} \quad \forall t \ge T_1$$

and the proof is completed. □

IV. Example

Consider the stochastic system

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = \left(x_1 + x_2^2 - \sin x_1\right)dt + \\ +\left(2 - \sin x_1\right)u dt + \left(1/2 + \sin x_1/6 + x_2/4\right)dw \\ y = x_1 \end{cases} \tag{35}$$

with initial conditions $x_1(0) = 0, x_2(0) = 0$ and reference signal $y_d(t) = 1 + \sin t$. Let the known lower bound $\underline{b} = 1$. From (35), one can calculate the (non-unique) bound $\|g\|^2 \le 8/9 + x_2^2/8$, from which we can construct the (non-unique) ψ_0, ψ_1 as follows

$$\begin{aligned} \|g\|^2 &\le 8/9 + x_2^2/8 = 8/9 + \left(z_2 - c_1 z_1 + y_d^{(1)}\right)^2/8 \\ &\le \underbrace{8/9 + \left(y_d^{(1)}\right)^2/4}_{\psi_0\left(y_d^{[1]}\right)} + \|z\|^2\underbrace{\left(1 + c_1^2\right)/4}_{\psi_1} \end{aligned} \tag{36}$$

Consider that the noise covariance bounds ψ_0, ψ_1 are given by (36). The controller given by (23) is used. An RBF NN with 4-inputs and $3^4 = 81$-nodes is employed. The adaptation and controller parameters are $c_1 = c_2 = 1$, $\Gamma = 10 I_{81}$, $\sigma = 5\times 10^{-3}$, $\xi = 0.2$.

In Fig.1 one can clearly see how the use of the NN improves the tracking performance; indeed, as time passes a remarkable tracking enhancement is achieved with the NN.

In Fig. 2, the approximation capability of the NN is illustrated. A very good approximation is achieved after a short initial period.

Finally, in Fig. 3 we examine two different cases for the design parameter ξ, i.e. for $\xi = 5$ and $\xi = 0.2$. The simulation results demonstrate that, as it is expected, a larger value of the parameter ξ reduces the tracking error.

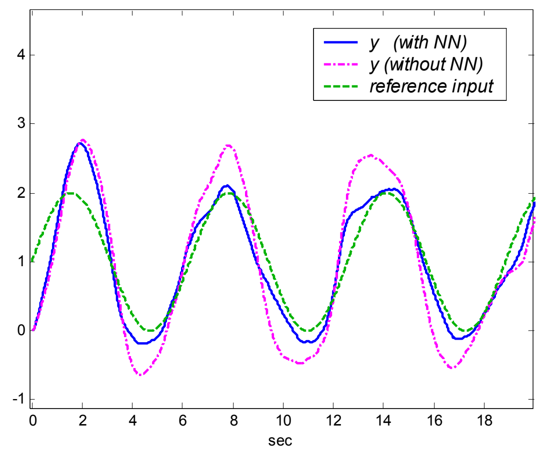

Fig. 1. Output response with and without NN.

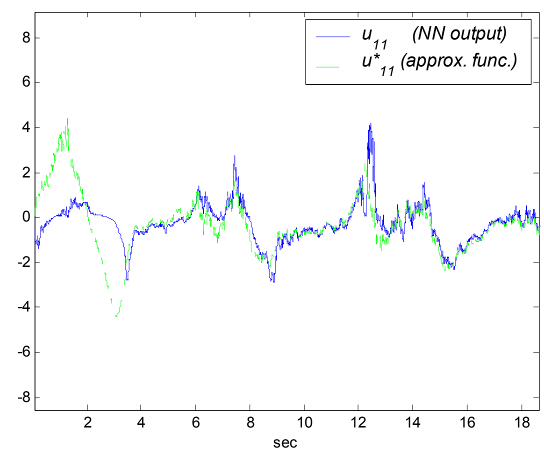

Fig. 2. NN approximation of the unknown function u^{*}_{11}.

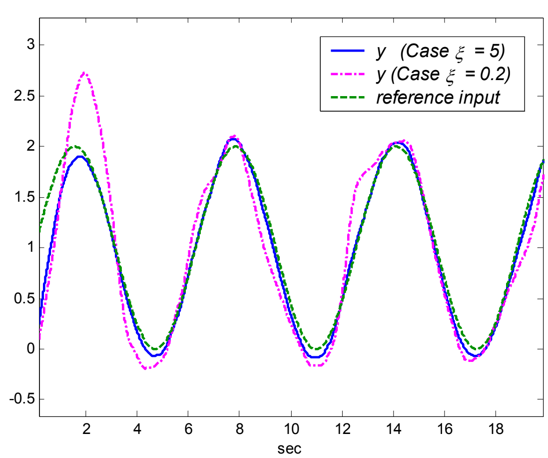

Fig. 3. Output response for different values of ξ.

V. Conclusion

In this paper, a suitable adaptive NN-based control method is proposed that guarantees effective output tracking for a class of SISO nonlinear systems in the face of system uncertainties and external stochastic disturbances. The theoretical results are analytically obtained and presented in the paper. These results are clearly confirmed by simulation studies that verify an effective tracking performance of the system.

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