Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# Further results on periodically time-varying behavioral systems

José Carlos Aleixo
Dept. Math.
University of Beira Interior
6201-001 Covilhã, Portugal
E-mail: *jcaleixo@mat.ubi.pt*

Jan Willem Polderman
Dept. Appl. Math.
University of Twente, P.O. Box 217
7500 AE Enschede, The Netherlands
E-mail: *j.w.polderman@math.utwente.nl*

Paula Rocha
Dept. Math.
University of Aveiro
3810-193 Aveiro, Portugal
E-mail: *procha@mat.ua.pt*

***Abstract*— We consider periodic behavioral systems as introduced in [1] and analyze two main issues: behavior representation and controllability. More concretely we study the equivalence and the minimality of representations. Moreover we relate the controllability of a periodic system with the controllability of an associated time-invariant system known as *lifted* system, and derive a controllability test. Finally we prove the existence of an autonomous/controllable decomposition similar to what happens for the time-invariant case.**

## I. Introduction

In the classical input-state-output framework, linear models with periodically time-varying coefficients can be found in numerous practical applications, such as satellite attitude control based on the periodicity of the earth magnetic field, control of rotating machinery, or sampled-data systems. An overview of the vast literature in the field of periodic systems can be found in the survey papers [2] and [3].

In this paper periodic systems are studied within the behavioral framework. The behavioral theory of linear, time-invariant systems has its roots in the mid eighties when J.C. Willems started his pioneering work. The main paradigms are the emphasis of the behavior, the set of possible trajectories of the system rather than on its mathematical representation and the absence of a prior partition of the signals into inputs and outputs.

In this paper we present some results on the behavioral theory of linear periodic systems. The paper builds on a framework that was developed Kuijper and Willems, see [1] and the references therein. An important tool is the *lifting technique* that associates to each periodic behavior a time-invariant behavior. The lifting map is the key tool in relating properties of the periodic behavior to similar properties of the lifted, time-invariant, behavior. Subsequently, known results for time-invariant behaviors are applied to the lifted behavior. Finally, these are translated back to the periodic behavior. In this way we obtain results about classification of representations of periodic behaviors, controllability, and autonomous periodic behaviors.

The paper is organized as follows. Section II contains the basic material such as the precise definition of periodic behavior, the lifting map, and a quick review of the results obtained in [1] that are relevant in our context. The main new results in Section III are an analysis how different representations of the same periodic behavior are related, and the characterization of minimal representations. Section IV is concerned with controllable and autonomous periodic behaviors. Results concerning the autonomous case have been obtained in [1]. Here we prove that a periodic behavior is controllable if and only if its lifted behavior is controllable. Furthermore a test for controllability is obtained. If a periodic behavior is neither controllable nor autonomous, then just like in the time-invariant case, the periodic behavior may be decomposed in a controllable and an autonomous part.

Throughout the paper we provide examples to illustrate the theory.

## II. Periodic behavioral systems

In the behavioral framework a dynamical system $\Sigma$ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ as the time set, $\mathbb{W}$ as the signal space and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ as the behavior. Here we focus on the discrete-time case, that is, $\mathbb{T} = \mathbb{Z}$, assuming furthermore that our space of external variables is $\mathbb{W} = \mathbb{R}^q$ with $q \in \mathbb{Z}_+$.

As is now well known, the behavior of a time-invariant systems is characterized by its invariance under the time shift, i.e., $\sigma\mathfrak{B} = \mathfrak{B}$, where $\sigma^\lambda : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$, defined by $(\sigma^\lambda w)(k) := w(k+\lambda)$, is called *backward $\lambda$-shift* in case $\lambda \in \mathbb{Z}_+$ or *forward $\lambda$-shift* in case $\lambda \in \mathbb{Z}_-$.

For $P$-periodic behaviors invariance is only required with respect to the $P$-th power of the shift, as stated in the next definition.

**Definition II.1.** [1] A system $\Sigma$ is said to be $P$-periodic (with $P \in \mathbb{N}$) if its behavior $\mathfrak{B}$ satisfies $\sigma^P\mathfrak{B} = \mathfrak{B}$. ◇

Note that this definition is not given in terms of the system mathematical description (system representation). However it has been shown in [1] and [4] that if $\mathfrak{B}$ is a $\sigma^P$-invariant linear closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$ (in the topology of point-wise convergence), then $\Sigma$ has a representation of the type

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = 0, \quad t = 1, \ldots, P, \ k \in \mathbb{Z}, \quad (1)$$

where $R_t \in \mathbb{R}^{g_t \times q}\left[\xi, \xi^{-1}\right]$. Remark that the Laurent-polynomial matrices $R_t$ need not have the same number of rows (in fact we could even have some $g_t$ equal to zero). Analogously to the time-invariant case, although with some abuse of language, we refer to (1) as a *kernel representation*.

**Example II.2.** Consider the 2-periodic system $\Sigma$ with behavior $\mathfrak{B} \subseteq \left(\mathbb{R}^2\right)^{\mathbb{Z}}$ defined by

$$\left\{ w \middle| \left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(2k+t) = 0, \ t = 1, 2, \ k \in \mathbb{Z} \right\},$$

with

$$R_1\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cc}\xi-1 & \xi^2\end{array}\right]\in\mathbb{R}^{1\times 2}\left[\xi,\xi^{-1}\right],$$
$$R_2\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cc}\xi^{-1}-1 & \xi^2-1\\ 2\xi & 1-\xi\end{array}\right]\in\mathbb{R}^{2\times 2}\left[\xi,\xi^{-1}\right].$$

This definition leads to the periodically time-varying difference equations

$$\left(\left[\begin{array}{cc}\sigma-1 & \sigma^2\end{array}\right]\left[\begin{array}{c}w_1\\ w_2\end{array}\right]\right)(2k+1)=0$$
$$\left(\left[\begin{array}{cc}\sigma^{-1}-1 & \sigma^2-1\\ 2\sigma & 1-\sigma\end{array}\right]\left[\begin{array}{c}w_1\\ w_2\end{array}\right]\right)(2k+2)=0. \qquad \diamond$$

In [1] two kinds of time-invariant systems are introduced that can be associated with a $P$-periodic system: the *lifted* and the *twisted* systems. In this paper we are concerned with the lifting option. Following [1], given a $P$-periodic system $\Sigma=(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$, the associated lifted system is defined as a time-invariant system $\Sigma^L=\left(\mathbb{Z},\mathbb{R}^{Pq},L\mathfrak{B}\right)$, with behavior defined by $L\mathfrak{B}=L(\mathfrak{B}):=\{\widetilde{w}|\ \widetilde{w}=Lw,\ w\in\mathfrak{B}\}\subseteq\left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}}$, where $L$ is the linear map $L:(\mathbb{R}^q)^{\mathbb{Z}}\to\left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}}$, defined by

$$(Lw)(k):=\left[\begin{array}{c}w(Pk+1)\\ \vdots\\ w(Pk+P)\end{array}\right].$$

Two important properties concerning this map are presented in the following proposition.

**Proposition II.3.** [1]

i) *$L\sigma^P=\sigma L$;*
ii) *$L$ is a homeomorphism; consequently $L$ is closed.*

These properties allow to relate a $P$-periodic system with the corresponding lifted system.

**Proposition II.4.** [1]

i) *$\Sigma$ is $P$-periodic if and only if $\Sigma^L$ is time-invariant;*
ii) *$\mathfrak{B}$ is linear if and only if $L\mathfrak{B}$ is linear;*
iii) *$\mathfrak{B}$ is closed if and only if $L\mathfrak{B}$ is closed.*

## III. Kernel representations

The main purpose of this section is to analyze the connection between different kernel representations of a given linear $P$-periodic behavior. We start by noting that, since

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=\left(\sigma^t R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk),$$

the kernel representation (1) can be written as

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk)=0,\quad k\in\mathbb{Z}, \tag{2}$$

where

$$R\left(\xi,\xi^{-1}\right):=\left[\begin{array}{c}\xi R_1\left(\xi,\xi^{-1}\right)\\ \xi^2 R_2\left(\xi,\xi^{-1}\right)\\ \vdots\\ \xi^P R_P\left(\xi,\xi^{-1}\right)\end{array}\right]\in\mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right], \tag{3}$$

with $g:=\sum_{t=1}^{P}g_t$. From now on we will refer to the matrix $R\left(\xi,\xi^{-1}\right)$ as a *representation matrix* of the corresponding behavior.

**Remark III.1.** Note that by considering the representation matrix $R$ (and the associated representation (2) for a $P$-periodic behavior) we are ignoring the partition which is initially given by the matrices $R_1,\ldots,R_P$. However, this partition is irrelevant, as can be seen in Example II.2. Indeed, in this example $P=2$, $R_1$ has one row and $R_2$ has two rows and the final description consists of three difference equations, which could as well be obtained by taking adequately defined $R_1$ and $R_2$ matrices with two and one row, respectively. $\diamond$

Now, decomposing $R\left(\xi,\xi^{-1}\right)$ as

$$\begin{aligned}R\left(\xi,\xi^{-1}\right)&=\xi R^1\left(\xi^P,\xi^{-P}\right)+\cdots+\xi^P R^P\left(\xi^P,\xi^{-P}\right)\\ &=R^L\left(\xi^P,\xi^{-P}\right)\Xi_{P,q}(\xi),\end{aligned} \tag{4}$$

with $\Xi_{P,q}(\xi):=\left[\begin{array}{ccc}\xi I_q & \cdots & \xi^P I_q\end{array}\right]^T$ and

$$R^L\left(\xi,\xi^{-1}\right)=\left[R^1\left(\xi,\xi^{-1}\right)R^2\left(\xi,\xi^{-1}\right)\cdots R^P\left(\xi,\xi^{-1}\right)\right], \tag{5}$$

and recalling the definition of the lifted trajectory $Lw$ associated to $w$, (2) can be written as

$$\left(R^L\left(\sigma,\sigma^{-1}\right)(Lw)\right)(k)=0,\quad k\in\mathbb{Z}.$$

**Example III.2.** Recall example II.2. By definition the matrix $R\left(\xi,\xi^{-1}\right)$ is given by

$$\left[\begin{array}{c}\xi R_1\left(\xi,\xi^{-1}\right)\\ \xi^2 R_2\left(\xi,\xi^{-1}\right)\end{array}\right]=\left[\begin{array}{cc}\xi^2-\xi & \xi^3\\ \xi-\xi^2 & \xi^4-\xi^2\\ 2\xi^3 & \xi^2-\xi^3\end{array}\right].$$

By decomposing $R$ as in (4), we obtain

$$R\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cc}R^1\left(\xi^2,\xi^{-2}\right) & R^2\left(\xi^2,\xi^{-2}\right)\end{array}\right]\Xi_{2,2}(\xi),$$

where

$$R^1\left(\xi^2,\xi^{-2}\right)=\left[\begin{array}{cc}-1 & \xi^2\\ 1 & 0\\ 2\xi^2 & -\xi^2\end{array}\right],\quad R^2\left(\xi^2,\xi^{-2}\right)=\left[\begin{array}{cc}1 & 0\\ -1 & \xi^2-1\\ 0 & 1\end{array}\right].$$

Therefore the matrix $R^L$ is given by

$$R^L\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cc|cc}-1 & \xi & 1 & 0\\ 1 & 0 & -1 & \xi-1\\ 2\xi & -\xi & 0 & 1\end{array}\right]. \qquad \diamond$$

Taking into account that this reasoning can be reversed, we obtain the following result.

**Lemma III.3.** [1] *A $P$-periodic behavior $\mathfrak{B}\subset(\mathbb{R}^q)^{\mathbb{Z}}$ is given by the kernel representation* (1)*, that is,*

$$\mathfrak{B}=\left\{w|\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=0,\ t=1,\ldots,P,\ k\in\mathbb{Z}\right\}$$

*if and only if the associated lifted behavior $L\mathfrak{B}$ is given by the kernel representation $L\mathfrak{B}=\left\{\widetilde{w}|R^L\left(\sigma,\sigma^{-1}\right)\widetilde{w}=0\right\}$, where $R^L\left(\xi,\xi^{-1}\right)\in\mathbb{R}^{g\times Pq}\left[\xi,\xi^{-1}\right]$, $g=\sum_{t=1}^{P}g_t$, is given as in* (5).

The following result is standard in time-invariant behaviors. A proof for the continuous time setting may be found in [5].

**Theorem III.4.** *Let two time-invariant behaviors* $\mathfrak{H}$ *and* $\mathfrak{H}'$ *be given. Then* $\mathfrak{H} = \ker H\left(\sigma, \sigma^{-1}\right) \subset \mathfrak{H}' = \ker H'\left(\sigma, \sigma^{-1}\right)$ *if and only if there exists a Laurent-polynomial matrix* $L\left(\xi, \xi^{-1}\right)$ *such that* $H'\left(\xi, \xi^{-1}\right) = L\left(\xi, \xi^{-1}\right) H\left(\xi, \xi^{-1}\right)$.

Invoking Theorem III.4 we can immediately conclude that for $P$-periodic behaviors $\mathfrak{B} \subset \mathfrak{B}'$ if and only if any matrices $R^L\left(\xi, \xi^{-1}\right)$ and $R'^L\left(\xi, \xi^{-1}\right)$ that represent the corresponding lifted behaviors are related by $R'^L\left(\xi, \xi^{-1}\right) = L\left(\xi, \xi^{-1}\right) R^L\left(\xi, \xi^{-1}\right)$ for some Laurent-polynomial matrix $L\left(\xi, \xi^{-1}\right)$. This constitutes an indirect characterization of behavior inclusion for the periodic case. Our next result provides a more direct condition, since it is stated in terms of the representation matrices of the periodic behaviors themselves.

**Theorem III.5.** *Let* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be two* $P$*-periodic behaviors with representation matrices* $R\left(\xi, \xi^{-1}\right)$ *and* $R'\left(\xi, \xi^{-1}\right)$*, respectively. Then* $\mathfrak{B} \subset \mathfrak{B}'$ *if and only if there exists a Laurent-polynomial matrix* $L\left(\xi, \xi^{-1}\right)$ *such that*

$$R'\left(\xi, \xi^{-1}\right) = L\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right). \tag{6}$$

*Proof.* Assume first that $\mathfrak{B} \subset \mathfrak{B}'$, then, by Theorem III.4, the matrices $R^L\left(\xi, \xi^{-1}\right)$ and $R'^L\left(\xi, \xi^{-1}\right)$ that represent the corresponding lifted behaviors are related by $R'^L\left(\xi, \xi^{-1}\right) = L\left(\xi, \xi^{-1}\right) R^L\left(\xi, \xi^{-1}\right)$, for some Laurent-polynomial matrix $L\left(\xi, \xi^{-1}\right)$, and (6) immediately follows from the relation (4).

Assume now that (6) holds. Taking into account the uniqueness of the decomposition (4), this implies that $R'^L\left(\xi^P, \xi^{-P}\right) = L\left(\xi^P, \xi^{-P}\right) R^L\left(\xi^P, \xi^{-P}\right)$. Since this equality still holds when $\xi^P$ is replaced by $\xi$, this means that $L\mathfrak{B} = \ker R^L\left(\sigma, \sigma^{-1}\right) \subset L\mathfrak{B}' = \ker R'^L\left(\sigma, \sigma^{-1}\right)$, which is equivalent to say that the corresponding inclusion also holds for the associated $P$-periodic behaviors, i.e., $\mathfrak{B} \subset \mathfrak{B}'$. □

¿From this theorem we can conclude that two representation matrices $R\left(\xi, \xi^{-1}\right)$ and $R'\left(\xi, \xi^{-1}\right)$ correspond to the same $P$-periodic behavior if and only if there exist two Laurent-polynomial matrices $L\left(\xi, \xi^{-1}\right)$ and $L'\left(\xi, \xi^{-1}\right)$ such that $R'\left(\xi, \xi^{-1}\right) = L\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right)$ and $R\left(\xi, \xi^{-1}\right) = L'\left(\xi^P, \xi^{-P}\right) R'\left(\xi, \xi^{-1}\right)$. Note that, in case the representation matrices $R^L$ and $R'^L$ of the corresponding lifted systems are not full row rank, the matrices $L$ and $L'$ are not unique.

**Example III.6.** Consider the 2-periodic systems $\Sigma$ and $\Sigma'$ with behaviors $\mathfrak{B}$ and $\mathfrak{B}'$ corresponding respectively to the representation matrices

$$R\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi^2 - \xi & \xi^3 \\ 2\xi & 1-\xi \\ \xi - \xi^2 & \xi^4 - \xi^2 \\ 2\xi^3 & \xi^2 - \xi^3 \end{bmatrix}$$

and

$$R'\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi^{-3} - \xi^{-2} & -\xi^{-1} \\ 2\xi & 1-\xi \\ 1-\xi^{-1} & \xi \\ \xi^3 - \xi^4 & \xi^6 - \xi^4 \end{bmatrix}.$$

It is easy to check that

$$\begin{aligned} R'\left(\xi, \xi^{-1}\right) &= L\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right) \\ R\left(\xi, \xi^{-1}\right) &= L'\left(\xi^P, \xi^{-P}\right) R'\left(\xi, \xi^{-1}\right), \end{aligned}$$

with

$$L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} -\xi^{-2} & -\alpha_1\xi & 0 & \alpha_1 \\ 0 & 1-\alpha_2\xi & 0 & \alpha_2 \\ \xi^{-1} & -\alpha_3\xi & 0 & \alpha_3 \\ 0 & -\alpha_4\xi & \xi & \alpha_4 \end{bmatrix} \tag{7}$$

$$L'\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \beta_1\xi - \xi^2 & 0 & \beta_1 & 0 \\ \beta_2\xi & 1 & \beta_2 & 0 \\ \beta_3\xi & 0 & \beta_3 & \xi^{-1} \\ \beta_4\xi & \xi & \beta_4 & 0 \end{bmatrix},$$

for any real values of $\alpha_i$ and $\beta_j$, $i, j = 1, \ldots, 4$, respectively. Thus $\mathfrak{B} \subset \mathfrak{B}'$ and $\mathfrak{B}' \subset \mathfrak{B}$, i.e., $\mathfrak{B} = \mathfrak{B}'$. ◇

If $R$ and $R'$ have the same number of rows, it is possible to prove that $L$ and $L'$ can be taken to be unimodular, [5]. This yields the following fundamental result. It is the counterpart for $P$-periodic behaviors of a similar result for time-invariant behaviors, [5, Theorem 3.6.2].

**Theorem III.7.** *Let* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be two* $P$*-periodic behaviors with representation matrices* $R\left(\xi, \xi^{-1}\right)$ *and* $R'\left(\xi, \xi^{-1}\right)$*, respectively, possessing the same number of rows. Then* $\mathfrak{B} = \mathfrak{B}'$ *if and only if there exists a unimodular matrix* $U\left(\xi, \xi^{-1}\right)$ *such that*

$$R'\left(\xi, \xi^{-1}\right) = U\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right). \tag{8}$$

**Example III.8.** Recall now example III.6. Taking in (7), $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$, we obtain a unimodular matrix

$$U\left(\xi, \xi^{-1}\right) = \begin{bmatrix} -\xi^2 & -\xi & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \xi^{-1} & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 \end{bmatrix},$$

such that (8) holds. ◇

An important issue in the representation of time-invariant systems is the question of minimality. Given a linear time-invariant system with behavior $\mathfrak{B}$ described by:

$$\left(R\left(\sigma, \sigma^{-1}\right) w\right)(k) = 0, \quad k \in \mathbb{Z}, \tag{9}$$

with $R\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$, we say that the representation (9) is minimal if the number of rows of the matrix $R\left(\xi, \xi^{-1}\right)$ is minimal (among all the other representations of $\mathfrak{B}$). This means that it is impossible to give a mathematical description of $\mathfrak{B}$ with less equations than in (9), and is equivalent to say that $R\left(\xi, \xi^{-1}\right)$ has full row rank (over $\mathbb{R}\left[\xi, \xi^{-1}\right]$).
In the periodically time-variant case, we adopt the same definition of minimality.

**Definition III.9.** A representation matrix $R \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$ of a $P$-periodic system $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ is said to be a minimal representation if for any other representation $R' \in \mathbb{R}^{g'\times q}\left[\xi,\xi^{-1}\right]$ of $\Sigma$, there holds $g \leq g'$. ◇

It is not difficult to check that a representation $R\left(\xi,\xi^{-1}\right)$ of a $P$-periodic system $\Sigma$ is minimal if and only if the same happens for the corresponding representation $R^L\left(\xi,\xi^{-1}\right)$ of the associated time-invariant lifted system $\Sigma^L$. Thus $R\left(\xi,\xi^{-1}\right)$ is minimal if and only if $R^L\left(\xi,\xi^{-1}\right)$ is full row rank over $\mathbb{R}\left[\xi,\xi^{-1}\right]$. The next lemma translates this in terms of the matrix $R\left(\xi,\xi^{-1}\right)$ itself.

**Lemma III.10.** *Let $R\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$ be the representation matrix of a $P$-periodic system and consider the corresponding matrix $R^L\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times Pq}\left[\xi,\xi^{-1}\right]$ given by* (4) *and* (5)*. Then, the following conditions are equivalent:*

i) *$R^L\left(\xi,\xi^{-1}\right)$ has full row rank over $\mathbb{R}\left[\xi,\xi^{-1}\right]$;*

ii) *$R\left(\xi,\xi^{-1}\right)$ has full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$ (i.e., if $r\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{1\times g}\left[\xi^P,\xi^{-P}\right]$ is such that $r\left(\xi^P,\xi^{-P}\right)R\left(\xi,\xi^{-1}\right) = 0 \in \mathbb{R}^{1\times q}\left[\xi,\xi^{-1}\right]$, then $r\left(\xi^P,\xi^{-P}\right)=0$).*

*Proof.* Recall that

$$R\left(\xi,\xi^{-1}\right)=R^L\left(\xi^P,\xi^{-P}\right)\Xi_{P,q}\left(\xi\right). \qquad (10)$$

¿From here we can immediately conclude that ii) ⇒ i). In order to see that i) ⇒ ii), assume that ii) does not hold, i.e., that there exists a nonzero row $r\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{1\times g}\left[\xi^P,\xi^{-P}\right]$ such that $r\left(\xi^P,\xi^{-P}\right)R\left(\xi,\xi^{-1}\right)=0$. Then, pre-multiplying both sides of (10) by $r\left(\xi^P,\xi^{-P}\right)$ yields that

$$r\left(\xi^P,\xi^{-P}\right)R^L\left(\xi^P,\xi^{-P}\right)\Xi_{P,q}\left(\xi\right)=0\in\mathbb{R}^{1\times q}\left[\xi,\xi^{-1}\right]. \qquad (11)$$

Since, as earlier mentioned, the decomposition of matrices $M\left(\xi,\xi^{-1}\right)$ over $\mathbb{R}\left[\xi,\xi^{-1}\right]$ as a product $M\left(\xi,\xi^{-1}\right) = M^L\left(\xi^P,\xi^{-P}\right)\Xi_{P,q}\left(\xi\right)$ is unique, (11) implies that $r\left(\xi^P,\xi^{-P}\right)R^L\left(\xi^P,\xi^{-P}\right)=0 \in \mathbb{R}^{1\times Pq}\left[\xi^P,\xi^{-P}\right]$, which in turn leads to $r\left(\xi,\xi^{-1}\right)R^L\left(\xi,\xi^{-1}\right) = 0 \in \mathbb{R}^{1\times Pq}\left[\xi,\xi^{-1}\right]$, thus showing that $R^L\left(\xi,\xi^{-1}\right)$ has not full row rank over $\mathbb{R}\left[\xi,\xi^{-1}\right]$. Therefore i) ⇒ ii). □

Together with the previous considerations, this result yields the following characterization of minimality.

**Theorem III.11.** *Let $R\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$ be the representation matrix of a $P$-periodic system. Then $R\left(\xi,\xi^{-1}\right)$ is a minimal representation if and only if it has full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$.*

**Example III.12.** The representation $R\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{3\times 2}\left[\xi,\xi^{-1}\right]$ of example (II.2) is minimal. Indeed, although it is clearly not full row rank over $\mathbb{R}\left[\xi,\xi^{-1}\right]$, it can be shown that it has full row rank over $\mathbb{R}\left[\xi^2,\xi^{-2}\right]$. ◇

## IV. Controllability and Autonomicity

We next consider some questions related to the behavioral controllability of a $P$-periodic system, such as the relationship between this property and the controllability of the associated lifted time-invariant system and the existence of a behavior autonomous/controllable decomposition (which is well known in the behavioral theory of time-invariant systems, [5]). We start by recalling the behavioral definition of controllability.

**Definition IV.1.** A system $\Sigma$ is said to be controllable if for all $w_1, w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$, there exists $w \in \mathfrak{B}$ and $k_1 \geq 0$ such that (12) holds,

$$w\left(k\right)=\begin{cases} w_1\left(k\right), & k \leq k_0 \\ w_2\left(k\right), & k > k_0+k_1 \end{cases}. \qquad (12)$$

◇

Theorem IV.2 below, states that the controllability of a $P$-periodic system is equivalent to the controllability of the corresponding lifted system.

**Theorem IV.2.** *A $P$-periodic system $\Sigma$ is controllable if and only if the associated lifted system $\Sigma^L$ is controllable.*

*Proof.*

(⇒): Assume that $\Sigma$ is controllable. Let $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ and $\widetilde{k}_0 \in \mathbb{Z}$. By construction there exist $w_1, w_2 \in \mathfrak{B}$ such that $Lw_i = \widetilde{w}_i,\ i=1,2$.

Take $k_0 := P\widetilde{k}_0+P$. Then, by the controllability of $\Sigma$, there exist $k_1 \in \mathbb{Z}$ and a trajectory $w \in \mathfrak{B}$ satisfying: $w\left(k\right) = w_1\left(k\right)$ for $k \leq k_0$ and $w\left(k\right) = w_2\left(k\right)$ for $k > k_0+k_1$. Take $\widetilde{k}_1 = \lceil\frac{k_1}{P}\rceil+2$. Then, the trajectory $\widetilde{w} := Lw \in L\mathfrak{B}$ coincides with $\widetilde{w}_1$ for instants $k \leq \widetilde{k}_0$ and with $\widetilde{w}_2$ for instants $k > \widetilde{k}_0+\widetilde{k}_1$, showing that $L\mathfrak{B}$ is controllable.

(⇐): Assume now that $\Sigma^L$ is controllable. Let $w_1, w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$. By construction there exist $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ such that $Lw_i = \widetilde{w}_i,\ i=1,2$.

Define $\widetilde{k}_0 = \lceil\frac{k_0}{P}\rceil-1$. Since $L\mathfrak{B}$ is controllable, there exist $\widetilde{k}_1 \in \mathbb{Z}$ (which can clearly always taken to be not less than 1) and a trajectory $\widetilde{w} \in L\mathfrak{B}$ such that $\widetilde{w}\left(k\right) = \widetilde{w}_1\left(k\right)$ for $k \leq \widetilde{k}_0$ and $\widetilde{w}\left(k\right) = \widetilde{w}_2\left(k\right)$ for $k > \widetilde{k}_0+\widetilde{k}_1$. Take $k_1 := P\left(\widetilde{k}_1-1\right)+1 \geq 0$, and let $w := L^{-1}\left(w\right) \in \mathfrak{B}$. Then, $w\left(k\right) = w_1\left(k\right)$ for $k \leq k_0$ and $w\left(k\right)=w_2\left(k\right)$ for $k > k_0+k_1$, which proves that $\mathfrak{B}$ is controllable.

□

This result, together with the characterization of behavioral controllability given in [4, Theorem V.2], allows to conclude the following.

**Proposition IV.3.** *Let $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ be a $P$-periodic system, represented by* (1)*, with representation matrix $R$ as in* (3)*. Then $\Sigma$ is controllable if and only if the corresponding* *matrix $R^L$ (see* (4) *and* (5)*) is such that $R^L(\lambda,\lambda^{-1})$ has constant rank over $\mathbb{C}\backslash\{0\}$.*

Note that if in addition the matrix $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ has full row rank, the condition that $R^L(\lambda,\lambda^{-1})$ has constant rank over $\mathbb{C}\backslash\{0\}$ is equivalent to say that $R^L(\xi,\xi^{-1})$ is left-prime, i.e, all its left divisors are unimodular matrices in $\mathbb{R}^{g\times g}[\xi,\xi^{-1}]$. It turns out that the left-primeness of $R^L(\xi,\xi^{-1})$ can be related with the following primeness property for $R(\xi,\xi^{-1})$.

**Definition IV.4.** A Laurent-polynomial matrix $R(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ with full row rank over $\mathbb{R}[\xi^P,\xi^{-P}]$ is said to be left-prime over $\mathbb{R}[\xi^P,\xi^{-P}]$, or simply $P$-left-prime, if whenever it is factored as $R(\xi,\xi^{-1}) = D(\xi^P,\xi^{-P})\overline{R}(\xi,\xi^{-1})$, with $D(\xi^P,\xi^{-P}) \in \mathbb{R}^{g\times g}[\xi^P,\xi^{-P}]$, then the factor $D(\xi^P,\xi^{-P})$ and, equivalently, $D(\xi,\xi^{-1})$, are unimodular (over $\mathbb{R}[\xi^P,\xi^{-P}]$ and $\mathbb{R}[\xi,\xi^{-1}]$, respectively). $\diamond$

**Lemma IV.5.** *Let $P \in \mathbb{N}$ and $R(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ have full row rank over $\mathbb{R}[\xi^P,\xi^{-P}]$. Consider the associated matrix $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ according to the decomposition* (4)*. Then, the following conditions are equivalent:*

i) *$R^L(\xi,\xi^{-1})$ is left-prime;*

ii) *$R(\xi,\xi^{-1})$ is $P$-left-prime.*

*Proof.* In order to prove that i) $\Rightarrow$ ii), assume that $R(\xi,\xi^{-1})$ is not $P$-left-prime. Then there exist a non-unimodular square matrix $D(\xi^P,\xi^{-P})$ and a matrix $\overline{R}(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ such that $R(\xi,\xi^{-1}) = D(\xi^P,\xi^{-P})\overline{R}(\xi,\xi^{-1})$. Letting $\overline{R}^L(\xi,\xi^{-1})$ be the matrix corresponding to $\overline{R}(\xi,\xi^{-1})$ according to the decomposition (4), we have that $R(\xi,\xi^{-1}) = D(\xi^P,\xi^{-P})\overline{R}^L(\xi^P,\xi^{-P})\Xi_{P,q}(\xi)$, which is a decomposition (4) for $R(\xi,\xi^{-1})$. Due to the uniqueness of such decomposition, we conclude that $R^L(\xi^P,\xi^{-P}) = D(\xi^P,\xi^{-P})\overline{R}(\xi^P,\xi^{-P})$ and hence $R^L(\xi,\xi^{-1}) = D(\xi,\xi^{-1})\overline{R}(\xi,\xi^{-1})$ with $D(\xi,\xi^{-1})$ non-unimodular, showing that $R^L(\xi,\xi^{-1})$ is not left-prime.
To derive implication ii) $\Rightarrow$ i) assume now that $R^L(\xi,\xi^{-1})$ is not left-prime. Then, there exist a non-unimodular square matrix $D(\xi,\xi^{-1})$ and a matrix $\overline{R}^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ such that $R^L(\xi,\xi^{-1}) = D(\xi,\xi^{-1})\overline{R}^L(\xi,\xi^{-1})$, which implies that $R(\xi,\xi^{-1}) = D(\xi^P,\xi^{-P})\overline{R}^L(\xi^P,\xi^{-P})\Xi_{P,q}(\xi)$. Hence $R(\xi,\xi^{-1})$ has a non-unimodular left factor $D(\xi^P,\xi^{-P})$ and is therefore not $P$-left-prime. $\square$

This leads to the following direct characterization of controllability.

**Theorem IV.6.** *A $P$-periodic system $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ is controllable if and only if its minimal representation matrices $R(\xi,\xi^{-1})$ are $P$-left-prime.*

As an opposite situation to controllability, which, roughly speaking, is the possibility of changing from one system trajectory to any other one, stands autonomicity, that corresponds to the impossibility of connecting a system trajectory with another different one.

**Definition IV.7.** The $P$-periodic system $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ is said to be autonomous if for all $k_0 \in \mathbb{Z}$ and all $w_1, w_2 \in \mathfrak{B}$

$$w_1(k) = w_2(k) \text{ for } k < k_0 \quad \Rightarrow \quad \{w_1 = w_2\}. \qquad \diamond$$

**Proposition IV.8.** [1] *Let $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ be a $P$-periodic system. Then $\mathfrak{B}$ is autonomous if and only if $L\mathfrak{B}$ is autonomous.*

It is well-known that every linear time-invariant behavior $\mathfrak{B}$ which is a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$ can be decomposed as the direct sum of an autonomous sub-behavior with a controllable sub-behavior, more concretely the following result holds true.

**Theorem IV.9.** [5] *Let $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ be linear, time-invariant and such that $\mathfrak{B}$ is a closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$. Then:*

i) *there exist an autonomous sub-behavior, $\mathfrak{B}^{\mathrm{a}}$, of $\mathfrak{B}$, and a controllable sub-behavior, $\mathfrak{B}^{\mathrm{c}}$, of $\mathfrak{B}$, such that $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$;*

ii) *if $\mathfrak{B}^{\mathrm{a}}$, autonomous, and $\mathfrak{B}^{\mathrm{c}}_1, \mathfrak{B}^{\mathrm{c}}_2$, controllable, are all sub-behaviors of $\mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}_1 = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}_2$, then $\mathfrak{B}^{\mathrm{c}}_1 = \mathfrak{B}^{\mathrm{c}}_2$.*

**Remark IV.10.** The controllable behavior $\mathfrak{B}^{\mathrm{c}}$ is defined by

$$\mathfrak{B}^{\mathrm{c}} := \overline{\mathfrak{B}^{\mathrm{cp}}}$$

where $\mathfrak{B}^{\mathrm{cp}}$ is the set of trajectories of $\mathfrak{B}$ with compact support and its closure is taken w.r.t. the topology of pointwise convergence. $\diamond$

We now prove a similar result for periodic behaviors.

**Theorem IV.11.** *Let $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ be $P$-periodic. Then there exist $\mathfrak{B}^{\mathrm{a}}, \mathfrak{B}^{\mathrm{c}} \subset \mathfrak{B}$ such that:*

i) *$\mathfrak{B}^{\mathrm{a}}$ is autonomous;*

ii) *$\mathfrak{B}^{\mathrm{c}}$ is controllable;*

iii) *$\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$.*

*Proof.* Define $\widetilde{\mathfrak{B}} := L\mathfrak{B}$. Then, there exists sub-behaviors $\widetilde{\mathfrak{B}}^{\mathrm{a}}$ and $\widetilde{\mathfrak{B}}^{\mathrm{c}}$ such that

$$\widetilde{\mathfrak{B}} = \widetilde{\mathfrak{B}}^{\mathrm{a}} \oplus \widetilde{\mathfrak{B}}^{\mathrm{c}}.$$

As stated in Proposition II.3 $L$ is a homeomorphism and therefore its inverse $L^{-1}$ is well defined. Let's use it to define $\mathfrak{B}^{\mathrm{a}} := L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{a}}$ and $\mathfrak{B}^{\mathrm{c}} := L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{c}}$. Remark that, see Proposition IV.8, $\mathfrak{B}^{\mathrm{a}}$ is autonomous and $\mathfrak{B}^{\mathrm{c}}$ is controllable (due to theorem IV.2).

Since $L$ is a homeomorphism, we then have that:

$$\begin{aligned}\mathfrak{B}^{\mathrm{a}} \cap \mathfrak{B}^{\mathrm{c}} &= L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{a}} \cap L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{c}} = L^{-1}\left(\widetilde{\mathfrak{B}}^{\mathrm{a}} \cap \widetilde{\mathfrak{B}}^{\mathrm{c}}\right)\\ &= L^{-1}(\{0\}) = \{0\}.\end{aligned}$$

Finally take $w \in \mathfrak{B}$. Let $\widetilde{w} := L(w)$ and take $\widetilde{w}_a$ and $\widetilde{w}_c$ to be such that

$$\widetilde{w} = \widetilde{w}_a + \widetilde{w}_c, \quad \widetilde{w}_a \in \widetilde{\mathfrak{B}}^{\mathrm{a}}, \; \widetilde{w}_c \in \widetilde{\mathfrak{B}}^{\mathrm{c}}.$$

Define $w_a := L^{-1}(\widetilde{w}_a)$, $w_c := L^{-1}(\widetilde{w}_c)$. Then,

$$\begin{aligned} w &= L^{-1}(\widetilde{w}) = L^{-1}(\widetilde{w}_a + \widetilde{w}_c) = L^{-1}(\widetilde{w}_a) + L^{-1}(\widetilde{w}_c) \\ &= w_a + w_c. \end{aligned}$$

□

**Example IV.12.** Let $\Sigma = (\mathbb{Z}, \mathbb{R}^2, \mathfrak{B})$ be the 2-periodic system with representation matrix

$$R(\xi, \xi^{-1}) = \begin{bmatrix} \xi^3 - \xi & \xi^5 - \xi \end{bmatrix} = R^L(\xi^2, \xi^{-2})\, \Xi_{2,2}(\xi),$$

with $R^L(\xi, \xi^{-1}) = \left[\begin{array}{cc|cc} \xi - 1 & \xi^2 - 1 & 0 & 0 \end{array}\right]$. It can be shown that the time-invariant lifted behavior $L\mathfrak{B}$, represented by $R^L(\xi, \xi^{-1})$, has the following autonomous/controllable decomposition:

$$L\mathfrak{B} = (L\mathfrak{B})^{\mathrm{a}} \oplus (L\mathfrak{B})^{\mathrm{c}},$$

where the autonomous behavior $(L\mathfrak{B})^{\mathrm{a}}$ is represented by

$$R_a^L(\xi, \xi^{-1}) = \begin{bmatrix} \xi - 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the controllable behavior $(L\mathfrak{B})^{\mathrm{c}}$ has a representation

$$R_c^L(\xi, \xi^{-1}) = \left[\begin{array}{cc|cc} 1 & \xi + 1 & 0 & 0 \end{array}\right].$$

We refer the reader to [5] for further details on how to obtain such a decomposition for time-invariant systems. This implies that $\mathfrak{B}$ can be decomposed as $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$, with $\mathfrak{B}^{\mathrm{a}} = L^{-1}((L\mathfrak{B})^{\mathrm{a}})$ represented by

$$R_a(\xi, \xi^{-1}) = R_a^L(\xi^2, \xi^{-2})\, \Xi_{2,2}(\xi) = \begin{bmatrix} \xi^3 - \xi & 0 \\ 0 & \xi \\ \xi^2 & 0 \\ 0 & \xi^2 \end{bmatrix}$$

and $\mathfrak{B}^{\mathrm{c}} = L^{-1}((L\mathfrak{B})^{\mathrm{c}})$ represented by

$$R_c(\xi, \xi^{-1}) = R_c^L(\xi^2, \xi^{-2})\, \Xi_{2,2}(\xi) = \begin{bmatrix} \xi & \xi^3 + \xi \end{bmatrix}. \quad \diamond$$

## V. Conclusions

In this paper we derived results about periodic behaviors concerning classification of representations, minimality of representations, and controllability. The main tool was the lifting technique that enabled us to exploit known results for time-invariant behaviors. Further results are underway. These concern topics like observability, stability, and controller design.


## Acknowledgments

This research was partially supported by the European Community Marie Curie Fellowship in the framework of the Control Training Site programme (grant no. HPMT-GH-01-00278-98) during the first author's stay at the Department of Applied Mathematics, University of Twente, The Netherlands, as well as by the *Unidade de Investigação Matemática e Aplicações* (UIMA), University of Aveiro, Portugal, through the *Programa Operacional "Ciência e Tecnologia e Inovação"* (POCTI) of the *Fundação para a Ciência e Tecnologia* (FCT), co-financed by the European Union fund FEDER.


## References


[1] M. Kuijper and J. C. Willems, "A behavioral framework for periodically time-varying systems," in *Proceedings of the 36th Conference on Decision & Control*, vol. 3, San Diego, California USA, 10-12 Dec. 1997, pp. 2013–2016.

[2] S. Bittanti and P. Colaneri, "Analysis of discrete-time linear periodic systems," in *Control and Dynamic Systems*, Academic Press, New York, 1996.

[3] ——, "Invariant representations of discrete-time periodic systems," *Automatica*, vol. 36, pp. 1777–1793, 2000.

[4] J. C. Willems, "Paradigms and puzzles in the theory of dynamical systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 3, pp. 259–294, March 1991.

[5] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, ser. Texts in Applied Mathematics. New York: Springer-Verlag, 1998, vol. 26.