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Stabilization of a Class of Planar Nonlinear Systems

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***Abstract*— In this paper we present necessary and sufficient conditions for stabilization of a nonlinear system made up of $M \geq 2$ second order homogeneous subsystems of the same degree. The analysis is based on the partition of all possible trajectories in two distinct sets: the first set concerns confined motions, the second one includes the trajectories rotating around the origin.**

I. INTRODUCTION

The stabilizability of a switching system made up of unstable nonlinear systems is a interesting control problem. The planar problem has been solved in [1] for the linear case.

The aim of this paper is to present a simple geometric procedure capable of facing directly the problem of stabilizability of planar switching systems made up of $M \geq 2$ subsystems of a particular class of nonlinear dynamics, i.e. the homogeneous systems of the same degree.

A key aspect of the procedure is the distinction between sequences yielding a confinement of the motion in a conic sector and sequences leaving the motion free to rotate in the whole state space. The same idea was developed in [2] to obtain necessary and sufficient conditions for the asympotic stability under arbitrary switching of planar linear switching systems. We start with the analysis of the first type of sequences in order to ascertain the presence of stabilizing motions. In this case we give also the conditon of reachability of the region where this motion occurs, from an arbitrary initial state. Only in absence of this kind of motions we analyze the second type of sequences. The latter task is accomplished building two particular switching sequences (one for each rotation direction) yielding a motion, which approach the origin as close as possible. From the analysis of these two sequences we can infer the stabilizability of the system.

The paper is organized as follows: in Section 2 we define homogeneous switching systems and give some necessary preliminary definitions. Section 3 deals with the analysis of confined motions. Section 4 addresses the construction of two best trajectories and the proof of the final stabilizability theorem. In Section 5 two exemples are given and finally, in Section 6 some conclusions are drawn.

II. DEFINITIONS

In accordance with the most common definition in literature we define a *switching system*, a system made up of different sub-systems and by a switching law specifying which sub-system is active at any given instant ([3], [1]). The sole assumption we make on the switching law is that there is a finite number of switchings in a finite time, thus preventing the manifestation of Zeno phenomena.

The model is given by:

$$\begin{aligned} \dot{x} &= \mathbf{f}_i(x) \\ i(t) &= \phi\left(x(t), i(t^-), t\right) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^2$, $\mathbf{f}_i(x) = \begin{bmatrix} f_{i1}(x) & f_{i2}(x) \end{bmatrix}^T$, $i(t) \in \{1, \ldots, M\} \equiv I$ is a set of indices, $\phi : \mathbb{R}^2 \times I \times \mathbb{R} \longrightarrow I$ is a piecewise constant function specifying which subsystem is active in each time instant. The components of vector fields $\mathbf{f}_i(x)$ are time invariant, Lipschitz and homogeneous functions of the same degree having the origin as unique unstable equilibrium point. A homogeneous function of degree k is defined as:

$$f(cx) = c^k f(x).$$

The Lipschitz condition in the origin requires $k \geq 1$([4]).

Definition 1: We define *growth angle* for the vector field $\mathbf{f}$ in the point $x \in \mathbb{R}^2 \setminus \{0\}$ the angle ϑ clockwise measured from the ray, going through the origin and x, to the field $\mathbf{f}$.

Definition 2: A vector field (or a dynamic system) in a point $x \in \mathbb{R}^2 \setminus \{0\}$ is said:

- *clockwise (cw)* if: $\vartheta \in (0, \pi)$;
- *counterclockwise (ccw)* if: $\vartheta \in (\pi, 2\pi)$;
- *radial-ingoing (ri)* if: $\vartheta = \pi$;
- *radial-outgoing (ro)* if: $\vartheta = 0 \equiv 2\pi$.

Recalling the properties of the cross product, we can determine the rotation direction of a vector field $\mathbf{f} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$ in a point $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \setminus \{0\}$ analyzing the sign of the component of the pseudovector produced:

$$m = (\mathbf{f} \wedge x) \cdot \hat{k} = (f_1 x_2 - f_2 x_1) \gtreqless 0 \rightarrow \begin{cases} cw \text{ if } m > 0 \\ ccw \text{ if } m < 0 \\ ro \text{ if } m = 0 \end{cases} \tag{2}$$

where $\hat{k}$ is the unit vector perpendicular to $\mathbf{f}$ and x, with the orientation determined by the right-hand rule. It is straightforward to prove that the previous relation is equivalent to the scalar product of $\mathbf{f}$ and the vector perpendicular to the radial direction $y = Rx$, where $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore, the determination of rotation directions amounts to solving the following inequality:

$$y \cdot \mathbf{f} = x^T R^T \mathbf{f} \geq 0 \tag{3}$$

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Definition 3: Given any two vectors v_1 and $v_2 \in \mathbb{R}^2$ with $v_1 \neq \lambda v_2$, $\lambda \in \mathbb{R}$, the *conic sector* centered in the origin and bounded by the rays $\ell_1 = \lambda_1 v_1$ and $\ell_2 = \lambda_2 v_2$ with $\lambda_1, \lambda_2 \geq 0$ is the set:

$$\Omega = \left\{q \in \mathbb{R}^2 : q = \alpha v_1 + \beta v_2,\ \alpha, \beta \geq 0\right\} \subset \mathbb{R}^2.$$

Definition 4: Given the state space $\mathbb{R}^2$, a state space *finite partition* $\mathcal{P}$ is a finite set of conic sectors $\{\Omega_i\}_{i=1,\ldots,n}$ such that:

i) $\bigcup_{i=1}^{n} \Omega_i = \mathbb{R}^2$;

ii) $\mathrm{int}\,\{\Omega_i\} \cap \mathrm{int}\,\{\Omega_j\} = \emptyset \ \ \forall i, j \in \{1, \ldots, n\} \ \ i \neq j$;

iii) On contiguous sectors different systems are active.

III. Confined motions

Confined motions can be an intrinsic characteristic of the subsystems (spontaneous confinement) or they can be produced by switching among them (induced confinement).

Consider the first case. Solving the equality associated to (3), if a point $\bar{x}$ is a solution, then all the points lying on the ray $l = \lambda \bar{x}$, with $\lambda \geq 0$, are also solutions. Generally, we can have a finite number of isoleted rays of solutions, an infinite number or no solutions at all. If the equation have no solutions, then all the motions produced by the field $\mathbf{f}$ are not confined, instead in the case of finite number of rays each of them define a particular trajectory of the field that is *ri* or *ro*. Each of these trajectories decomposes the plane into sectors such that any trajectory originating in one of them remains inside it forever [4]. However, for our purpose, we are interesting only in convergent confined motions, that is motions going towards the origin. According to a classical classification (see again [4]) we have three kinds of sectors: hyperbolic, node and elliptic sector. Hyperbolic sectors are bounded by a *ri* and a *ro* trajectories and produce only divergent motions. Node sectors are bounded by two *ri* trajectories and produce convergent motions. Elliptic sectors are bounded by a *ri* and a *ro* trajectories and produce convergent motions if we exclude the *ro* ray. However the trajectories of the elliptic sector can go arbitrarily far from the origin before converging on it. To distinguish elliptic from hyperbolic sectors an analysis of the rotation direction is sufficient. Due to the fact that the rotation direction is constant in each sector, it is straightforward to see that if we span the sector from the *ro* ray to the *ri* ray in a *cw* direction and the field is *cw* in that sector, then the sector is elliptic, else it is hyperbolic. The same is true for *ccw* direction.

If the number of rays of solutions is infinite, there exist sectors whose trajectories are all rays through the origin. We are interested only in those of them with *ri* direction. Hence for stabilizability we consider node sectors, elliptic sectors without the *ro* ray and *ri* sectors not limited to a single ray.

As concerns induced confined motions, recall that they can be produced by forcing an inversion of the rotation direction alternating subsystems with different rotation directions. This way, equivalent directions of motion other than those of the single subsystems can be produced. These new directions of motion are induced by *pseudosliding phenomena*. Recalling the Filippov definition [5] of sliding motion:

$$\mathbf{f}(x) = \alpha \mathbf{f}_1(x) + (1-\alpha)\mathbf{f}_2(x) \qquad \alpha \in [0,1]$$

we can note that also for the homogeneous subsystems, if the sliding condition is verified in a point $x \in \mathbb{R}^2$, then it holds for the whole ray passing through the origin and x. In fact, we have:

$$\begin{aligned}\mathbf{f}(cx) &= \alpha \mathbf{f}_1(cx) + (1-\alpha)\mathbf{f}_2(cx)\\ &= c^k\left[\alpha \mathbf{f}_1(x) + (1-\alpha)\mathbf{f}_2(x)\right] = c^k \mathbf{f}(x) \qquad \alpha \in [0,1]\end{aligned}$$

because all subsystems have the same degree of homogeneity. This allows us to consider any equivalent direction as another homogeneous system. Therefore all the directions included in the vector cone bounded by the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are possible.

With the no Zeno phenomena assumption, the existence of a real sliding motion is excluded. There is however the possibility of building admissible switching sequences, which are potentially profitable by alternating the two fields producing equivalent directions of motion close to that of pure sliding (see for instance the *time average control* of [6]). From now on, we will use the term pseudosliding to refer to this type of motions. It can be proven that convergent pseudosliding motions, i.e. pseudosliding motions going towards the origin, are possible if and only if the vector cone generated by $\mathbf{f}_1$ and $\mathbf{f}_2$ includes the *ri* direction. Then, we prove that any kind of induced confined convergent motions is possible if and only if the previous condition holds.

Before proceeding, it is necessary to introduce the following important definition:

Definition 5: We define *best switching law* a switching strategy that associates to every point $x \in \mathbb{R}^2$ the subsystem whose vector field has the minimum[1] cosine of the growth angle in x.

Using the previous association (see [1]), we select the subsystem that drives the system as close as possible to the origin. Furthermore it is not difficult to prove that given two vector fields $\mathbf{f}_1$ and $\mathbf{f}_2$ both *cw* (*ccw*) in $x \in \mathbb{R}^2 \setminus \{0\}$ and defining $\vartheta_1 = \widehat{x\mathbf{f}_1}$ and $\vartheta_2 = \widehat{x\mathbf{f}_2}$ as the growth angles of $\mathbf{f}_1$ and $\mathbf{f}_2$ (according to Definition 1) we have that $\cos\vartheta_1 \leq \cos\vartheta_2$ iff $(\mathbf{f}_1 \wedge \mathbf{f}_2)\cdot\hat{k} \geq 0$ ($(\mathbf{f}_2 \wedge \mathbf{f}_1)\cdot\hat{k} \geq 0$), or equivalently:

$$\begin{aligned}&(\mathbf{f}_1 \wedge \mathbf{f}_2)\cdot\hat{k} \geq 0 \sim (R\mathbf{f}_2)\cdot\mathbf{f}_1 = \mathbf{f}_2^T R^T \mathbf{f}_1 > 0,\\ &\qquad \forall x \text{ s.t. } \mathbf{f}_1,\ \mathbf{f}_2\ cw,\\ &(\mathbf{f}_2 \wedge \mathbf{f}_1)\cdot\hat{k} \geq 0 \sim (R\mathbf{f}_1)\cdot\mathbf{f}_2 = \mathbf{f}_1^T R^T \mathbf{f}_2 > 0,\\ &\qquad \forall x \text{ s.t. } \mathbf{f}_1,\ \mathbf{f}_2\ ccw.\end{aligned} \tag{4}$$

Remark 1: We discard the case $\cos(\vartheta) = -1$ (*ri* direction) due to the previous remark about spontaneous confinement.

We are now ready to establish the following result:

Theorem 1: An induced convergent confined motion is possible for the switching system (1) iff at least a pair of

[1] If more than one subsystem satisfies the minimum criterion, then any of them can be chosen.

subsystems $\mathbf{f}_1$ and $\mathbf{f}_2$ produce a vector cone including the *ri* direction in a point $x \in \mathbb{R}^2$.

Proof: ($\Rightarrow$) If $\mathbf{f}_1$ and $\mathbf{f}_2$ produce in a point $x \in \mathbb{R}^2$ a vector cone including the *ri* direction, then due to the homogeneity of systems, the same condition holds for λx, $\lambda \in \mathbb{R}$. This means a convergent pseudosliding motion takes place and thus an induced convergent confined motion. ($\Leftarrow$) Consider a sector $\Omega \subset \mathbb{R}^2$ bounded by two rays ℓ_1 and ℓ_2 with $\ell_1 \neq \ell_2$ where fields with opposite rotation directions are present and consider a point $x_{0i} \in \ell_1$. If one leaves the system (1) free to evolve according to the minimum cosine criterion applied to cw subsystems in Ω (the ccw case is analogous) the motion eventually reaches a point $x_{Fi} \in \ell_2$. Denote with $\gamma^i_{\min cw}$ this trajectory and with $\mathcal{A}_i$ the compact connected region enclosed by the portions of the radial rays passing through the origin and x_{0i} and x_{Fi} and by $\gamma^i_{\min cw}$. For each $x_{0i} \in \ell_1$ we can define such a region and all these regions define a family $\mathcal{F}_A$. Due to the homogeneity of subsystems the regions are all homothetic between each other. Suppose now there is a not spontaneous convergent confined (in Ω) trajectory γ_d, choosing arbitrarily a point $x_0 \in \gamma_d$ and a region $\mathcal{A}_k \in \mathcal{F}_A$ such that $x_0 \notin \mathcal{A}_k$, we can see that for γ_d to be convergent whilst remaining confined in Ω it has to enter the region $\mathcal{A}_k$ crossing $\gamma^k_{\min cw}$. By construction, $\gamma^k_{\min cw}$ is such that each *cw* system applied to $x \in \gamma^k_{\min cw}$ is directed outwards from $\mathcal{A}_k$ or is tangent to $\gamma^k_{\min cw}$. Therefore the only chance that γ_d has to enter $\mathcal{A}_k$ is by crossing $\gamma^k_{\min cw}$ in a point $\bar{x}$ by means of a *ccw* system (say $\mathbf{f}_2$)[2]. If $\mathbf{f}_1$ is the *cw* system tangent to $\gamma^k_{\min cw}$ in $\bar{x}$, then the vector cone generated by $\mathbf{f}_1$ and $\mathbf{f}_2$ includes the radial direction. What is left to determine, is whether the radial direction is outgoing or ingoing. The system $\mathbf{f}_2$ belongs to the half plane defined by the tangent to $\gamma^k_{\min cw}$ in $\bar{x}$ and including the inwards normal to $\mathcal{A}_k$ in $\bar{x}$. The *ri* direction also belongs to this half plane and therefore the vector cone generated by $\mathbf{f}_1$ and $\mathbf{f}_2$ includes the *ri* direction. ■

It is not difficult to verify that given two vector fields $\mathbf{f}_1$ *cw* in $x \in \mathbb{R}^2 \setminus \{0\}$ and $\mathbf{f}_2$ *ccw* in the same point, the *ri* direction is included in the vector cone generated by $\mathbf{f}_1$ and $\mathbf{f}_2$ iff $(\mathbf{f}_1 \wedge \mathbf{f}_2) \cdot \hat{k} < 0$. As before, we have that $(\mathbf{f}_1 \wedge \mathbf{f}_2) \cdot \hat{k} = f_{11} f_{22} - f_{12} f_{21}$ where $\mathbf{f}_1 = \begin{bmatrix} f_{11} & f_{12} \end{bmatrix}^T$ and $\mathbf{f}_2 = \begin{bmatrix} f_{21} & f_{22} \end{bmatrix}^T$, is equivalent to the scalar product between the vector $\tilde{\mathbf{f}}_2 = R\mathbf{f}_2$ and the vector $\mathbf{f}_1$ where R is defined as above. Hence we have:

$$(\mathbf{f}_1 \wedge \mathbf{f}_2) \cdot \hat{k} < 0 \sim \tilde{\mathbf{f}}_2 \cdot \mathbf{f}_1 = \mathbf{f}_2^T R^T \mathbf{f}_1 < 0, \quad \forall x \text{ s.t. } \mathbf{f}_1 \text{ } cw \text{ and } \mathbf{f}_2 \text{ } ccw \tag{5}$$

What remains to prove is the reachability of conic sectors where convergent confined motions take place. Denoting with $\Omega_{\mathrm{spont}i}$ a generic conic sector where a spontaneous convergent confined motion is present, we define the *spontaneous domain* as follows:

$$\mathcal{D}_{\mathrm{spont}} = \bigcup_i \Omega_{\mathrm{spont}i}.$$

[2]The radial ingoing direction is discarted

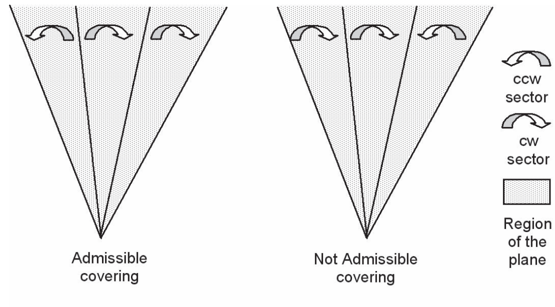

Fig. 1. Examples of covering.

Analogously, for the induced convergent confined motion we define the *pseudosliding domain*:

$$\mathcal{D}_{\mathrm{pseudo}} = \bigcup_i \Omega_{\mathrm{pseudo}i}.$$

Finally the *convergent confined domain* is defined as follows:

$$\mathcal{D}_{\mathrm{conv}} = \mathcal{D}_{\mathrm{spont}} \cup \mathcal{D}_{\mathrm{pseudo}}.$$

We denote instead with Ω_{cw} (Ω_{ccw}) a conic sector bounded by two rays ℓ_1 and ℓ_2 *cw* (*ccw*) numbered where it is possible to define a *cw* (*ccw*) trajectory, that is a trajectory starting from ℓ_1 and reaching ℓ_2.

Definition 6: We define *admissible covering* of a conic region of the plane, a covering obtained with Ω_{cw} and Ω_{ccw} sectors and such that the ray between contiguous sectors does not behave like an actractive domain (see Fig. 1).

The complement set $\mathcal{S} = \mathbb{R}^2 \setminus \mathcal{D}_{\mathrm{conv}}$ consists of conic sectors whose delimiting rays are the external rays of $\mathcal{D}_{\mathrm{spont}}$ and $\mathcal{D}_{\mathrm{pseudo}}$. Finally the reachability condition is:

Proposition 1: It is possible to reach the domain $\mathcal{D}_{\mathrm{conv}}$ from an arbitrary point $x \in \mathbb{R}^2$ iff an admissible covering of $\mathcal{S}$ exists.

IV. NOT CONFINED MOTIONS

If the switching system does not present convergent confined motions, we can continue the analysis with **not** confined motions. Firstly, one can reduce the analysis to two best trajectories monotonically rotating around the origin i.e. trajectories never inverting their rotation direction. In fact, Theorem 2 shows that, once a rotation direction is fixed, there is no advantage in switching on systems with opposite rotation direction. Recalling Definition 5, it is straigthforward to see that the best trajectories are obtained choosing the system having in each point the minimum cosine among those systems having the same rotation directions. For this, it is necessary to introduce the following definition:

Definition 7: Given a conic sector Ω_j bounded by two rays ℓ_1 and ℓ_2 *cw* (*ccw*) numbered, an *overall clockwise (counterclockwise)* trajectory is a trajectory originating in $x_0 \in \Omega_j$ and leaving the sector Ω_j in a point $x_u \in \ell_2$ with a *cw* (*ccw*) vector field.

We are now ready to prove the following theorem.

Theorem 2: Denote with $\gamma_{\min cw}$ ($\gamma_{\min ccw}$) the trajectory originating in a point x_0 belonging to any conic sector Ω_j bounded by rays ℓ_1 and ℓ_2 *cw* (*ccw*) numbered and associating every $x \in \mathbb{R}^2$ the *cw* field having the minimum cosine among the *cw* (*ccw*) systems. Then, naming $x_f \in \ell_2$ the exit point of $\gamma_{\min cw}$ ($\gamma_{\min ccw}$) from Ω_j and $x_u \in \ell_2$ the exit point of any overall *cw* (*ccw*) trajectory γ generated by x_0, we have that $\|x_u\| \geq \|x_f\|$ if there is no convergent pseudosliding.

Proof: The proof is provided only for *cw* case (*ccw* case is similar). As for Theorem 1 we consider the family $\mathcal{F}_A$ of regions $\mathcal{A}_i$ bounded by portions of radial rays ℓ_1 and ℓ_2 and $\gamma^i_{\min cw}$. There exists a k such that $\gamma_{\min cw}$ is a portion of $\gamma^k_{\min cw}$ associated to the region $\mathcal{A}_k$ ($x_0 \in \gamma^k_{\min cw}$). Given the hypothesis of no convergent pseudosliding, the Theorem 1 holds and therefore any overall *cw* trajectory originating in x_0 and leaving Ω_j in a point $x_u \in \ell_2$ lies entirely outside $\mathcal{A}_k$. This implies $\|x_u\| \geq \|x_f\|$. ∎

Finally, we can apply the minimum cosine criterion among systems with the same rotation direction thus producing two best state space partitions $\mathcal{P}_{cw}$ and $\mathcal{P}_{ccw}$.

We now define switching sequence consistent with the partitions as follows:

Definition 8: Given a finite partition of the state space $\mathbb{R}^2$, the *switching sequence consistent with the partition* is the sequence which associates each sector with its corresponding system.

We can define two switching sequences consistent with $\mathcal{P}_{cw}$ and $\mathcal{P}_{ccw}$ and as a result obtain two families of periodic (around the origin) motions to be analyzed for stability.

The stability analysis makes use of the the fact that a trajectory which starts from a point in the state space will eventually intersect a ray passing through the origin and the initial point after a complete rotation (*cw* or *ccw*). If this intersection point has a smaller norm than the original starting point then the system is asymptotically stable, if its norm is greater then the system is unstable and if the two points coincide, then the system is (simply) stable (the trajectory is a constant oscillation). This approach would require a repetition of the same procedure for each trajectory starting from each initial point of the state space. Fortunately, it is possible to prove that in order to derive the stability of the system, it is sufficient to analyze only one trajectory generating from an arbitrary point. This is the matter of the following theorem:

Theorem 3: Given a $\mathcal{P}_{cw}$ or $\mathcal{P}_{ccw}$ partition of the state space $\mathbb{R}^2$ in conic sectors, chosen arbitrarily a ray ℓ and a point $x_0 \in \ell$ and named x_f the point where the trajectory originating from x_0 intersects ℓ for the first time after a turn around the origin when the switching system evolves according to a switching law consistent with the partition, then $\frac{\|x_f\|}{\|x_0\|} = \eta \in \mathbb{R}^+$, η being a constant independent of the choice of x_0.

Proof: The proof is based essentially on the homotheticity of the trajectories. We must show that, given a sector Ω bounded by rays ℓ_1 and ℓ_2 on which only one subsystem is active, the trajectory $\gamma(t, x_0)$, generated by a

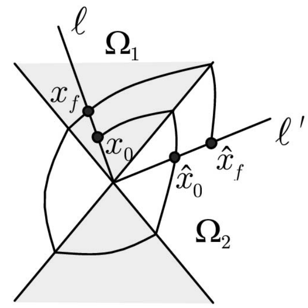

Fig. 2. Example of $\mathcal{P}_{cw}$ partition and related trajectory.

point $x_0 \in \ell_1$ and reaching ℓ_2 in a point x_f is homothetic to any other trajectory of the same subsystem originating from a point $\tilde{x}_0 \in \ell_1$. Recalling that two curves $p(t)$ and $r(s)$ are homothetic of ratio λ with respect to the origin if $\forall \bar{t}$, $\exists \bar{s}$ such that $p(\bar{t}) = \lambda r(\bar{s})$ and, chosen $\tilde{x}_0 = \lambda x_0$, we have the following well known property of differential systems with homogeneous right side [4]:

$$\gamma(t, \lambda x_0) = \lambda \gamma(\lambda^{k-1} t, x_0) = \lambda \gamma(s, x_0). \quad (6)$$

Let us consider a partition $\mathcal{P}_{cw}$ or $\mathcal{P}_{ccw}$ and suppose, without lack of generality, that $x_0 \in \ell \subset \Omega_1$ and x_f as said in the statement of the theorem. We define $\eta = \frac{\|x_f\|}{\|x_0\|}$ and $\hat{x}(t)$ the trajectory from x_0 to x_f. First we have to prove that η is a constant independent of the choice of the initial condition $\hat{x}_0$ if it belongs to $\hat{x}(t)$. To this aim consider the ray ℓ' that goes through $\hat{x}_0$ and identify with $\hat{x}_f$ the point where the trajectory originating from $\hat{x}_0$ intersects ℓ' for the first time after a turn around the origin (see Fig. 2). Let us prove that $\frac{\|\hat{x}_f\|}{\|\hat{x}_0\|} = \eta$. If we put x_f as the initial condition of a new trajectory which arrives on ℓ', we have that $x_f = \eta x_0$. For the linearity of the subsystems all stretches of trajectory contained in a given sector are homothetic among them. Hence[3] if $x_0 \longrightarrow \hat{x}_0$ then $\eta x_0 = x_f \longrightarrow \hat{x}_f = \eta \hat{x}_0$. Finally we have only to prove that any point $\tilde{x}_0$ not necessarily belonging to $\hat{x}(t)$ originates a point $\tilde{x}_f$ such that $\frac{\|\tilde{x}_f\|}{\|\tilde{x}_0\|} = \eta$. To prove it let us take a point $\tilde{x}_0 = \beta \hat{x}_0$ and recall that $\tilde{x}_0 \longrightarrow \tilde{x}_f = \beta \hat{x}_f$ obtaining $\frac{\|\tilde{x}_f\|}{\|\tilde{x}_0\|} = \frac{\beta}{\beta} \frac{\|\hat{x}_f\|}{\|\hat{x}_0\|} = \eta$. ∎

After the construction of $\gamma_{\min cw}$ and $\gamma_{\min ccw}$, we derive the corrisponding η_{cw} and η_{ccw}. The stability is insured if at least one of these constants is less than 1.

Theorem 4: Given a $\mathcal{P}_{cw}$ ($\mathcal{P}_{ccw}$) partition of the state space $\mathbb{R}^2$ in conic sectors, then the associated best trajectory $\gamma_{\min cw}$ ($\gamma_{\min ccw}$) is asymptotically stable iff the corresponding constant η_{cw} (η_{ccw}) is less than 1.

Proof: This theorem can be proved in a manner similar to Theorem 1 of [1]. However, a different proof can be found by rearranging the proof given in [2]. ∎

[3]With the notation $x \longrightarrow y$ we mean that the state x reaches the state y living the switching system evolving according to a switching law consistent with the partition.

As a final result of inspection of confined and not confined motions, it follows:

Theorem 5: The switching system (1) is asymptotically stabilizable iff at least one of the following three conditions holds:

1) $\mathcal{D}_{\text{conv}} \neq \varnothing \wedge \exists$ an admissible covering of $\mathcal{S} = \mathbb{R}^2 \setminus \mathcal{D}_{\text{conv}}$;
2) $\gamma_{\min cw}$ is asymptotically stable;
3) $\gamma_{\min ccw}$ is asymptotically stable.

V. Examples

In this section we show three examples: in Example 1 we consider a system with stable confined motion due to spontaneous confinement, in Example 2 we have a system with stable confined motion by virtue of pseudosliding, in Example 3 we have a system with stable not confined motion.

Example 1: Consider a switching system made up of the following unstable homogeneous subsystems of degree 4:

$$\mathbf{f}_1 = \begin{bmatrix} -4x_1^3x_2 + x_2^4 \\ -(x_1^2 - x_2^2)^2 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} (x_1^2 + 4x_2^2)^2 \\ \sqrt{(x_1^8 + 3x_1^4x_2^4 + 7x_2^8)} \end{bmatrix},$$
$$\mathbf{f}_3 = \begin{bmatrix} -(x_1 + x_2)^2 (3x_1 + 2x_2)^2 \\ x_1^4 \end{bmatrix}.$$

Solving the equality associated to (3) and letting $w_1 = \begin{bmatrix} 1 & 0.4168 \end{bmatrix}^T$, $w_2 = \begin{bmatrix} 1 & 1.491 \end{bmatrix}^T$, $w_3 = \begin{bmatrix} -1 & 0.4117 \end{bmatrix}^T$, we have that the vector field $\mathbf{f}_1$ is *ri* for every $x \in \alpha w_1 \cup \beta w_2 \cup \gamma w_3$, with $\alpha, \beta, \gamma > 0$. The region of spontaneous convergence is:

$$\mathcal{D}_{\text{spont}} = \left\{x \in \mathbb{R}^2 : x = \alpha w_1 + \beta w_3 \text{ with } \alpha, \beta > 0\right\} \longrightarrow \mathbf{f}_1.$$

Letting $v_1 = \begin{bmatrix} 1 & 0.2027 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} -1 & -0.0585 \end{bmatrix}^T$, $v_3 = \begin{bmatrix} -1 & -1.1578 \end{bmatrix}^T$, an admissible covering is:

$$\begin{aligned} x &= \alpha v_1 + \beta w_1 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_3 \\ x &= \alpha v_1 + \beta v_3 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_2 \\ x &= \alpha v_2 + \beta v_3 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_3 \\ x &= \alpha v_2 + \beta w_3 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_2 \end{aligned}$$

Choosing as initial state the point $x_0 = \begin{bmatrix} -5 & -1 \end{bmatrix}^T$, the evolution of the system is given in Fig. 3.

Example 2: Consider a switching system made up of the following unstable homogeneous subsystems of degree 3:

$$\mathbf{f}_1 = \begin{bmatrix} \frac{x_1^5}{x_1^2 + x_2^2} \\ -x_1^2x_2 - x_2^3 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} \sqrt{(x_1^6 + 3x_2^6)} \\ x_1x_2^2 \end{bmatrix},$$
$$\mathbf{f}_3 = \begin{bmatrix} (x_1 + x_2)^3 \\ -x_1^3 \end{bmatrix}.$$

Solving inequality (3) we have that the vector field $\mathbf{f}_3$ is *cw* for every $x \in \mathbb{R}^2$, the vector field $\mathbf{f}_2$ is *cw* in the I° and II° quadrant and *ccw* in the III° and IV° quadrant. The vector field $\mathbf{f}_1$ is *cw* in the I° and III° quadrant and *ccw* in the II° and IV° quadrant. Solving inequality (5) we ascertain the presence of pseudosliding and we have:

$$\mathcal{D}_{\text{pseudo}} = \left\{x \in \mathbb{R}^2 : x_1 < 0\right\}.$$

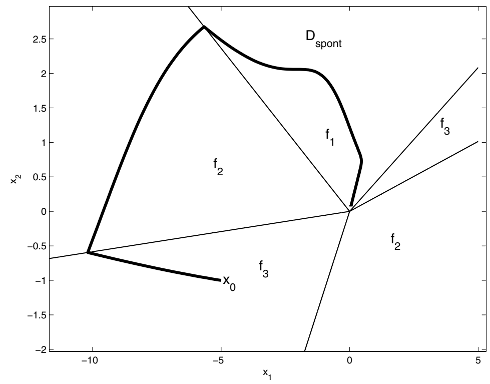

Fig. 3. Convergent spontaneous motion.

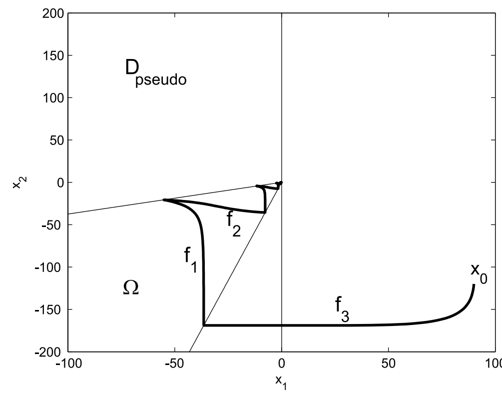

Fig. 4. Convergent pseudosliding (the $\mathcal{D}_{\text{pseudo}}$ domain is the left half plane).

Arbitrarily choosing a conic sector $\Omega \subset \mathcal{D}_{\text{pseudo}}$ and $x_0 = \begin{bmatrix} 90 & -120 \end{bmatrix} \notin \mathcal{D}_{\text{pseudo}}$ as starting point, the reachability is obtained by means of $\mathbf{f}_3$. The evolution of the switching system is shown in Fig. 4.

Note that, the time spent by $\mathbf{f}_1$ and $\mathbf{f}_2$ to cross the sector Ω decreases approaching the origin, but it is zero only in the origin. Hence the no Zeno phenomena assumption is still satisfied.

Example 3: Consider a switching system made up of the following unstable homogeneous subsystems of degree 2:

$$\mathbf{f}_1 = \begin{bmatrix} -x_1^2 + x_1x_2 - x_2^2 \\ x_1^2 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} (x_1 + x_2)^2 \\ -3x_1^2 \end{bmatrix},$$
$$\mathbf{f}_3 = \begin{bmatrix} -x_1x_2 + \frac{1}{4}x_2^2 \\ x_1^2 + \frac{1}{2}x_1x_2 \end{bmatrix}.$$

Solving inequality (3) and letting $v_1 = \begin{bmatrix} -0.8815 & 0.4720 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} -0.4252 & 0.9051 \end{bmatrix}^T$ and $v_3 =$

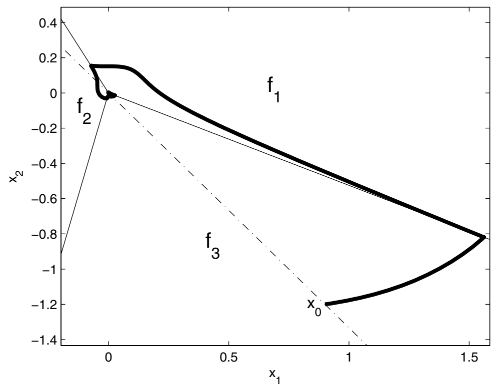

Fig. 5. Convergent not confined motion.

$\begin{bmatrix}0.1994 & 0.9799\end{bmatrix}^T$ we have:

$$\begin{aligned}&\mathbf{f}_1 \text{ is } ccw\ \forall x > \alpha v_1,\ cw \text{ elsewhere}\\&\mathbf{f}_2 \text{ is } ccw\ \forall x > \alpha v_2,\ cw \text{ elsewhere}\\&\mathbf{f}_3 \text{ is } ccw\ \forall x > \alpha v_3,\ cw \text{ elsewhere}\end{aligned}$$

for $\alpha \in \mathbb{R}$.[4] Solving the inequalities (4) we can build the following $\mathcal{P}_{ccw}$ partition[5]:

$$\mathcal{P}_{ccw} : \begin{cases} x = \alpha v_2 - \beta v_1 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_1 \\ x = \alpha v_2 - \beta v_3 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_2 \\ x = -\alpha v_1 - \beta v_3 \quad \text{with } \alpha,\beta > 0 \longrightarrow \mathbf{f}_3 \end{cases} .$$

The evolution of the switching system is stable (see Fig. 5).

VI. Conclusions

Necessary and sufficient conditions for stabilizability of a class of non-linear planar systems are presented. The procedure consists in the division between confined motions and those which are not confined. The first type of convergence is connected to the existence of spontaneous or induced confined motions. The second type involves the study of two particular monotonic (in direction of rotation) trajectories. With this analysis we can infer the stabilizability of the whole system.

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[4]With the inequality $\forall x > \alpha v$, we refer to the half-plane 'above' the straight line passing through the vector v.

[5]It is worth noting that for even degree of homogeneity the partitions can be asymmetrical.