Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Control of underactuated mechanical systems by the transverse function approach

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***Abstract*— An approach for the control of a class of underactuated mechanical systems on Lie groups, including many systems previously studied in the control literature, like underactuated planar manipulators and rigid bodies (spacecrafts, hovercrafts, etc), is proposed. The main outcome of the paper is the derivation, based on the transverse function approach initially proposed by the authors for the control of nonholonomic (driftless) mechanical systems, of smooth feedback control laws which stabilize, in a practical sense, *any* (possibly non-admissible) reference trajectory in the configuration space.**

I. Introduction

This paper addresses the control of underactuated (mechanical) systems the dynamics of which can be modeled in the form

$$\begin{cases} \dot{g} = X(g)v := \sum_{i=1}^{n} X_i(g)v_i \\ \dot{v} = \varphi(v) + \sum_{i=1}^{m} e_i u_i \quad (m \leq n) \end{cases} \tag{1}$$

with g the system's configuration (e.g. position and orientation) belonging to an n-dimensional connected Lie group G, $\{X_1, \ldots, X_n\}$ a *left-invariant* basis of the group's Lie algebra $\mathfrak{g}$, $v \in \mathbb{R}^n$ a vector of *instantaneous velocities*, φ a smooth vector-valued function (typically containing terms associated with Coriolis and centrifugal forces), $\{e_1, \ldots, e_m\}$ independent vectors of $\mathbb{R}^n$, and $u = (u_1, \ldots, u_m)$ the vector of control inputs produced by actuators. Such a system is *invariant* on the Lie group G in the sense that, given an initial velocity $v(0)$ then, whatever the input function $t \mapsto u(t)$ $(t \geq 0)$ applied to the system, the associated trajectory originated at some point g_1 is the same as the one originated at another point g_2, modulo a *fixed* translation on the group. This property is a consequence of the non-dependence of the system's dynamical equations (the second set of equations) upon the system's configuration g. When $m = n$, the system is said to be completely, or fully, actuated. In this case, it is conceptually possible to simplify the second set of equations into $\dot{v} = u$. This corresponds to pre-compensating the drift vector φ. Otherwise, when $m < n$, the system is *underactuated*. In this latter case, some of the coupling Coriolis forces cannot be directly compensated by the actuators. The system's state is the couple (g, v). It is simple to verify that a fully actuated system is small-time locally controllable (STLC) in the sense of Sussmann [1], whereas an underactuated system may, or may not, possess this property (see [2], [3] for more details).

A reference trajectory $g_r(t)$ $(t \geq 0)$ on G is said to be *admissible* if it satisfies the system's equations for some velocity $v_r(t)$ and input $u_r(t)$. Traditionally, trajectory stabilization for underactuated systems has focused on admissible trajectories. A particular case of an admissible trajectory of System (1) is a fixed configuration on G. By application of Brockett's theorem [4], it is well known that the asymptotic stability of such a configuration cannot be achieved with a continuous pure-state feedback when $m < n$. It can however be obtained with a continuous time-varying feedback [5] when classical sufficient conditions for the system to be STLC are satisfied, and explicit feedback laws of this type have been proposed for a certain number of mechanical systems (see e.g. [6], [7], [8], [9]). Hybrid (continuous/discrete) feedback laws have also been considered [10], [3], [11]. The asymptotic stabilization of *specific non-constant* admissible trajectories has been addressed in several studies [12], [13], [14], but it has been proven in [15], under mild assumptions, that the property of admissibility is not by itself sufficient to ensure the existence of a continuous (possibly time-varying) asymptotical stabilizer. This result points out the difficulty/impossibility to guarantee the convergence of the tracking error to zero when other properties of the reference trajectory (in terms of persistent excitation, for instance) cannot be asserted in advance.

The difficulties evoked above, for the asymptotic stabilization of admissible reference trajectories, are not specific to underactuated mechanical systems. They are also encountered with non-holonomic (driftless) systems. In this latter context, we have proposed in [16] a control approach which circumvents them by slightly weakening the objective of asymptotic stabilization. The smooth feedback control laws derived with this approach yield the *practical* stability of any (not necessarily admissible) reference trajectory, and ensure the ultimate boundedness of the tracking error by a pre-specified arbitrary small value. Note that, in the case of a non-admissible trajectory, this is as good a result as one can hope for. While underactuated mechanical systems are significantly different from (and more difficult to control than) non-holonomic systems, we show in this paper that the approach proposed in [16], based on the use of transverse functions, can be adapted to them in order to practically stabilize *any* trajectory on G with a smooth control law. The property of smoothness is important because it is related to the issue of good numerical conditioning and, more generally, to the one of robustness with respect to a certain number of adverse conditions (measurement noise, modeling errors, etc...). To our knowledge, the published work closest to the control approach and results described here is [17]. There are, however, important differences with what we have done: in this reference, the concept of transverse function is absent (a notion of a dynamic oscillator is used instead), the properties of systems on Lie groups are not explicitely exploited, and only the case of admissible trajectories is considered.

II. Notation and recalls

A. Systems on Lie groups

Let G denote a connected Lie group of dimension n, and $\bullet$ the associated group operation. The neutral element for this operation is denoted as e, i.e. $\forall g \in G \;:\; g \bullet e = e \bullet g = g$. The inverse g^{-1} of $g \in G$ is the (unique) element in G such that $g \bullet g^{-1} = g^{-1} \bullet g = e$. The left (resp. right) translation operator on G is denoted as l (resp. r), i.e. $\forall(\sigma,\tau) \in G^2 \;:\; l_\sigma(\tau) = r_\tau(\sigma) = \sigma \bullet \tau$. A v.f. X on G is left-invariant iff $\forall(\sigma,\tau) \in G^2,\; dl_\sigma(\tau)X(\tau) = X(\sigma \bullet \tau)$, with df denoting the differential of the function f. The Lie algebra of the group G –of left-invariant v.f.– is denoted as $\mathfrak{g}$. The adjoint representation of G equipped with $\bullet$ is denoted as Ad, i.e. $\forall \sigma \in G,\; \mathrm{Ad}(\sigma) := dI_\sigma(e)$, with $I_\sigma : G \to G$ defined by $I_\sigma(g) := \sigma \bullet g \bullet \sigma^{-1}$. By extension of the definition of Ad, we define $\mathrm{Ad}(\sigma)X(g) := dl_g(e)\mathrm{Ad}(\sigma)X(e)$. If $X \in \mathfrak{g}$, $\exp(tX)$ is the solution at time t of $\dot{g} = X(g)$ with the initial condition $g(0) = e$. A driftless control system $\dot{g} = \sum_{i=1}^m X_i(g)v_i$ on G is said to be left-invariant on G if the control v.f. X_i are left-invariant. Given a family $Y := \{Y_1,\ldots,Y_p\}$ of vector fields on G and a vector $v \in \mathbb{R}^p$, we denote by $Y(g)v$ the vector field $\sum_{i=1}^p Y_i(g)v_i$ (this notation is already used in Eq. (1)).

Let $X = \{X_1,\ldots,X_n\}$ denote a basis of $\mathfrak{g}$. If $(g_1(t),v_1(t))$ and $(g_2(t),v_2(t))$ $(t \geq 0)$ are two solutions to $\dot{g} = X(g)v$, then (by omitting the time index)

$$\frac{d}{dt}(g_1 \bullet g_2^{-1}) = X(g_1 \bullet g_2^{-1})\mathrm{Ad}^X(g_2)(v_1 - v_2) \quad (2)$$

with Ad^X the (invertible) matrix-valued function defined by $\forall \sigma \in G,\; \forall u \in \mathbb{R}^n,\; \mathrm{Ad}(\sigma)X(e)u = X(e)\mathrm{Ad}^X(\sigma)u$. According to this definition, $\mathrm{Ad}^X(e) = I_n$, with I_n the identity matrix associated with $\mathbb{R}^n$. We have also

$$\frac{d}{dt}(g_1^{-1} \bullet g_2) = X(g_1^{-1} \bullet g_2)(u_2 - \mathrm{Ad}^X(g_2^{-1} \bullet g_1)u_1) \quad (3)$$

Let $\mathrm{d}_G : (g_1,g_2) \mapsto \mathrm{d}_G(g_1,g_2)$ denote a distance on G, left-invariant w.r.t. the group operation $\bullet$, i.e. such that $\forall g_1 \in G,\; \mathrm{d}_G(g_2,g_3) = \mathrm{d}_G(g_1 \bullet g_2, g_1 \bullet g_3)$. Then, for any $\gamma \geq 0$, we denote by $\mathrm{B}_G(\gamma) := \{g \in G : \mathrm{d}_G(g,e) \leq \gamma\}$ the closed ball of radius γ and center e in G.

B. Transverse Functions

Let
- $\mathbb{T}^k$ denote the k-dimensional torus, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,
- $X = \{X_1,\ldots,X_n\}$ denote a basis of $\mathfrak{g}$,
- f denote a smooth function from $\mathbb{T}^{n-m}$ $(m < n)$ to a neighbourhood $U \subset G$ of e.

Then, there exists a matrix-valued function C such that, along any differentiable path $\theta(t)$ on $\mathbb{T}^{n-m}$, one has

$$\begin{aligned}\dot{f}(\theta) &= X(f(\theta))C(\theta)\dot{\theta} \\ &= X^1(f(\theta))C^1(\theta)\dot{\theta} + X^2(f(\theta))C^2(\theta)\dot{\theta}\end{aligned} \quad (4)$$

with $X^1 = \{X_1,\ldots,X_m\}$ and $X^2 = \{X_{m+1},\ldots,X_n\}$. The function f is said to be *transversal to the v.f.* $X_1,\ldots,X_m$ iff $C^2(\theta)$ is invertible $\forall \theta \in \mathbb{T}^{n-m}$. The transverse function theorem given in [16] asserts the existence of such functions, whatever the size of U, provided that the Lie algebra generated by the family X^1 is equal to $\mathfrak{g}$. It also provides a general expression for a family of such functions.

III. Control design

The control of fully-actuated mechanical systems has been extensively studied in the past via various approaches (static feedback linearization, passivity,...) and it is not the object of the present study. However, we find it useful to give a control design result in this case (with no claim of originality at this level), prior to treating the more difficult underactuated case, in order to progressively introduce the solution that we propose for the latter case, and help the reader appreciate the similarities and differences between the two control solutions.

IV. Asymptotic stabilization in the full-actuation case

The system's equations are given by

$$\left\{\begin{array}{rcl}\dot{g} &=& X(g)v \\ \dot{v} &=& u\end{array}\right. \quad (5)$$

Consider a trajectory of reference configurations $g_r(t)$, and denote by $v_r(t)$ the associated velocity vector (assumed differentiable), i.e. $\forall t > 0,\quad \dot{g}_r(t) = X(g_r(t))v_r(t)$. The element $\tilde{g}(t) := g_r(t)^{-1} \bullet g(t)$ characterizes the tracking error at time t. By using (3) one obtains the following *error system*:

$$\left\{\begin{array}{rcl}\dot{\tilde{g}} &=& X(\tilde{g})(v - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \\ \dot{v} &=& u\end{array}\right. \quad (6)$$

and $(\tilde{g},v) = (e,v_r)$ is a solution to this control system, associated with the control input $u = \dot{v}_r$. The control problem is now to stabilize this solution. Let V denote a twice differentiable positive function on G, such that for some constants $\gamma, \alpha_m, \alpha_M, \beta_m, \beta_M > 0$, and for any $g \in \mathrm{B}_G(\gamma)$,

$$\begin{array}{lcl}\mathrm{P1} &:& \alpha_m \mathrm{d}_G^2(g,e) \leq V(g) \leq \alpha_M \mathrm{d}_G^2(g,e) \\ \mathrm{P2} &:& \beta_m V(g) \leq \sum_{i=1}^n (dV(g)X_i(g))^2 \leq \beta_M V(g)\end{array} \quad (7)$$

Let us remark that such a function always exists, for instance in the form of a quadratic function when working with a system of coordinates.

Proposition 1 *Let*

$$\begin{aligned} u := &- k(v - \mathrm{Ad}^X(\tilde{g}^{-1})v_r - v^\star(\tilde{g})) + \mathrm{Ad}^X(\tilde{g}^{-1})\dot{v}_r \\ &+ d(F_{v_r} + v^\star)(\tilde{g})\left(X(\tilde{g})(v - \mathrm{Ad}^X(\tilde{g}^{-1})v_r)\right) \end{aligned} \quad (8)$$

with $k > 0$*,* $F_{v_r}(\tilde{g}) := \mathrm{Ad}^X(\tilde{g}^{-1})v_r$*, and*

$$v_i^\star(\tilde{g}) := -k_i dV(\tilde{g})X_i(\tilde{g}) \quad (k_i > 0;\ i = 1, \ldots, n) \quad (9)$$

Then, the feedback control (8) applied to the system (6) exponentially stabilizes the solution $(\tilde{g}, v) = (e, v_r)$*.*

Proof: The control (8) is built following a classical backstepping procedure. More precisely,

$$\tilde{\xi} := v - \mathrm{Ad}^X(\tilde{g}^{-1})v_r - v^\star(\tilde{g}) \quad (10)$$

satisfies the equality $\dot{\tilde{\xi}} = -k\tilde{\xi}$ along any solution of the controlled system. Therefore, $\tilde{\xi}$ exponentially converges to zero and, in view of (6), $\dot{\tilde{g}} \approx X(\tilde{g})v^\star(\tilde{g})$, with $v^\star$ itself chosen in order to yield the exponential stabilization of $\tilde{g} = e$ when this relation is a strict equality. A more complete and rigorous proof consists in showing that the function defined by $\tilde{V}(\tilde{g}, \tilde{\xi}) := V(\tilde{g}) + \mu\|\tilde{\xi}\|^2$, with $\mu > 0$ large enough, is a Lyapunov function for the controlled system, and that $\tilde{V}$ decreases uniformly exponentially to zero along the solutions of this system.

V. Practical stabilization of a class of underactuated systems

In what follows, G is a 3-dimensional Lie group ($n = 3$) (like $\mathbb{R}^3$, $SE(2)$, and $SO(3)$, for example) and we consider systems with two control inputs such that, for some basis $X = \{X_1, X_2, X_3\}$ of $\mathfrak{g}$ (and some possible change of control inputs) System (1) is given by

$$\left\{ \begin{array}{rcl} \dot{g} &=& X(g)v \\ \dot{v}_1 &=& u_1 \\ \dot{v}_2 &=& u_2 \\ \dot{v}_3 &=& av_1v_2 \end{array} \right. \quad (11)$$

with $a \neq 0$. It is not difficult to verify, by application of [1], that these systems are STLC. We will show further, via a selection of (classical) examples, that several underactuated mechanical systems can be modeled by these equations. With the notation of Section IV, the associated error system w.r.t. a trajectory of reference configurations g_r is

$$\left\{ \begin{array}{rcl} \dot{\tilde{g}} &=& X(\tilde{g})(v - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \\ \dot{v}_1 &=& u_1 \\ \dot{v}_2 &=& u_2 \\ \dot{v}_3 &=& av_1v_2 \end{array} \right. \quad (12)$$

and the problem is to determine a feedback control law which (practically) stabilizes the point $\tilde{g} = e$ for this system. Let us first introduce two auxiliary equations whose solutions will be used in the control design

$$\left\{ \begin{array}{rcll} \dot{p}_1 &=& \vartheta_1 & (p_1 \in \mathbb{R}) \\ \dot{h}_1 &=& X_1(h_1)\vartheta_1 & (h_1 \in G) \end{array} \right. \quad (13)$$

The solutions to these equations are given by

$$\begin{array}{rcl} p_1(t) &=& p_1(0) + \int_0^t \vartheta_1(s)\,ds \\ h_1(t) &=& h_1(0) \bullet \exp\left((p_1(t) - p_1(0))X_1\right) \end{array} \quad (14)$$

The last relation indicates that it suffices that $h_1(0)$ be "close" to e and $|\int_0^t \vartheta_1(s)ds|$ be uniformly bounded by a small positive number for $h_1(t)$ to remain "close" to e ($\forall t$). Let us define $\bar{g} := \tilde{g} \bullet h_1^{-1}$. For $\tilde{g}$ to remain close to e, it suffices that h_1 and $\bar{g}$ stay close to e. We show next how to design a smooth feedback control law which achieves this, whatever the reference trajectory. In view of (12), (13), and (2), the time derivative of $\bar{g}$ is given by

$$\dot{\bar{g}} = X(\bar{g})\mathrm{Ad}^X(h_1)\left(\begin{pmatrix} \bar{v}_1 \\ v_2 \\ v_3 \end{pmatrix} - \mathrm{Ad}^X(\tilde{g}^{-1})v_r\right) \quad (15)$$

with $\bar{v}_1 := v_1 - \vartheta_1$. Consider also the set of equations

$$\left\{ \begin{array}{rcl} \dot{p}_1 &=& v_1 - \bar{v}_1 \\ \dot{v}_2 &=& u_2 \\ \dot{v}_3 &=& av_1v_2 \end{array} \right. \quad (16)$$

By setting $y := (p_1, v_2, v_3)^T$, $Y_1(y) := (1, 0, ay_2)^T$, and $Y_2 := (0, 1, 0)^T$, these equations can also be written as

$$\dot{y} = Y_1(y)v_1 + Y_2u_2 + (-\bar{v}_1, 0, 0)^T \quad (17)$$

By noticing that Y_1 and Y_2 coincide with the control v.f. of the 3-dimensional chained system with two inputs (up to the parameter a which is not necessarily equal to one), and by interpreting v_1 and u_2 as control inputs, the system (17) may be seen as a chained system subjected to an additive perturbation (the last term in the right-hand side of (17)). It is also well known (and easy to verify) that these v.f. are left-invariant w.r.t. the group operation $\circ$ on $\mathbb{R}^3$ defined by

$$\forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad x \circ y := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + ay_1x_2 \end{pmatrix} \quad (18)$$

Moreover, the v.f. Y_1, Y_2, and $Y_3 = [Y_2, Y_1] = (0, 0, a)^T$ form a basis of the Lie algebra generated by Y_1 and Y_2. By application of the transverse function theorem [16], there exists a function f, from $\mathbb{T}$ to $\mathbb{R}^3$, which is transversal to Y_1 and Y_2, i.e.

$$\dot{f}(\theta) = Y(f(\theta))c(\theta)\dot{\theta} = \sum_{i=1}^{3} Y_i(f(\theta))c_i(\theta)\dot{\theta} \quad (19)$$

with $c_3(\theta) \neq 0$, $\forall \theta \in \mathbb{T}$. Such a function is defined e.g. by

$$\begin{array}{rcl} f(\theta) &=& \exp((\varepsilon_1 \sin\theta)Y_1 + (\varepsilon_2 \cos\theta)Y_2) \\ &=& \left(\varepsilon_1 \sin\theta, \varepsilon_2 \cos\theta, \frac{a\varepsilon_1\varepsilon_2}{4} \sin 2\theta\right)^T \end{array} \quad (20)$$

with $\varepsilon_1, \varepsilon_2 > 0$. Indeed, one easily verifies from this expression that the relation (19) is satisfied with

$$c_1(\theta) = \varepsilon_1 \cos\theta, \quad c_2(\theta) = -\varepsilon_2 \sin\theta, \quad c_3 = -(\varepsilon_1\varepsilon_2)/2$$

The application, to the system (17), of the approach proposed in [16] for the control of driftless systems invariant on Lie groups then yields to define the new variable

$$\begin{aligned} z &:= y \circ (f(\theta))^{-1} \\ &= \begin{pmatrix} y_1 - f_1(\theta) \\ y_2 - f_2(\theta) \\ y_3 - f_3(\theta) - a f_1(\theta)(y_2 - f_2(\theta)) \end{pmatrix} \end{aligned} \tag{21}$$

Either by application of the relation (18) in [16], or by direct calculation, the time-derivative of z is given by

$$\dot{z} = \Delta(f_1(\theta), z) Y(f(\theta)) \begin{pmatrix} v_1 - \bar{v}_1 - c_1(\theta)\dot{\theta} \\ u_2 - c_2(\theta)\dot{\theta} \\ -c_3\dot{\theta} + \bar{v}_1 v_2 \end{pmatrix} \tag{22}$$

with

$$\Delta(f_1, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a z_2 & -a f_1 & 1 \end{pmatrix}$$

or, equivalently,

$$\left\{ \begin{aligned} \dot{z}_1 &= v_1 - \bar{v}_1 - c_1(\theta)\dot{\theta} \\ \dot{z}_2 &= u_2 - c_2(\theta)\dot{\theta} \\ \dot{z}_3 &= -a c_3 \dot{\theta} + a \bar{v}_1 v_2 - a f_1(\theta)(u_2 - c_2(\theta)\dot{\theta}) \\ &\quad + a y_2 (v_1 - \bar{v}_1 - c_1(\theta)\dot{\theta}) \end{aligned} \right. \tag{23}$$

Let us set $p_1 \equiv f_1$, then $z_1(t) = 0$ ($\forall t$) and the above set of equations yields

$$\left\{ \begin{aligned} \dot{\bar{v}}_1 &= \tfrac{d}{dt}(v_1 - c_1(\theta)\dot{\theta}) = u_1 - c_1'(\theta)\dot{\theta}^2 - c_1(\theta)\ddot{\theta} \\ \dot{z}_2 &= u_2 - c_2(\theta)\dot{\theta} \\ \dot{z}_3 &= -a c_3 \dot{\theta} + a \bar{v}_1 v_2 - a f_1(\theta)(u_2 - c_2(\theta)\dot{\theta}) \end{aligned} \right.$$

Once this choice is made, having in mind that (z_2, z_3) is related to (v_2, v_3) via the relations $z_2 = v_2 - f_2$ and $z_3 = v_3 - f_3 - a f_1(v_2 - f_2)$, the control inputs u_1, u_2, and $\dot{\theta}$ can be used to monitor the vector $(\bar{v}_1, v_2, v_3)^T$ appearing in the right-hand side of (15). The control strategy that we propose here is to exponentially stabilize

$$\bar{\xi} := \begin{pmatrix} \bar{v}_1 \\ z_2 \\ z_3 \end{pmatrix} - \mathrm{Ad}^X(\bar{g}^{-1}) v_r - v^\star(\bar{g}) \tag{24}$$

to zero (compare with (10)), with $v^\star$, given by (9), the function considered in the case of full actuation. One easily verifies that the time-derivative of $\bar{\xi}$ along the system's solutions has the following structure:

$$\left\{ \begin{aligned} \dot{\bar{\xi}}_1 &= u_1 + \varepsilon_1 \dot{\theta}^2 \sin\theta - \varepsilon_1 \ddot{\theta} \cos\theta + r_1 + \bar{\xi}_1 s_1 \\ \begin{pmatrix} \dot{\bar{\xi}}_2 \\ \dot{\bar{\xi}}_3 \end{pmatrix} &= M(\theta) \begin{pmatrix} u_2 \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} r_2 \\ r_3 \end{pmatrix} + \bar{\xi}_1 \begin{pmatrix} s_2 \\ s_3 \end{pmatrix} \end{aligned} \right. \tag{25}$$

with $M(\theta)$ the invertible matrix defined by

$$M(\theta) := \begin{pmatrix} 1 & \varepsilon_2 \sin\theta \\ -a\varepsilon_1 \sin\theta & \frac{a\varepsilon_1\varepsilon_2}{2}\cos 2\theta \end{pmatrix}$$

and r_i, s_i $(i = 1, 2, 3)$, some functions depending upon $\bar{g}, \bar{\xi}_2, \bar{\xi}_3, \theta, v_r$, and $\dot{v}_r$, but not upon $\bar{\xi}_1$. There are obviously many ways to exploit this structure in order to stabilize $\bar{\xi}$ to zero. One of them is pointed out in the following lemma.

Lemma 1 *Consider the smooth feedback control defined by*

$$\left\{ \begin{aligned} \begin{pmatrix} u_2 \\ \dot{\theta} \end{pmatrix} &:= (M(\theta))^{-1} \left(-k \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_3 \end{pmatrix} - \begin{pmatrix} r_2 \\ r_3 \end{pmatrix} \right) \\ u_1 &:= -\varepsilon_1 \dot{\theta}^2 \sin\theta + \varepsilon_1 \theta^{(2)} \cos\theta - r_1 \\ &\quad -\bar{\xi}_1 s_1 - k\bar{\xi}_1 - \bar{\xi}_2 s_2 - \bar{\xi}_3 s_3 \end{aligned} \right. \tag{26}$$

with $k > 0$, $\theta(0)$ equal to any value, and $\theta^{(2)}$ the function depending upon $\bar{g}, \bar{\xi}, \theta, v_r, \dot{v}_r$, and $\ddot{v}_r$, whose value coincides with the time-derivative of the control input $\dot{\theta}$ along any solution of the controlled system.

Then, the application of this control to the system (11) yields the following equality

$$\frac{1}{2}\frac{d}{dt}\|\bar{\xi}\|^2 = -k\|\bar{\xi}\|^2 \tag{27}$$

and thus the exponential stabilization of $\bar{\xi} = 0$.

The proof is straightforward.

We now show that the feedback control law defined in the previous lemma also ensures, under certain conditions, the ultimate boundedness of the distance between g and the reference situation g_r. By using the definitions of y, z, and $\bar{\xi}$, the equation (15) can be rewritten as

$$\begin{aligned} \dot{\bar{g}} = &X(\bar{g}) \mathrm{Ad}^X(h_1) \Big(T(f_1)(v^\star(\bar{g}) + \bar{\xi}) + (0, f_2, f_3)^T + \\ &\big(T(f_1) \mathrm{Ad}^X(\bar{g}^{-1}) - \mathrm{Ad}^X(\tilde{g}^{-1}) \big) v_r \Big) \end{aligned} \tag{28}$$

with

$$T(f_1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a f_1 & 1 \end{pmatrix} \tag{29}$$

By (14), and since $p_1 = f_1$, $h_1(t) = h_1(0) \bullet \exp((f_1(\theta(t)) - f_1(\theta(0)))X_1)$. Let us assume (for the sake of simplification) that $h_1(0)$ is chosen equal to e. Then, when $\max_\theta \|f(\theta)\|$ is small, System (28) may be seen as an "approximation" of

$$\dot{\tilde{g}} = X(\tilde{g}) v^\star(\tilde{g}) \tag{30}$$

for which the point $\tilde{g} = e$ is exponentially stable. One may thus hope that it retains some of the stability properties of (30). The following proposition, which is the main result of this paper, gives concrete form to this hope.

Proposition 2 *Let $h_1(0) := e$, $\theta(0) = \pm\pi/2$, and let η denote a class-$\mathcal{K}$ function such that $\max_\theta(\|f(\theta)\| + \mathrm{d}_G(h_1, e) + \|I_3 - \mathrm{Ad}^X(h_1)\|) \le \eta(\varepsilon)$ with $\varepsilon := \|(\varepsilon_1, \varepsilon_2)\|$. Then, for any constant K_r, there exists $\varepsilon_0, \gamma_g, \gamma_v, \beta > 0$ such that, for any reference trajectory g_r such that $\|v_r\| \le K_r$, and for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\left. \begin{aligned} \mathrm{d}_G(\tilde{g}(0), e) &\le \gamma_g \\ \|(v - v_r)(0)\| &\le \gamma_v \end{aligned} \right\} \Rightarrow \mathrm{d}_G(\tilde{g}, e) \text{ is u.b. by } \beta\eta(\varepsilon) \tag{31}$$

where "u.b." *means* "ultimately bounded"*. Moreover, if $\|\dot{v}_r(t)\|$ and $\|\ddot{v}_r(t)\|$ are bounded, then $\|v(t)\|$ and the control inputs $u_1(t)$, $u_2(t)$, and $\dot{\theta}(t)$, are bounded.*

The important points of this proposition are *i)* the existence of an ultimate bound for the closed-loop tracking error, *ii)* the (theoretical) possibility of reducing this bound as much as desired by choosing ε_1 and ε_2 small enough, and *iii)* the possibility of specifying an attraction domain *uniform* w.r.t. the reference trajectory (for a given bound on $\|v_r\|$), and w.r.t. $\varepsilon \in (0, \varepsilon_0]$.

Proof: From (28), $\dot{\bar{g}} = L_1 + L_2 + L_3 + L_4$ with

$$\begin{array}{ll} L_1 = & X(\bar{g})v^\star(\bar{g}) \\ L_2 = & X(\bar{g})(\mathrm{Ad}^X(h_1)T(f_1) - I_3)v^\star(\bar{g}) \\ L_3 = & X(\bar{g})\mathrm{Ad}^X(h_1)(T(f_1)\mathrm{Ad}^X(\bar{g}^{-1}) - \mathrm{Ad}^X(\tilde{g}^{-1}))v_r \\ & +X(\bar{g})\mathrm{Ad}^X(h_1)(0, f_2, f_3)^T \\ L_4 = & X(\bar{g})\mathrm{Ad}^X(h_1)T(f_1)\bar{\xi} \end{array}$$

Let γ denote any constant such that the properties P_1 and P_2 in (7) are satisfied for $g \in \mathrm{B}_G(\gamma)$. Then, for $\bar{g} \in \mathrm{B}_G(\gamma)$, the derivatives $dV(\bar{g})L_i$ of $V(\bar{g})$ along L_i $(i = 1, \dots, 4)$ satisfy the following relations:

$$\begin{array}{l} dV(\bar{g})L_1 \leq -\beta_m k_m V(\bar{g}) \\ dV(\bar{g})L_2 \leq \alpha_2(\eta + \eta^2)(\varepsilon)V(\bar{g}) \\ dV(\bar{g})L_3 \leq \alpha_3(\eta + \eta^2)(\varepsilon)\left(1 + K_r\zeta(\bar{g})\right)V^{\frac{1}{2}}(\bar{g}) \\ dV(\bar{g})L_4 \leq \alpha_4(1 + \eta)^2(\varepsilon)\|\bar{\xi}(0)\| \exp(-kt)V^{\frac{1}{2}}(\bar{g}) \end{array} \tag{32}$$

with ζ a smooth function, and where $\alpha_1, \dots, \alpha_4$ denote some constants. The first inequality in (32) follows from (7) and (9). The second inequality follows from (7), (9), the definition of η, and the definition (29) of $T(f_1)$. The third inequality is also based on these relations, the fact that $\|v_r\| \leq K_r$, and the relation $\tilde{g} = \bar{g}h_1$ which implies that

$$\mathrm{Ad}^X(\tilde{g}^{-1}) = \mathrm{Ad}^X(h_1^{-1})\mathrm{Ad}^X(\bar{g}^{-1})$$

Finally, the last inequality follows from (7) and (27). By using the assumption $\theta(0) = \pm\pi/2$, and the fact that

$$\mathrm{d}_G(\bar{g}, e) \leq \mathrm{d}_G(\tilde{g}, e) + \mathrm{d}_G(h_1, e) \leq \mathrm{d}_G(\tilde{g}, e) + \eta(\varepsilon) \tag{33}$$

one shows from (24) that, when $\tilde{g}(0)$ and $v(0)$ satisfy the majorations in (31),

$$\|\bar{\xi}(0)\| \leq \alpha_5\left((1 + K_r)\gamma_g + (1 + \eta)\gamma_v + (1 + K_r + \eta)\eta\right)$$

Since η vanishes at $\varepsilon = 0$, one deduces from the above inequality, and in view of (32), that, for $\varepsilon_0, \gamma_g, \gamma_v$ small enough, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\{\bar{g} \in \mathrm{B}_G(\gamma) \text{ and } V(\bar{g}) = \alpha_m\gamma^2\} \Longrightarrow \dot{V}(\bar{g}) < 0 \tag{34}$$

with

$$\dot{V}(\bar{g}) := \sum_{i=1}^{4} dV(\bar{g})L_i$$

By reducing ε_0 and γ_g further, if necessary, one deduces from (7) and (33) that

$$\begin{array}{l} \mathrm{d}_G(\tilde{g}(0), e) \leq \gamma_g \Longrightarrow \mathrm{d}_G(\bar{g}(0), e) \leq \gamma\sqrt{\alpha_m/\alpha_M} \\ \qquad \Longrightarrow V(\bar{g}(0)) \leq \alpha_m\gamma^2 \text{ and } \bar{g}(0) \in \mathrm{B}_G(\gamma) \end{array}$$

so that, by (7) and (34), $\bar{g}(t) \in \mathrm{B}_G(\gamma)$ for all t and $\mathrm{d}_G(\bar{g}(t), e)$ is bounded. Therefore $\mathrm{d}_G(\tilde{g}(t), e)$ is also bounded. The ultimate bound of $\mathrm{d}_G(\tilde{g}, e)$ pointed out by (31) is then obtained by using the fact that $\eta(0) = 0$ (so that $\eta^2(\varepsilon) \ll \eta(\varepsilon)$ when ε is small), and by using (7), (32), and the inequality

$$\mathrm{d}_G(\tilde{g}, e) \leq \mathrm{d}_G(\bar{g}, e) + \mathrm{d}_G(h_1, e) \leq \mathrm{d}_G(\bar{g}, e) + \eta(\varepsilon)$$

The last part of the proposition is obvious. Indeed, the boundedness of $\mathrm{d}_G(\bar{g}, e)$, $\|\bar{\xi}\|$, $\|v^\star\|$, and $\|v_r\|$ yields the boundedness of v_2 and v_3. Since $\|\dot{v}_r\|$ is bounded, u_2, $\dot{\theta}$, and v_1 are also bounded. Finally, the boundedness of u_1 follows from all these bounds and the boundedness of $\|\ddot{v}_r\|$.

VI. EXAMPLES

In this section, examples of systems that can be modeled by (11) are pointed out. The control approach developed in the previous section applies to them directly.

A. Second-order chained system

In the same way as first-order chained systems are used to model the kinematic equations of various nonholonomic mechanisms, the following second-order chained system

$$\left\{ \begin{array}{lcl} \ddot{x}_1 & = & u_1 \\ \ddot{x}_2 & = & u_2 \\ \ddot{x}_3 & = & u_1 x_2 \end{array} \right. \tag{35}$$

can be used to model the dynamics of a certain number of underactuated mechanical systems, like planar PPR and RRR manipulators, idealized surface vessels and underwater vehicles [18], [19], [20], [21], [22]. It is simple to verify that this second-order chained system belongs to the set of systems (11), with $G = \mathbb{R}^3$, $g = x$, $X_1(x) = (1, 0, x_2)^T$, $X_2 = (0, 1, 0)^T$, $X_3 = (0, 0, 1)^T$, and $a = -1$. These v.f. are left-invariant w.r.t. the group operation $\bullet$ defined by $x \bullet y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + y_1 x_2)^T$ (i.e. the same as for the first-order chained system). The next examples illustrate that, as for nonholonomic systems, the transformation of mechanical equations into the chained form is not necessary for control design purposes.

B. Planar PPR manipulator

The system and the equations are those described in [18]:

$$\left\{ \begin{array}{lcl} m_x\ddot{x} - m_3 l\ddot{\alpha}\sin\alpha - m_3 l\dot{\alpha}^2\cos\alpha & = & \tau_1 \\ m_y\ddot{y} + m_3 l\ddot{\alpha}\cos\alpha - m_3 l\dot{\alpha}^2\sin\alpha & = & \tau_2 \\ I\ddot{\alpha} - m_3 l\ddot{x}\sin\alpha + m_3 l\ddot{y}\cos\alpha & = & 0 \end{array} \right. \tag{36}$$

with $m_x > m_y > m_3$, and $I = I_3 + m_3 l^2$. One can verify that the above system of equations is equivalent to

$$\left\{ \begin{array}{lcl} \ddot{\bar{x}} & = & \frac{\tau_1}{m_3} - (\frac{m_x}{m_3} - 1)\ddot{x} \\ \ddot{\bar{y}} & = & \frac{\tau_2}{m_3} - (\frac{m_y}{m_3} - 1)\ddot{y} \\ \ddot{\alpha} & = & \delta\ddot{\bar{x}}\sin\alpha - \delta\ddot{\bar{y}}\cos\alpha \end{array} \right.$$

with $\delta = \frac{m_3 l}{I_3}$, $\bar{x} := x + l\cos\alpha$, and $\bar{y} := y + l\sin\alpha$. From there, it is not difficult to show that this system can also be written as

$$\begin{array}{lcl} \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\alpha} \end{pmatrix} & = & \begin{pmatrix} R(\alpha) & 0_{2\times1} \\ 0_{1\times2} & 1 \end{pmatrix} \bar{v} \\ \dot{\bar{v}} & = & \begin{pmatrix} \bar{u}_1 + \bar{v}_2\bar{v}_3 \\ \bar{u}_2 - \bar{v}_1\bar{v}_3 \\ -\delta\bar{u}_2 \end{pmatrix} \end{array} \tag{37}$$

with $R(\alpha)$ the rotation matrix in the plane of angle α, and $(\bar{u}_1, \bar{u}_2)$ new control variables such that (τ_1, τ_2) is equal to some function (defined everywhere) of $(\bar{u}_1, \bar{u}_2, \alpha, \dot{\alpha})$. This system can in turn be rewritten as (11) with $g_1 := \bar{x} + \frac{\cos\alpha}{\delta} = x + (l + \frac{1}{\delta})\cos\alpha$, $g_2 := \alpha$, $g_3 := \bar{y} + \frac{\sin\alpha}{\delta} = y + (l + \frac{1}{\delta})\sin\alpha$, $u_1 := \bar{u}_1 + \bar{v}_2\bar{v}_3$, $u_2 := -\delta\bar{u}_2$, $v_1 := \bar{v}_1$, $v_2 := \bar{v}_3$, $v_3 = \bar{v}_2 + \bar{v}_3/\delta$, $a = -1$, and

$$X(g) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

The associated Lie group is $G = SE(2)$ and the group operation is defined (with a slight abuse of notation) by $g \bullet g' := g + X(g)g'$.

C. Planar rigid body (hovercraft)

Consider a planar body with center of mass C capable of gliding above the ground with no friction, and denote by $(\bar{x}, \bar{y})$ the coordinates of C w.r.t. a fixed frame. A force is applied to this body at a point P located at a distance l ($\neq 0$) from C, and the direction of $\overrightarrow{PC}$ characterizes the body's orientation α. The components of the force in the body's frame are (f_1, f_2), with f_1 the projection of the force on $\overrightarrow{PC}$. The asymptotic stabilization of the situation of this system has been studied e.g. in [9] and [3]. One can verify that the equations modeling the motion of this underactuated system are the same as those of the planar PPR manipulator. They are given by (37), with $\bar{u}_1 := \frac{f_1}{m}$, $\bar{u}_2 := \frac{f_2}{m}$, and $\delta = \frac{lm}{J}$ (with m and J the body's mass and inertia).

D. Underactuated satellite (with thrusters)

Let us assume that two (sets of) thrusters produce torques to modify the orientation of a rigid body floating in space, and, for the sake of simplification, that the directions of these torques are aligned with the first two principal axes of the satellite. The asymptotic stabilization of the satellite's attitude has previously been studied, for instance, in [6],[7], [8], or [3]. The well-known equations of this system are

$$\left\{ \begin{array}{lcl} \dot{R} & = & RS(v) \\ J\dot{v} & = & Jv \times v + (\tau_1, \tau_2, 0)^T \end{array} \right. \qquad (38)$$

with $S(v)$ the matrix associated with the vector product in $\mathbb{R}^3$, i.e. such that $S(v)x := v \times x$, $J = \mathrm{Diag}(j_1, j_2, j_3)$. We further assume that $a := \frac{j_1 - j_2}{j_3} \neq 0$, so that the system is STLC. A rewriting of this system in the form of (11) is obtained by setting $g = R$, $X_i(R) := RS(e_i)$, $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$, $u_1 := \frac{1}{j_1}(\tau_1 + (j_2 - j_3)v_2v_3)$, and $u_2 := \frac{1}{j_2}(\tau_2 + (j_3 - j_1)v_1v_3)$. For this system, $G = SO(3)$, and the group operation is the classical matrix product.

VII. Concluding remarks

A new control approach for a class of STLC underactuated mechanical systems has been proposed. It is based on the use of transverse functions, yields smooth feedback laws, and allows to practically stabilize *any* (admissible or non-admissible) reference trajectory of configurations with pre-defined precision. Possible prolongations of this work are multiple: extension to systems of higher dimensions, generalization of the approach to a larger class of systems (not necessarily STLC), detailed study of the stabilization of fixed situations and, more generally, of admissible trajectories, robustness issues, etc.

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