Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

An analytically tractable model for combustion instability

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***Abstract*— A system of two coupled Van der Pol equations is considered as a model for combustion instability. This system is analyzed using the Krylov-Bogoliubov method. The results of the analysis are compared with simulation results and with data obtained from a combustion system.**

***Index Terms*— modelling, combustion instability, nonlinear oscillating systems, Krylov-Bogoliubov method.**

I. Introduction

Combustion instabilities in gas turbine engines and power plants are the focus of a significant number of current studies, see [1] for a recent survey and [2] for an overview of a recent research program funded by the USA Office of Navy Research in the area. Major research activities are conducted *inter alia* in USA, France and UK [3], [4], [5], [6]. The heart of the issue is that such engines combust more coolly and therefore pollute less at low equivalence ratio (fuel-to-air ratio). As the equivalence ratio, ϕ, decreases however, a combustion instability appears due to the nonlinear interaction of heat release with combustion chamber acoustics, and the appearance of this instability reverses any gain from lower ϕ. The instability manifests itself through the occurrence of a limit cycle in most variables; pressure, heat release rate, flame surface, etc, producing an audible tone, which leads to its being called *howling*, *hooting* or *screaming*.

Active control has been proposed as a method to deal with this instability, with the actuation taking place primarily through modulation of a fraction of the fuel flow into the combustion chamber [2]. A recent article [7] reports the successful application of periodic forcing at the limit cycle frequency with slowly adaptive tuning of the relative phase. This method does not rely on a model of the system, only on a structure of the controller. Additionally [2], [8], experimental reports indicate that non-harmonic (including sub-harmonic) forcing also can be effective in reducing limit cycle magnitude, although without much guidance as to how to select the forcing parameters.

One of the features of active control of combustion instabilities is the difficulty in developing a reliable model which captures the dominant dynamical effects in realistic operation. The phenomenon has been scientifically studied in the laboratory since Rayleigh and Rijke [9], [10] and has been described much earlier [11]. However, the modelling still is not fully resolved and certainly has not been resolved well enough to permit the design of controllers. Part of the difficulty has been the natural arising of nonlinear delay differential equations through the transcription of compartmental phenomenological models, which capture the known physical subprocesses but are not amenable to tractable analytical tools other than brute simulation. Our objective in this paper is to present, motivate and study an alternative for the widely used model of Perrachio and Proscia [12] and their descendants based on a coupled Van der Pol Equation. This latter dynamic system is then analyzed using the methods of Krylov and Bogoliubov [13], which is amenable to the development of formal stabilization design studies.

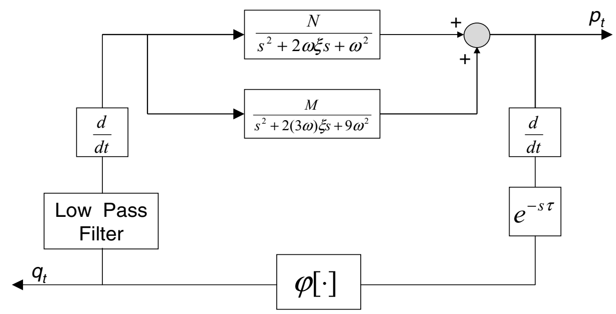

Fig. 1. Block diagram of the model [14]

In [14] a model for the combustion instability has been proposed, which is an extension of [12] through the inclusion of and additional third-harmonic resonance. The block diagram of the model is shown in Figure 1, where p_t is the downstream pressure perturbation at the burning plane, q_t is the flame heat release rate, $\varphi[\cdot]$ is a static non-linear mapping, τ is a transport time delay from nozzle to flame surface, ω is the fundamental acoustic mode of combustor, ξ is the associated damping ratio and N and M are the gains of second-order oscillators with the natural radian frequency equal to ω and 3ω, respectively. The model arises by considering the acoustics of the combustion chamber interacting with the sensitivity of the location of the flame surface. A number of competing physical explanations are available for instability phenomenon based on variations in the length direction alone or in terms of vortex shedding. Our aim is to find a model suited to control; single-input via an actuated variable, single output in terms of a measurable variable (Pressure is one of the few quantities able to be measured in an environment as hostile as an operating combustion chamber.), manageable complexity, and capturing the important dynamical phenomena.

The model is characterized by an inherent feedback structure resulting from the thermo-acoustic feedback [14]. The feedforward path is characterized by two linear resonators in feedback with a nonlinear path which create a negative damping in certain situation leading to the occurrence of the oscillations. It should be noted that this model is composed of a strictly linear forward path in feedback with a memoryless nonlinearity, making it immediately a candidate for describing function analysis, if one ignores the infinite-dimensionality introduced by the delay element, the non-lowpass nature of the forward path, etc. In [14], [15] computational bifurcation analysis of the nonlinear delay-differential was performed and shown to extend the simplified describing function analysis. A rapprochement of the bifurcation diagram with the experimental evidence led to a level of confidence in the capacity of this model to reflect the data.

This delay-differential model is not, however, tractable for feedback control design and stability analysis. Accordingly, we seek to replace it by a similarly performing delay-free nonlinear model based on coupled Van der Pol systems presented below. The equations governing this system belong to the class of equations called *near-conservative autonomous systems* which are described by equations of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \epsilon f\left(x, \frac{dx}{dt}\right), \tag{1}$$

where ϵ is a small positive quantity and f may be a power series in ϵ whose coefficients are polynomial in x and $\frac{dx}{dt}$.

Since in general one cannot find the exact solutions for this type of differential equation, approximation procedures for the analysis of this type of equations have to be considered.

The Krylov-Bogoliubov (K-B) method [13], [16], [17], [18], [19], [20] is without doubt one of the most efficient procedure of analyzing oscillating systems governed by equations of the form (1). In brief the K-B method is looking for solutions of the form

$$x(t) = a(t)\cos\psi(t),$$

where a is the time-varying magnitude of the fundamental oscillation term and ψ is the instantaneous total phase. They obey, in the single resonator case, the differential equations

$$\begin{cases} \frac{da}{dt} = -\frac{1}{2\omega\pi}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi d\psi, \\ \frac{d\psi}{dt} = \omega - \frac{1}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi d\psi. \end{cases}$$

Note that ψ can be written in the form

$$\psi(t) = \omega t + \theta(t),$$

where θ is the instantaneous phase.

It is this approach which will be used for analyzing the behavior of the model of the combustion instability system. From the analysis point of view the combustion instability model presents a number of difficulties among which we mention :

- Presence of two coupled resonators.
- Complicated dynamics in the nonlinear path due to the cascade of a differentiator and delay.

This present work will focus on the analysis of the effect of two-coupled-resonators structure, without the necessity of their being harmonically related.

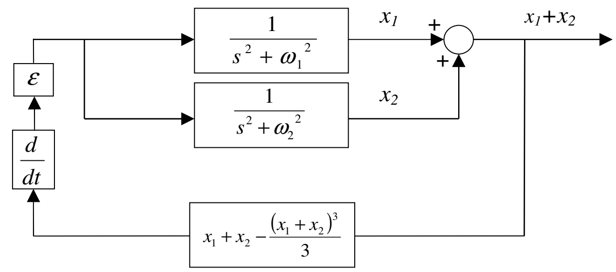

Fig. 2. Combustion instability model based on coupled Van der Pol equations

The nonlinear term will be approximated by a simpler one, but nevertheless representative for these type of oscillations. Specifically the nonlinearity encountered in Van der Pol equations will be considered. Therefore the model of the combustion instability will be approached by a system of two coupled Van der Pol equations (two coupled Van der Pol generators)

$$\begin{cases} \frac{d^2x_1}{dt^2} + \omega_1^2 x_1 = \epsilon\frac{d}{dt}\left((x_1+x_2) - \frac{1}{3}(x_1+x_2)^3\right), \\ \frac{d^2x_2}{dt^2} + \omega_2^2 x_2 = \epsilon\frac{d}{dt}\left((x_1+x_2) - \frac{1}{3}(x_1+x_2)^3\right), \end{cases} \tag{2}$$

where ω_1 and ω_2 are the natural radian frequencies of the first and second equations respectively and which can have arbitrary values with some modest provisions to be developed, ϵ is a small positive quantity and the corresponding block diagram is shown in Figure 2. A study involving a small-parameter linearized analysis of the Dunstan model operating with noise excitation in a regime immediately before the appearance of the limit cycle has been conducted in [14], [21]. This bears strong resemblance to the this Van der Pol model. At a fundamental level, the presence of the differentiator, appearing in [12], in the right-hand path has been questioned from a physical perspective. Because of this uncertainty, the presence of small loop gain ϵ, and the resonant forward path, the removal of the cascade differentiator followed by a time delay should be manageable provided the requisite phase match is preserved.

While this type of system has not yet been studied in the literature, one can mention that the single resonator Van der Pol equation has been successfully analyzed using the K-B method [13], [16], [17], [18], [19], [20].

The system (2) will be analyzed by the K-B method and systematically compared with the results of the simulation of system (2). This will allow us to see to what extent, the K-B method gives results close to the exact solutions. In the last part of the paper the results obtained will be compared with the phenomena observed with the combustion instability in [14].

II. First K-B approximation for autonomous multi-resonator systems

Consider a system with n resonators which are be described by equations of the form

$$\frac{d^2x_k}{dt^2} + \omega_k^2 x_k = \epsilon f_k\left(x, \frac{dx}{dt}\right), \quad (k = 1, 2, \ldots, n) \tag{3}$$

where $x = \{x_1, \ldots, x_n\}$, $\frac{dx}{dt} = \{\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\}$ and ϵ is a small parameter.

To summarize (for more details see Chapter 2 of [18]), for the resonator j, the first K-B approximation proposes the solution

$$x_j = a_j \cos(\psi_j), \tag{4}$$

where $\psi_j = \omega_j t + \theta_j$, a_j and θ_j are slowly time-varying functions obeying the equations

$$\left\{ \begin{array}{l} \frac{da_j}{dt} = -\frac{\epsilon}{2\omega_j} H_{jj}(a_1, \ldots, a_n, \theta_1, \ldots, \theta_n), \\ \frac{d\theta_j}{dt} = -\frac{\epsilon}{2\omega_j a_j} G_{jj}(a_1, \ldots, a_n, \theta_1, \ldots, \theta_n). \end{array} \right. \tag{5}$$

with H_{jj} and G_{jj} are obtained from the function $f_j\left(x, \frac{dx}{dt}\right)$ by substituting

$$\left\{ \begin{array}{l} x_k = a_k \cos(\omega_k t + \theta_k), \\ \frac{dx_k}{dt} = -a_k \omega_k \sin(\omega_k t + \theta_k), \end{array} \right. \quad (k = 1, 2, \ldots, n) \tag{6}$$

and by setting it in the form

$$\begin{aligned} & f_j\,(a_1 \cos(\omega_1 t + \theta_1), \ldots, a_n \cos(\omega_n t + \theta_n), \\ & \qquad - a_1 \omega_1 \sin(\omega_1 t + \theta_1), \ldots, -a_n \omega_n \sin(\omega_n t + \theta_n)) \\ & = H_{jj} \sin(\omega_j t + \theta_j) + G_{jj} \cos(\omega_j t + \theta_j) \\ & \quad + \sum_{\omega_j \not\approx \omega_\ell}^{r} (H_{\ell j} \sin(\omega_\ell t + \theta_\ell) + G_{\ell j} \cos(\omega_\ell t + \theta_\ell)), \end{aligned} \tag{7}$$

where ω_ℓ and θ_ℓ are the linear combinations of $\omega_1, \ldots, \omega_n$ and $\theta_1, \ldots, \theta_n$, respectively, and r is the number of possible linear combinations of $\omega_1, \ldots, \omega_n$ different from ω_j. Furthermore for x_j, the coefficients of fundamental term in (7) are used and the all other terms are eliminated.

III. K-B APPROXIMATION OF TWO COUPLED VAN DER POL EQUATIONS

Consider the equations system (2) and the form (3), in this case

$$\begin{aligned} f_1 = f_2 &= f(x_1, x_2, \tfrac{dx_1}{dt}, \tfrac{dx_2}{dt}) \\ &= \left(1 - (x_1 + x_2)^2\right) (\tfrac{dx_1}{dt} + \tfrac{dx_2}{dt}). \end{aligned} \tag{8}$$

Introducing

$$\left\{ \begin{array}{l} x_i = a_i \cos(\omega_i t + \theta_i), \\ \frac{dx_i}{dt} = -a_i \omega_i \sin(\omega_i t + \theta_i), \end{array} \right. \quad (i = 1, 2)$$

into (8), one gets

$$\begin{aligned} & f\,(a_1 \cos(\omega_1 t + \theta_1), a_2 \cos(\omega_2 t + \theta_2), \\ & \quad -a_1 \omega_1 \sin(\omega_1 t + \theta_1), -a_2 \omega_2 \sin(\omega_2 t + \theta_2)) \\ & = -\left(1 - (a_1 \cos(\omega_1 t + \theta_1) + a_2 \cos(\omega_2 t + \theta_2))^2\right) \\ & \quad \times (a_1 \omega_1 \sin(\omega_1 t + \theta_1) + a_2 \omega_2 \sin(\omega_2 t + \theta_2)). \end{aligned} \tag{9}$$

To approximate the solution of (2), it is necessary to set (9) in the form (7). In [22], one gives the details of computation leading to the expression

$$\begin{aligned} & f\,(a_1 \cos(\omega_1 t + \theta_1), a_2 \cos(\omega_2 t + \theta_2), \\ & \quad -a_1 \omega_1 \sin(\omega_1 t + \theta_1), -a_2 \omega_2 \sin(\omega_2 t + \theta_2)) \\ & = -\omega_1 a_1 \left(1 - \tfrac{a_1^2}{4} - \tfrac{a_2^2}{2}\right) \sin(\omega_1 t + \theta_1) \\ & \quad - \omega_2 a_2 \left(1 - \tfrac{a_2^2}{4} - \tfrac{a_1^2}{2}\right) \sin(\omega_2 t + \theta_2) \\ & \quad + \omega_1 \tfrac{a_1^3}{4} \sin(3(\omega_1 t + \theta_1)) + \omega_2 \tfrac{a_2^3}{4} \sin(3(\omega_2 t + \theta_2)) \\ & \quad + (2\omega_1 + \omega_2) \tfrac{a_1^2 a_2}{2} \sin((2\omega_1 + \omega_2)t + 2\theta_1 + \theta_2) \\ & \quad + (\omega_1 + 2\omega_2) \tfrac{a_1 a_2^2}{2} \sin((\omega_1 + 2\omega_2)t + \theta_1 + 2\theta_2) \\ & \quad + (2\omega_1 - \omega_2) \tfrac{a_1^2 a_2}{4} \sin((2\omega_1 - \omega_2)t + 2\theta_1 - \theta_2) \\ & \quad + (2\omega_2 - \omega_1) \tfrac{a_2^2 a_1}{4} \sin((2\omega_2 - \omega_1)t + 2\theta_2 - \theta_1), \end{aligned} \tag{10}$$

from which one can see the existence of the frequency set

$$\begin{aligned} W = \{ & \omega_1, \omega_2, 3\omega_1, 3\omega_2, 2\omega_1 + \omega_2, \omega_1 + 2\omega_2, \\ & 2\omega_1 - \omega_2, 2\omega_2 - \omega_1 \}. \end{aligned} \tag{11}$$

This set is very important for finding the possible operation regimes of the system, i.e. for x_1 (respectively x_2), the remaining terms from (10) after application of the K-B approximation will only be the terms with the frequency ω from W such as $\omega \approx \omega_1$ (respectively ω_2). Consequently, one has the following classification, which will be elaborated and explained shortly :

1) $\omega_1 \not\approx \left\{\omega_2, 3\omega_2, \frac{\omega_2}{3}\right\}$-two generators with competitive quenching
2) $\omega_1 \approx \omega_2$-mutual synchronization with close frequencies
3) $\omega_1 \approx 3\omega_2$ (respectively $\omega_2 \approx 3\omega_1$)-mutual synchronization with multiple frequencies

A. Two generators with competitive quenching

Consider the case where the frequencies ω_1 and ω_2 respect Condition 1 above. In this case, there is no interconnection effect between the both frequencies and the K-B approximation uses only the fundamental oscillations terms of $f\,(a_1 \cos(\omega_1 t + \theta_1), a_2 \cos(\omega_2 t + \theta_2),$ $-a_1 \omega_1 \sin(\omega_1 t + \theta_1), -a_2 \omega_2 \sin(\omega_2 t + \theta_2))$. Consequently the approximate solutions of (2) are (for details see [22])

$$x_i = a_i \cos(\omega_i t + \theta_i), \quad (i = 1, 2) \tag{12}$$

with

$$\left\{ \begin{array}{l} \frac{da_1}{dt} = \epsilon \frac{a_1}{2} \left(1 - \frac{a_1^2}{4} - \frac{a_2^2}{2}\right), \\ \frac{da_2}{dt} = \epsilon \frac{a_2}{2} \left(1 - \frac{a_2^2}{4} - \frac{a_1^2}{2}\right), \\ \frac{d\theta_1}{dt} = 0, \\ \frac{d\theta_2}{dt} = 0. \end{array} \right. \tag{13}$$

Let us find steady-state solutions of (13). In this case, (13) possesses four steady-state solutions

$$a_1 = 0 \text{ and } a_2 = 0, \tag{14}$$

$$a_1 = \tfrac{2}{\sqrt{3}} \text{ and } a_2 = \tfrac{2}{\sqrt{3}}, \tag{15}$$

$$a_1 = 2 \text{ and } a_2 = 0, \tag{16}$$

$$a_1 = 0 \text{ and } a_2 = 2. \tag{17}$$

Both former solutions (14) and (15) are unstable, and both latter solutions (16) and (17) are stable. Therefore, the amplitudes of x_1 and x_2 converge to one of both

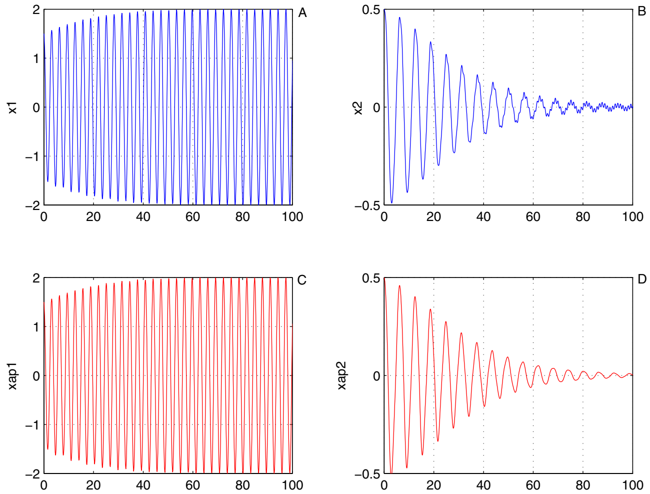

Fig. 3. (A) x_1 simulated from (2), (B) x_2 simulated from (2), (C) x_1 approximated by (12) and (13), (D) x_2 approximated by (12) and (13)

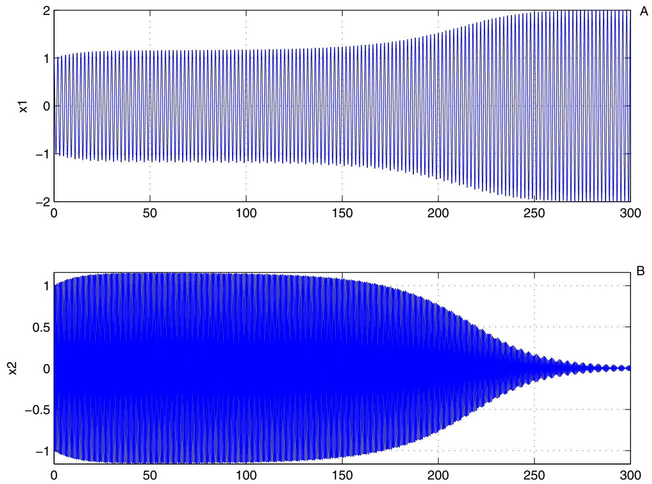

Fig. 4. (A) x_1 simulated from (2), (B) x_2 simulated from (2)

possible stationary states (16) and (17). Depending on the initial condition, one of the generators is excited, while the oscillations of the other generator are entirely quenched. Such quenching of the oscillations of one of the generators, caused by the sufficiently large non-linear coupling between them, is know as *competitive quenching*.

It was noted that, if $a_1(0) > a_2(0)$, x_1 is excited and the oscillations of x_2 are entirely quenched, and the converse effect occurs when $a_1(0) < a_2(0)$. Figure 3 presents a simulation test with $\omega_1 = 2$, $\omega_2 = 1$, $\epsilon = 0.1$, $a_1(0) = 1.5$ and $a_2(0) = 0.5$, the upper part shows the outputs of (2) and lower part shows the outputs approximated by (12) and (13). Also, when $a_1(0) = a_2(0)$ (but not equal to zero), it was noted that:

- In (13), the magnitudes a_1 and a_2 converge to $\frac{2}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ respectively, which correspond to the unstable steady-state (15).
- In (2), the magnitudes a_1 and a_2 converge to $\frac{2}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ respectively, and remain temporarily, but after a long time (if one compares it to the convergence dynamics) these the magnitudes will converge necessarily to one of the steady-states (16) and (17).

This implies that in certain conditions, both frequencies can coexist for a long time before the entry into the competitive quenching regime. To illustrate this phenomenon, Figure 4 presents a simulation test with $\omega_1 = \pi$, $\omega_2 = 3.5\omega_1 = 3.5\pi$, $\epsilon = 0.1$, $a_1(0) = 1$ and $a_2(0) = 1$, the upper part is the output x_1 of (2) and lower part is the output x_2 of (2).

B. Mutual synchronization with close frequencies

Consider the case where the frequencies ω_1 and ω_2 are close. For x_1 (respectively x_2), the application of K-B approximation implies the conservation of all coefficients of sinusoidal terms in $f(a_1\cos(\omega_1 t+\theta_1), a_2\cos(\omega_2 t+\theta_2), -a_1\omega_1\sin(\omega_1 t+\theta_1), -a_2\omega_2\sin(\omega_2 t+\theta_2))$ with a frequency close to ω_1 (respectively ω_2) and the elimination of all other terms. Consequently the approximate solutions of (2) are (for details see [22])

$$x_i = a_i\cos(\omega_i t+\theta_i), \quad (i = 1, 2) \tag{18}$$

with a_1, a_2, θ_1 and θ_2 are governed by

$$\left\{\begin{aligned}
\frac{da_1}{dt} &= \epsilon\left\{\frac{a_1}{2}\left(1-\frac{a_1^2}{4}-\frac{a_2^2}{2}\right)\right.\\
&\quad+\left[\frac{a_2\omega_2}{2\omega_1}\left(1-\frac{a_1^2+a_2^2}{4}\right)-\frac{a_2a_1^2}{4}\right]\cos(\Delta\psi)\\
&\quad\left.+(\omega_1-2\omega_2)\left(\frac{a_1a_2^2}{8\omega_1}\right)\cos(2\Delta\psi)\right\},\\
\frac{da_2}{dt} &= \epsilon\left\{\frac{a_2}{2}\left(1-\frac{a_2^2}{4}-\frac{a_1^2}{2}\right)\right.\\
&\quad+\left[\frac{a_1\omega_1}{2\omega_2}\left(1-\frac{a_1^2+a_2^2}{4}\right)-\frac{a_1a_2^2}{4}\right]\cos(\Delta\psi)\\
&\quad\left.+(\omega_2-2\omega_1)\left(\frac{a_2a_1^2}{8\omega_2}\right)\cos(2\Delta\psi)\right\},\\
\frac{d\theta_1}{dt} &= -\epsilon\left\{\frac{a_2^2}{8\omega_1}(\omega_1-2\omega_2)\sin(2\Delta\psi)+\left[\frac{a_2a_1}{8\omega_1}(2\omega_1-\omega_2)\right.\right.\\
&\quad\left.\left.+\frac{a_2\omega_2}{2a_1\omega_1}\left(1-\frac{a_2^2}{4}-\frac{a_1^2}{2}\right)\right]\sin(\Delta\psi)\right\},\\
\frac{d\theta_2}{dt} &= \epsilon\left\{\frac{a_1^2}{8\omega_2}(\omega_2-2\omega_1)\sin(2\Delta\psi)+\left[\frac{a_2a_1}{8\omega_2}(2\omega_2-\omega_1)\right.\right.\\
&\quad\left.\left.+\frac{a_1\omega_1}{2a_2\omega_2}\left(1-\frac{a_1^2}{4}-\frac{a_2^2}{2}\right)\right]\sin(\Delta\psi)\right\},
\end{aligned}\right. \tag{19}$$

where $\Delta\psi = \psi_1 - \psi_2 = (\omega_1-\omega_2)t + \theta_1 - \theta_2$.

This result is very important, because in parallel with differential equation (2), it is possible to compute from (19) the amplitude and the phase evolutions of the output and to compare both to signals measured in practice.

The integration and study of stationary solutions of (19) are very difficult. However, to find the stationary solutions when $\omega_1 = \omega_2 = \omega$, one can adopt the following steps.

Using $y = x_1 + x_2$, if we add the both equations of (2) we obtain

$$\begin{aligned}
\frac{d^2(x_1+x_2)}{dt^2}+\omega^2(x_1+x_2) &= \epsilon\left(1-(x_1+x_2)^2\right)\frac{d(x_1+x_2)}{dt}\\
\Rightarrow\ \frac{d^2y}{dt^2}+\omega^2 y &= \epsilon\left(1-y^2\right)\frac{dy}{dt}.
\end{aligned} \tag{20}$$

It is seen that (20) corresponds to classical Van der Pol equation. It is well to known that for the classical Van der Pol

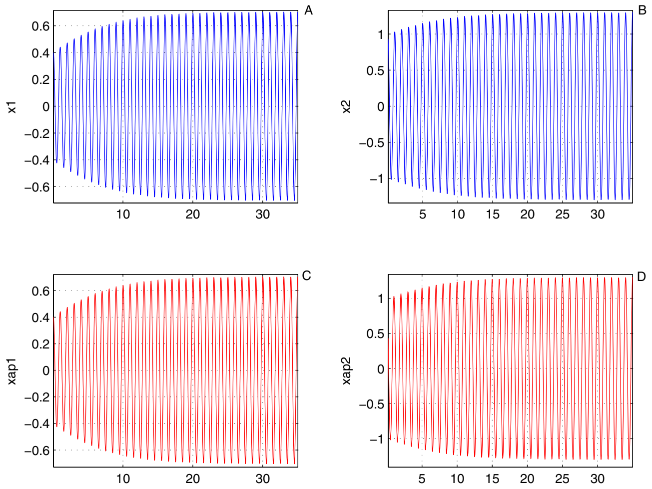

Fig. 5. (A) x_1 simulated from (2), (B) x_2 simulated from (2), (C) x_1 approximated by (18) and (19), (D) x_2 approximated by (18) and (19)

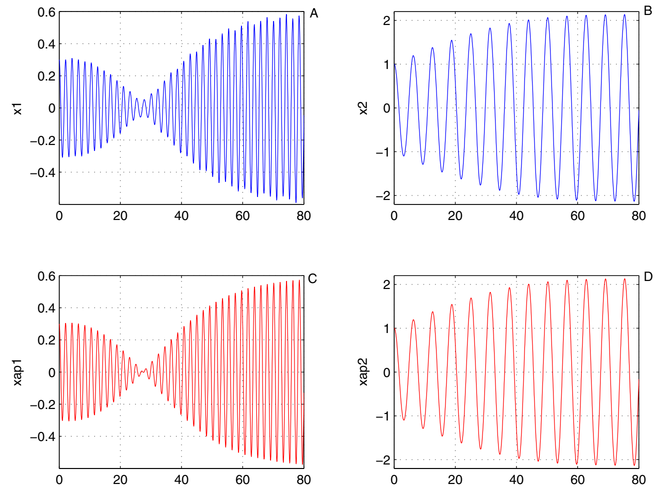

Fig. 6. (A) x_1 simulated from (2), (B) x_2 simulated from (2), (C) x_1 approximated by (24) and (25), (D) x_2 approximated by (24) and (25)

equation, the K-B approximation gives a stationary solution [19], [13], [16], [17], [18], [20]

$$y = 2\cos(\omega t + \theta). \tag{21}$$

$$\Rightarrow x_1 + x_2 = 2\cos(\omega t + \theta), \tag{22}$$

where θ is the arbitrary instantaneous phase, which satisfies

$$a_1^2 + a_2^2 + 2a_1 a_2 \cos(\theta_1 - \theta_2) = 4 \tag{23}$$

One notes that there exist an infinity of steady-state points and that the convergence of the amplitude and phase depends essentially on the initial state of x_1 and x_2. Therefore, to get the same result between (2) and (19), one must initialize (19) with the appropriate values of initial amplitude and phase.

Figure 5 shows a simulation test with $\omega_1 = \omega_2 = 2\pi$, $a_1(0) = 0.4$, $a_2(0) = 1$, $\epsilon = 0.1$ and $\theta_1(0) = \theta_1(0) = \frac{\pi}{2}$, the upper part is the outputs of (2) and lower part is the outputs approximated by (18) and (19).

C. Mutual synchronization with multiple frequencies

Consider the case where the frequency ω_1 is close to $3\omega_2$. In this case the terms with frequencies ω_1 and $3\omega_2$ are used for x_1 approximation, and the terms with frequencies ω_2 and $(2\omega_2 - \omega_1)$ are used for x_2 approximation. Therefore, one finds (for details see [22])

$$x_i = a_i \cos(\omega_i t + \theta_i), \quad (i = 1, 2) \tag{24}$$

with

$$\left\{ \begin{aligned} \frac{da_1}{dt} &= \epsilon \left[\frac{a_1}{2}\left(1 - \frac{a_1^2}{4} - \frac{a_2^2}{2}\right) - \frac{\omega_2 a_2^3}{8\omega_1} \cos(\Delta\psi) \right], \\ \frac{da_2}{dt} &= \epsilon \left[\frac{a_2}{2}\left(1 - \frac{a_2^2}{4} - \frac{a_1^2}{2}\right) \right. \\ &\quad \left. + \frac{a_1 a_2^2}{8\omega_2}(2\omega_2 - \omega_1)\cos(\Delta\psi) \right], \\ \frac{d\theta_1}{dt} &= \epsilon \frac{\omega_2 a_2^3}{8\omega_1 a_1} \sin(\Delta\psi), \\ \frac{d\theta_2}{dt} &= -\epsilon \frac{a_1 a_2}{8\omega_2}(\omega_1 - 2\omega_2)\sin(\Delta\psi), \end{aligned} \right. \tag{25}$$

where $\Delta\psi = \psi_1 - 3\psi_2 = (\omega_1 - 3\omega_2)t + \theta_1 - 3\theta_2$,

Let us find steady-state solution of (25). One stable steady-state point can be computed analytically from (25)

$$a_1 = 2 \quad \text{and} \quad a_2 = 0. \tag{26}$$

Others points need numerical solution of (25). Furthermore, one introduces

$$p = \frac{\omega_1 - 3\omega_2}{\omega_2}. \tag{27}$$

When $p = 0$ and for any value of ω_2 (or ω_1), it exist one other stable steady-state point

$$a_1 = 0.593, \quad a_2 = 2.136 \quad \text{and} \quad \Delta\psi = \pi. \tag{28}$$

For $p \neq 0$, it exists one other stable steady-state point which depend from the value of p and ω_2 (or ω_1).

From this, one can see that if ω_1 is close to $3\omega_2$ (respectively $\frac{\omega_2}{3}$), it is possible to have two phenomena depending on the initial condition. In the first phenomenon, the generator with frequency ω_1 is excited and the other generator with frequency ω_2 is quenched. In the second phenomenon, one has synchronization regime. By a synchronization regime is meant that the oscillation frequency of the second generator, which is equal to $\omega_2 + \dot{\theta}_2$, is exactly a third of the oscillation frequency of the first generator, which is equal to $\omega_1 + \dot{\theta}_1$.

Figure 6 presents a simulation test with $\omega_1 = 3\omega_2 = 3$, $\epsilon = 0.1$, $a_1(0) = 0.3$, $a_2(0) = 1$ and $\theta_1(0) = \theta_1(0) = \frac{\pi}{2}$, the upper part is the outputs of (2) and lower part is the outputs approximated by (24) and (25).

IV. Summary of the Analysis Results

We have identified the following three situations relating the proximity of the natural frequencies of the individual oscillators. From a practical point of view one can say that the system is characterized in steady state either by a single oscillating frequency (which correspons to one of the resonance frequencies of the linear oscillators) or by stable simultaneous oscillations (which correspond to synchronized oscillations of both generators).

This single oscillation phenomena which is known as *competitive quenching phenomena*, occurs when the ratio of resonance frequencies of the two resonators are different from 3, 1 and $\frac{1}{3}$, and occurrence of one of frequencies (among the two) will depend on the initial conditions.

Stable simultaneous oscillations with two distinct frequencies will occur only when $\omega_1 \approx 3\omega_2$ or $\omega_2 \approx 3\omega_1$ and the initial condition is sufficiently good so that both generators are excited.

V. Comparison with experimental results

Representative experimental results are discussed in a [23], [15] which demonstrate salient dynamical phenomena, such as the simultaneous presence of two sinusoidal components; a strong dominant tone at 210Hz, and a lesser but persistent non-harmonic tone at 714Hz. The capacity of a model to display the coexistence of these modes is regarded as an important corroboration of the model.

In [15], [14] a describing function analysis (slightly extended) shows that equilibrium period solutions at these frequencies should exist with 210Hz being stable and 714Hz having a low-dimensional unstable manifold. This would normally lead to the extinction of the 714Hz mode except for a measure-zero set of initial conditions. A bifurcation analysis shows that, contrary to the prediction from describing functions, the 714Hz mode of the Dunstan model is in fact stable, but that noise induced perturbations can induced jumping to and from the stable 210Hz mode, thereby creating the coexistence.

It is important to note that the new model proposed here also demonstrates presence of two stable limit cycles with non-measure-zero basins of attraction. The ability of the model to capture this unusual and testable aspect of the system dynamics, is a strong indicator of its possible strength for system design. The tractability of the K-B method for analysis is a major advantage of this model versus that of [14].

VI. Conclusion

The aim of this paper has been to show that it is possible to go further in the analysis of the instability combustion model proposed in [14]. The two coupled Van der Pol equations considered in this paper may be an effective choice to approach the combustion instability model. The analysis method is based on the use of Krylov-Bogoliubov approach for oscillatory systems. Indeed, this approach allows one to overcome one of the difficulties related to the combustion instability model, i.e. the presence of two coupled resonators. The simulation tests have illustrated the precision of the Krylov-Bogoliubov approximation.

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