Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Conservativity and Time-Flow Invertibility of Boundary Control Systems

Jarmo Malinen
Institute of Mathematics, TKK
Box 1100 FI-02015
Helsinki University of Technology, Finland.
Jarmo.Malinen@tkk.fi

Abstract— **We give sufficient and necessary conditions for a boundary control system (in the sense of Salamon) to define a Livšic – Brodskiĭ operator node; i.e. a linear (scattering) conservative system. This appears to be a special case of a more general result involving time-flow invertible linear systems. Finally, an example involving the wave equation is considered.**

I. Introduction

In this paper, we give sufficient and necessary conditions for the conservativity of linear boundary control systems. Such systems are often described by differential equations of the form

$$\begin{cases} \dot z(t) = Lz(t), \\ Gz(t) = u(t), \\ y(t) = Kz(t) \quad \text{for all } t \geq 0 \end{cases} \tag{1}$$

where the operators comprising the *boundary node* $\Xi = (G, L, K)$ satisfy the additional conditions of Definition 1.

In this paper, the linear systems are described by *system nodes* (or, more generally, *operator nodes*) $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on the Hilbert spaces[1] U, Y, X as defined in [12, Section 2]. Following e.g. [12, Proposition 2.5], assume that the functions $u(\cdot) \in C^2(\mathbb{R}_+; U)$, $z(\cdot) \in C^1(\mathbb{R}_+; X)$, $y(\cdot) \in C(\mathbb{R}_+; Y)$ satisfy the differential equation associated to S:

$$\begin{cases} \dot z(t) &= A_{-1} z(t) + Bu(t), \\ y(t) &= C\&D \left[\begin{smallmatrix} z(t) \\ u(t) \end{smallmatrix}\right] \quad \text{for all } t \geq 0; \end{cases} \tag{2}$$

here $A \in \mathcal{L}(X_1, X)$, $A_{-1} \in \mathcal{L}(X, X_{-1})$ and $B \in \mathcal{L}(U; X_{-1})$ are the main operator, the *Yoshida extended* main operator and the input operator of S, respectively, and $X_{-1} := \mathrm{dom}\,(A^*)^d$. We shall show in Theorem 1 that any *internally well-posed* boundary node Ξ can be translated to a unique system node S satisfying the conditions of Definition 2, so that the differential equations (1) and (2) have the same, unique solutions for same $u(\cdot) \in C^2(\mathbb{R}_+; U)$ and (compatible) initial values $z(0) = z_0$.

What is a (scattering) conservative linear system, then? We say that the system node S is *energy preserving* if for any input $u(\cdot) \in C^2(\mathbb{R}_+; U)$ and any (compatible) initial state $z(0) = z_0$, the unique solution of (2) satisfies the *energy balance equation* $\frac{d}{dt}\|x(t)\|_X^2 = \|u(t)\|_U^2 - \|y(t)\|_Y^2$, see [12, Definition 3.1]. That S is (a) *conservative* (system) means that both S and S^d are energy preserving, where S^d denotes the *dual system node* of S as described in [12, Proposition 2.3]. This notion of conservativity is the “right one” because it connects directly to the classical Livšic – Brodskiĭ (operator) nodes. A rich theory exists for these nodes (including a good selection of canonical realizations and a state space isomorphism theorem), see [1], [4], [5], [6], [12], [16], [17], [18], [19].

Unfortunately, this definition of conservativity refers directly to S^d, and it is less than obvious how to relate S^d to the operators G, L, K appearing in (1) – the given data of a typical boundary control problem. The purpose of this paper is to solve these complications with the aid of main Theorems 5 and 6. Such tools are required, when applying conservativity-related operator theory techniques (such as the one proposed in [7] for numerical input/output approximation) to practical problems (such as the wave equation in $\Omega \subset \mathbb{R}^n$, described in Section VI). The boundary control related results of this paper can also be found in [11].

II. Boundary nodes and operator nodes

We develop the required background results for boundary nodes, and show that they induce operator nodes (of boundary control type). We then review the solvability of (1). Let us start with two definitions.

Definition 1: Assume that U, X and Y are (separable) Hilbert spaces. Assume that Z is a Hilbert space, such that $Z \subset X$ with a bounded and dense inclusion.

(i) Let $L \in \mathcal{L}(Z; X)$, $G \in \mathcal{L}(Z; U)$ and $K \in \mathcal{L}(Z; Y)$ be operators such that the following conditions hold for some $\alpha \in \overline{\mathbb{C}_+}$:
  (a) $U = \mathrm{Ran}\ G$,
  (b) $\mathrm{Ker}\ G$ is dense in X,
  (c) $(\alpha - L)\mathrm{Ker}\ G = X$, and
  (d) $\mathrm{Ker}\ (\alpha - L) \cap \mathrm{Ker}\ G = \{0\}$.

  Then the triple $\Xi = (G, L, K)$ is called *a boundary node* on spaces U, X, Z and Y. The space Z is *the solution space* of Ξ.

(ii) If both $\Xi = (G, L, K)$ and $\Xi^{\leftarrow} := (K, -L, G)$ are boundary nodes, then Ξ is called a *doubly* boundary node.

[1] All Hilbert spaces in this paper are separable.

(iii) If $L|\mathrm{Ker}\ G$ generates a C_0-semigroup, Ξ is then called *internally well-posed*.

There are a number of essentially equivalent definitions and names for boundary nodes. See e.g. [14], [15] that unfortunately contain a mistake[2]. An earlier approach is [2].

Definition 2: Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an operator node on spaces U, X and Y as in [12, Definition 2.2]. Then S is of *boundary control type* (in the sense of Salamon), if $\rho(A) \cap \overline{\mathbb{C}_+} \neq \emptyset$, Ker $B = \{0\}$ and $BU \cap X = \{0\}$.

The spaces U, X, and Y are called *input*, *state*, and *output* spaces of both Ξ and S, respectively. It is sometimes said that B is *strictly unbounded* if $BU \cap X = \{0\}$.

Any boundary node is canonically associated to an operator node of boundary control type.

Theorem 1: Let $\Xi = (G, L, K)$ be a boundary node on Hilbert spaces U, X and Y with the solution space Z. Define

$$\begin{aligned} &X_1 := \mathrm{Ker}\ G \quad \text{ and } A := L|X_1 : X_1 \to X, \\ &X_{-1} := \mathrm{dom}\,(A^*)^d \text{ and } A_{-1} : X \to X_{-1} \text{ as usual}, \\ &BGz := Lz - A_{-1}z \quad \text{ for all } z \in Z, \text{ and} \\ &V := \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right] : A_{-1}x + Bu \in X\}. \end{aligned}$$

Then $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node of boundary control type (on spaces U, X, and Y) with $\mathrm{dom}\,(S) = V$, where for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$

$$A\&B \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := A_{-1}x + Bu \text{ and } C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := Kx. \tag{3}$$

Also $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ and $Z = X_1 \dot{+} (\alpha - A_{-1})^{-1} BU$ hold for any $\alpha \in \rho(A)$. Moreover, S is a system node if and only if Ξ is internally well-posed.

Proof: See [10, Theorem 1, Proposition 3]. ■

We proceed to solve the Cauchy problem for differential equation (1) by using Theorem 1 and [12, Proposition 2.5].

Lemma 1: Assume that $\Xi = (G, L, K)$ is an internally well-posed boundary node on Hilbert spaces U, X, Z and Y. Let $u \in C^2([0, \infty); U)$ and $z_0 \in Z$ be such that the compatibility condition $Gz_0 = u(0)$ is satisfied.

(i) Then equations (1) has a unique classical solution $z(\cdot) \in C([0, \infty); Z) \cap C^1([0, \infty); X)$, such that $z(0) = z_0$ and $y(\cdot) \in C([0, \infty); Y)$.

(ii) The same functions $u(\cdot)$, $z(\cdot)$ and $y(\cdot)$ satisfy (2), too.

Proof: By $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ denote the system node that is related to Ξ as in Theorem 1. Define the norm for $V = \mathrm{dom}\,(S)$ by setting

$$\left\| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|_V^2 := \|x\|_X^2 + \|u\|_U^2 + \|A_{-1}x + Bu\|_X^2.$$

Since A is the generator of a C_0-semigroup, it follows from [12, Proposition 2.5] that there exists a unique $z(\cdot) \in C^1([0, \infty); X) \cap C^2([0, \infty); X_{-1})$ such that (2) holds and $\left[\begin{smallmatrix} z(\cdot) \\ u(\cdot) \end{smallmatrix}\right] \in C([0, \infty); V)$. Since the inclusion $V \subset \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$ is bounded and $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ by [10, Proposition 3], it follows that $z(\cdot) \in C([0, \infty); Z)$ and $u(t) = Gz(t)$ for all $t \geq 0$. Since $L = A_{-1}|Z + BG$, (2) implies that for all $t \geq 0$

$$\dot{z}(t) = A_{-1}z(t) + Bu(t) = (A_{-1}|Z + BG)\, z(t) = Lz(t).$$

Since $C\&D$ and K are connected by (3), we conclude that $z(\cdot)$ solves (1). ■

Lemma 1 gives a working interpretation to differential equation (1). Note that the trajectory $z(\cdot)$ is continuous in Z, but the derivative $\dot{z}(\cdot)$ is computed (as a limit of a differential quotient) in the weaker norm of X.

We remark that the converse of Theorem 1 holds, too. Indeed, given an operator node S of boundary control type, we can construct L, G, and K, so that $\Xi = (G, L, K)$ is a boundary node, see [10, Theorem 2]. Then the relation of Ξ to S is the same as described in Theorem 1 above.

III. Conservativity and time-flow inverses

Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a conservative system with input operator B and output operator $C := C\&D|X_1$. Such systems satisfying the additional requirements

$$\mathrm{Ker}\ B = \{0\}, \qquad \mathrm{Ker}\ C^* = \{0\}$$

are called *tory systems*[3] in [12]. In some sense, tory systems have no "redundant" or "wasted" subspaces in U and Y, so that all the information is circulated through the state space. Such systems have been characterized in [12, Theorem 4.4] by using as few assumptions as possible. Any conservative system can be represented as a "cartesian product" of a tory system and an isometric isomorphism from Ker B onto Ran $C^\perp$, see [12, Theorem 4.5]. The purpose of this section is to give yet another characterisation – Theorem 2 – for tory systems. The proofs for Theorems 5 and 6 depend on this theorem.

For some system nodes $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, equations (2) can be solved backwards in time, if the input and output are interchanged, too. For bounded B, C, D, and D^{-1}, the inverse dynamics can be obtained easily:

$$\begin{cases} \dot{z}(t) = \left(-A + BD^{-1}C\right)_{-1} z(t) - BD^{-1}y(t), \\ u(t) = -D^{-1}Cz(t) + D^{-1}y(t). \end{cases}$$

The general case is unfortunately more technical:

Definition 3: Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an operator node on spaces U, X and Y with $\mathrm{dom}\,(S) = V$. We say that S is *time-flow invertible*, if there exists an operator node $S^\leftarrow = \left[\begin{smallmatrix} [A\&B]^\leftarrow \\ [C\&D]^\leftarrow \end{smallmatrix}\right]$ on spaces Y, X and U, with $\mathrm{dom}\,(S^\leftarrow) = V^\leftarrow$ and the main operator $A^\leftarrow$ such that

(i) both $\rho(A) \cap \overline{\mathbb{C}_+} \neq \emptyset$ and $\rho(A^\leftarrow) \cap \overline{\mathbb{C}_+} \neq \emptyset$,

(ii) $\begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix} : V \to V^\leftarrow$ is a bounded bijection, and

(iii) we have on all of $V^\leftarrow$

$$S^\leftarrow = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1}. \tag{4}$$

When these conditions hold for S and $S^\leftarrow$, we say that $S^\leftarrow$ is the *time-flow inverse* of S.

Time-flow invertibility has been treated in depth in [18], [20]. We point out that $(S^\leftarrow)^\leftarrow = S$ whenever S is time-flow invertible. Time-flow invertibility and conservativity of S go

[2]This rather small mistake was independently discovered (at least) by G. Weiss and the author.

[3]Such operator nodes are also known as *Julia colligations*.

hand in hand as the following two propositions will remind us. (Recall that S^d stands for the dual operator node of S.)

Proposition 1: Let S be a system node. Then S is conservative if and only if it is time-flow invertible and $S^d = S^{\leftarrow}$.

Proposition 2: An energy preserving system node S is conservative if and only if it is time-flow invertible.

For proofs of these propositions, see either [10, Propositions 5 and 6], or [18, Lemma 11.2.4], [20, Theorem 7.2].

Proposition 3: Assume $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a time-flow invertible system node. Let $A^{\leftarrow} : X_1^{\leftarrow} \to X$ be the main operator and $C^{\leftarrow}$ the output operator of the time-flow inverse $S^{\leftarrow}$. Assume that the dual cross-term equation holds

$$C\&D \begin{bmatrix} I \\ B^* \end{bmatrix} = 0 \quad \text{on} \quad X_1^d, \tag{5}$$

and $A^* = A^{\leftarrow}$ (with equal domains). Then

$$A_{-1} + A^* + BB^* = 0 \quad \text{on} \quad X_1^d$$

and $C^{\leftarrow} = B^*$ on X_1^d.

Proof: Because $A^* = A^{\leftarrow}$, we have $X_1^{\leftarrow} = X_1^d$. Hence $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}$ for all $x \in X_1^d$. Because $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] : V \to V^{\leftarrow}$ is a bounded bijection (see Definition 3), there exists for any $x \in X_1^d$ a *unique* vector $\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right] \in V$ such that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}.$$

By using the assumed dual cross-term equation (5), we see that in fact $x_1 = x$ and $u_1 = B^* x$. Hence, for any $x \in X_1^d$

$$\begin{bmatrix} A^{\leftarrow} x \\ C^{\leftarrow} x \end{bmatrix} = S^{\leftarrow} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ B^* x \end{bmatrix} = \begin{bmatrix} -A_{-1}x - BB^*x \\ B^*x \end{bmatrix}.$$

But $A^{\leftarrow} x = A^* x$ by assumption, and the claim follows. ■

We shall next characterize tory systems using the time-flow inverse $S^{\leftarrow}$ instead of the dual system S^d, as is usual. The proof of the following theorem is based on Proposition 3, [12, Proposition 2.4 and Theorem 4.4].

Theorem 2: Assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a time-flow invertible operator node. By $A^{\leftarrow}$ denote the main operator of the time-flow inverse $S^{\leftarrow}$. Then S is tory if and only if

(i) $\operatorname{Ker} B = \{0\}$,

(ii) $A + A_{-1}^* = -C^*C$ on X_1,

(iii) $C\&D \begin{bmatrix} I \\ B^* \end{bmatrix} = 0$ on X_1^d, and

(iv) we have $A^{\leftarrow} = A^*$ (with equal domains).

Proof: Conditions (i) – (iii) are necessary for toryness by [12, Theorem 4.4]. By Proposition 1, tory systems satisfy $S^d = S^{\leftarrow}$, and (iv) follows, too.

Assume that conditions (i) – (iv) hold. Then the dual Liapunov equation is given by Proposition 3, and S is tory by [12, Theorem 4.4] provided we can show that $\operatorname{Ker} C^* = \{0\}$. Following [12, Proposition 2.4], decompose the space Y orthogonally $Y = \left[\begin{smallmatrix} Y_1 \\ Y_0 \end{smallmatrix}\right]$ where $Y_1 = \overline{\operatorname{Ran} C}$ and $Y_0 = Y_1^{\perp}$. The induced decomposition of S is then given by

$$S = \begin{bmatrix} [A\&B]_r \\ [C\&D]_r \\ 0 \quad D_{01} \end{bmatrix} : V \to \begin{bmatrix} X \\ Y_1 \\ Y_0 \end{bmatrix} \quad \text{with} \quad S_r := \begin{bmatrix} [A\&B]_r \\ [C\&D]_r \end{bmatrix};$$

here S_r is the reduced operator node with output space Y_1, the domains satisfy $V = \operatorname{dom}(S) = \operatorname{dom}(S_r)$, and $D_{01} \in \mathcal{L}(U; Y_0)$ is nonzero if and only if Y_0 is nontrivial. Since $B = B_r$, $C = \left[\begin{smallmatrix} C_r \\ 0 \end{smallmatrix}\right]$, and $C^* = \begin{bmatrix} C_r^* & 0 \end{bmatrix}$, we conclude (using Proposition 3) that $A + A_{-1}^* = -C_r^* C_r$ on X_1, together with $A_{-1} + A^* = -B_r B_r^*$ and $[C\&D]_r \left[\begin{smallmatrix} I \\ B_r^* \end{smallmatrix}\right] = 0$ on X_1^d.

It follows from [12, Theorem 4.4] that S_r is a tory node, and it is thus time-flow invertible with $S_r^{\leftarrow} = S_r^d = \left[\begin{smallmatrix} [A\&B]_r^d \\ [C\&D]_r^d \end{smallmatrix}\right]$; see Proposition 1. In particular, $\left[\begin{smallmatrix} I & 0 \\ [C\&D]_r \end{smallmatrix}\right] : V \to V_r^d = \operatorname{dom}\left(S_r^d\right)$ is a bijection with the inverse $\left[\begin{smallmatrix} I & 0 \\ [C\&D]_r^d \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \quad 0 \\ [C\&D]_r \\ 0 \quad D_{01} \end{smallmatrix}\right]$. Because also S is time-flow invertible, we get

$$V^{\leftarrow} = \begin{bmatrix} I \quad 0 \\ [C\&D]_r \\ 0 \quad D_{01} \end{bmatrix} V = \begin{bmatrix} \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 0 & D_{01} \end{smallmatrix}\right] \begin{bmatrix} I \quad 0 \\ [C\&D]_r \end{bmatrix}^{-1} \end{bmatrix} V_r^d \tag{6}$$
$$= \begin{bmatrix} \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \\ D_{01}\, [C\&D]_r^d \end{bmatrix} V_r^d$$

and

$$\begin{bmatrix} I & 0 \\ C\&D \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & 0 \\ [C\&D]_r^d & & 0 \end{bmatrix}\Bigg|_{V^{\leftarrow}}.$$

But now we obtain $S^{\leftarrow} = \left[\begin{smallmatrix} [A\&B]_r^d & 0 \\ [C\&D]_r^d & 0 \end{smallmatrix}\right]$ on all of $V^{\leftarrow}$ by (4). Because both S^d and $S^{\leftarrow}$ are operator nodes, it follows that $V^{\leftarrow} = \left[\begin{smallmatrix} V_r^d \\ Y_0 \end{smallmatrix}\right]$ which contradicts (6) unless $D_{01} = 0$. This completes the proof. ■

IV. Construction of the Time-Flow Inverse

We show next that the time-flow invertibility of an operator node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (in the sense of Definition 3) *almost* follows if it is known that S is of boundary control type in the sense of Definition 2.

In this section, we assume that $\Xi = (G, L, K)$ is a boundary node, and S is an operator node associated to Ξ as in Theorem 1. We further write

$$V := \begin{bmatrix} I \\ G \end{bmatrix} Z \quad \text{and} \quad V^{\leftarrow} := \begin{bmatrix} I \\ K \end{bmatrix} Z. \tag{7}$$

Note that $V = \operatorname{dom}(S)$ by [10, Proposition 3]. Assuming that $\operatorname{Ran} K = Y$ and $\operatorname{Ker} K \subset X$ is dense, [10, Proposition 10] shows that the linear mapping

$$S^{\leftarrow} := \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ U \end{bmatrix} \tag{8}$$

is densely defined with $\operatorname{dom}(S^{\leftarrow}) = V^{\leftarrow}$. To show the time-flow invertibility of S, it remains to prove that $S^{\leftarrow}$ is an operator node, see Definition 3.

Definition 4: Assume that $\Xi = (G, L, K)$ is a boundary node, and let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be the operator node associated to Ξ by Theorem 1. Assume that $\operatorname{Ran} K = Y$, $\operatorname{Ker} K$ is dense in X, and $\rho(-L|\operatorname{Ker} K) \cap \overline{\mathbb{C}_+} \neq \emptyset$.

(i) The mapping $A^{\leftarrow} : \operatorname{Ker} K \to X$ is defined by $A^{\leftarrow} := -L|\operatorname{Ker} K$.

(ii) Denote by $X_{-1}^{\leftarrow}$ the completion of X in norm $\|x\|_{X_{-1}^d} := \|(\alpha - A^{\leftarrow})^{-1} x\|$ for $\alpha \in \rho(A^{\leftarrow}) \cap \overline{\mathbb{C}_+}$.

(iii) Define $B^{\leftarrow} : Y \to X_{-1}^{\leftarrow}$ by setting for all $x \in Z$

$$B^{\leftarrow}Kx := -Lx - A_{-1}^{\leftarrow}x,$$

where $A_{-1}^{\leftarrow} \in \mathcal{L}(X; X_{-1}^{\leftarrow})$ is the Yoshida extension of $A^{\leftarrow}$.

(iv) The mapping $C^{\leftarrow} : \mathrm{Ker}\ K \to Y$ is defined by $C^{\leftarrow} := G|\mathrm{Ker}\ K$.

Theorem 3: Assume that $\Xi = (G, L, K)$ is a boundary node, and let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be the operator node associated to Ξ by Theorem 1. Assume that $\mathrm{Ran}\ K = Y$, $\mathrm{Ker}\ K$ is dense in X, and $\rho(-L|\mathrm{Ker}\ K) \cap \overline{\mathbb{C}_+} \neq \emptyset$. Define $V^{\leftarrow}$ and $S^{\leftarrow}$ by (7) and (8). Define the operators $A^{\leftarrow}$, $B^{\leftarrow}$, and $C^{\leftarrow}$ by Definition 4. Then the following holds:

(i) $S^{\leftarrow} : \mathrm{dom}\,(S^{\leftarrow}) \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ is an operator node with $\mathrm{dom}\,(S^{\leftarrow}) = V^{\leftarrow}$.
The main operator of $S^{\leftarrow}$ is $A^{\leftarrow}$ with domain $\mathrm{dom}\,(A^{\leftarrow}) = \mathrm{Ker}\ K$. The operator $B^{\leftarrow}$ is the input operator of $S^{\leftarrow}$, and the combined feedthrough/output operator $[C\&D]^{\leftarrow}$ of $S^{\leftarrow}$ satisfies

$$[C\&D]^{\leftarrow} \begin{bmatrix} x \\ y \end{bmatrix} = Gx \quad \text{for all} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in V^{\leftarrow}.$$

(ii) The operator node S is time-flow invertible, and its time-flow inverse equals $S^{\leftarrow}$.

Proof: All this follows from a chain of technical propositions, see [10, Propositions 8, 9, 10 and 11]. ■

We remark that under the assumptions of the previous theorem, also $S^{\leftarrow}$ itself is of boundary control type, see [10, Corollary 1]. The following reformulation of the previous theorem is given in [10, Theorem 6]:

Theorem 4: Let $\Xi = (G, L, K)$ be a doubly boundary node, and assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the associated operator node given by Theorem 1. Then

(i) S is time-flow invertible, and the time-flow inverse $S^{\leftarrow}$ is of boundary control type, and

(ii) the time-flow inverse $S^{\leftarrow}$ is associated to the boundary node $\Xi^{\leftarrow} := (K, -L, G)$ in the sense of Theorem 1.

By Theorem 4, it is reasonable to call any *doubly* boundary node $\Xi = (G, L, K)$ time-flow invertible. Then the time-flow inverse of Ξ would be $\Xi^{\leftarrow} = (K, -L, G)$, as is to be expected from equations (1).

V. Time-flow invertibility and conservativity of boundary nodes

We are now ready to apply all the previous results to conservative boundary control systems. First comes an adaptation of Theorem 2 to the boundary control context.

Lemma 2: Let $\Xi = (G, L, K)$ be a boundary node with $\mathrm{Ran}\ K = Y$, and assume $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node of boundary control type, as given by Theorem 1. Then S is tory if and only if

(i) the primal Liapunov equation $A + A_{-1}^{*} = -C^{*}C$ holds on X_1,

(ii) we have $Gx = B^{*}x$ for all $x \in X_1^d := \mathrm{dom}\,(A^{*})$, and

(iii) the identity $-L|\mathrm{Ker}\ K = A^{*}$ holds (with equal domains).

Proof: We start from the more interesting "sufficiency" part. It is clear that condition (i) of Theorem 2 always holds for boundary control systems. Conditions (ii) and (iv) of Theorem 2 are same as condition (i) and (iii) of this lemma. By condition (iii), we have $X_1^d = \mathrm{Ker}\ K \subset Z$. By condition (ii) we have $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] x \subset \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z = V = \mathrm{dom}\,(S)$ for all $x \in X_1^d$, and hence $C\&D \left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x \in Y$ is well defined. Since K and $C\&D$ are connected by equation (3), we obtain $C\&D \left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x = Kx = 0$ for all $x \in X_1^d$. Thus condition (iii) of Theorem 2 holds. Time-flow invertibility of S follows from condition (iii) and Theorem 3 since $-L|\mathrm{Ker}\ K = A^{*}$ and $\rho(A) \cap \overline{\mathbb{C}_+} \neq \emptyset$.

To prove the "necessity" part, assume that S is tory. Such S is time-flow invertible and $S^{\leftarrow} = S^d$ by Proposition 1. Thus all the conditions of Theorem 2 hold; hence also conditions (i) and (iii) of this lemma hold, too.

By [12, Theorem 4.4], the dual Liapunov equation

$$\begin{bmatrix} A_{-1} & B \end{bmatrix} \begin{bmatrix} I \\ B^{*} \end{bmatrix} x = -A^{*}x \in X$$

holds for all $x \in X_1^d = \mathrm{Ker}\ K$, and hence $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset V$. Since $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$, the inclusion $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ implies condition (ii) of this lemma. ■

We have actually proved above that condition (ii) of Lemma 2 can be replaced by the inclusion $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset V$.

The main result of this paper is the following:

Theorem 5: Let $\Xi = (G, L, K)$ be a doubly boundary node, and assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the associated operator node given in Theorem 1. Then S is conservative (hence, tory) if and only if

(i) $2\Re \langle x, Lx \rangle_X = -\|Kx\|_Y^2$ for all $x \in \mathrm{Ker}\ G$,

(ii) $\langle z, Lx \rangle_X + \langle Lz, x \rangle_X = \langle Gz, Gx \rangle_U$ for all $z \in Z$ and $x \in \mathrm{Ker}\ K$.

Proof: Since Ξ is is a doubly boundary node, the time-flow inverse $S^{\leftarrow}$ exists by Theorem 4, and it is of boundary control type. For the usual spaces and operators involving S and $S^{\leftarrow}$, we have the identities $X_1 = \mathrm{Ker}\ G$, $A = L|\mathrm{Ker}\ G$, $C = K|\mathrm{Ker}\ G$, $X_1^{\leftarrow} = \mathrm{Ker}\ K$, $A^{\leftarrow} = -L|\mathrm{Ker}\ K$, and $C^{\leftarrow} = G|\mathrm{Ker}\ K$. Then (i) is same as $2\Re \langle x, Ax \rangle_X = -\|Cx\|_Y^2$ for all $x \in X_1$, which is (by polarisation) equivalent to condition (i) of Lemma 2. Condition (ii) of Lemma 2 holds if and only if

$$\begin{aligned} -\langle z, A^{*}x \rangle_X + \langle Lz, x \rangle_X &= \langle Gz, Gx \rangle_U \\ \text{for all } z \in Z \text{ and } & x \in \mathrm{dom}\,(A^{*}), \end{aligned} \tag{9}$$

since $\mathrm{Ran}\ G = U$ and $BGz = -A_{-1}z + Lz$. This together with condition (iii) of Lemma 2 imply condition (ii).

Because X_1 is dense in X, condition (iii) of Lemma 2 holds if and only if $X_1^{\leftarrow} = \mathrm{dom}\,(A^{*})$ and $\langle z, A^{\leftarrow}x \rangle_X = \langle z, A^{*}x \rangle_X$ for all $z \in X_1, x \in \mathrm{dom}\,(A^{*})$ if and only if

$$\langle z, Lx \rangle_X + \langle Lz, x \rangle_X = 0 \tag{10}$$

for all $z \in \mathrm{Ker}\ G$ and $x \in \mathrm{Ker}\ K$. Clearly (ii) implies (10), and hence it implies condition (iii) of Lemma 2, too. Finally note that (ii) together with condition (iii) of Lemma 2 imply (9) and thus condition (ii) of Lemma 2. ■

There is another, slightly weaker variant of Theorem 5 whose formulation is more symmetric.

Theorem 6: Let $\Xi = (G, L, K)$ be a doubly boundary node, and assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the associated operator node given in Theorem 1. Then S is conservative (hence, tory) if and only if the *Green–Lagrange identity*

$$2\Re \langle z_0, L z_0 \rangle_X = \|G z_0\|_U^2 - \|K z_0\|_Y^2 \tag{11}$$

holds for all $z_0 \in Z$.

Proof: By the polarisation identity, (11) implies for all $z_1, z_2 \in Z$ the identity $\langle z_1, L z_2 \rangle_X + \langle L z_1, z_2 \rangle_X = \langle G z_1, G z_2 \rangle_U - \langle K z_1, K z_2 \rangle_U$. It is trivial that both the conditions (i) and (ii) of Theorem 5 follow from this.

Conversely, assume that S is conservative. Let $z_0 \in Z$ be arbitrary and $u \in C^2([0,\infty); U)$ such that $G z_0 = u(0)$. By Lemma 1, there exists a solution $z(\cdot) \in C([0,\infty); Z) \cap C^1([0,\infty); X)$ of (1) that satisfies $z(0) = z_0$ and $\frac{d}{dt}\|z(t)\|_X^2 = \|u(t)\|_U^2 - \|y(t)\|_Y^2$. Differentiating and using (1) gives

$$\begin{aligned} &\langle z(t), L z(t) \rangle_X + \langle L z(t), z(t) \rangle_X \\ &= \langle G z(t), G z(t) \rangle_U - \langle K z(t), K z(t) \rangle_Y \end{aligned}$$

for all $t > 0$. Since all the operators L, G and K are bounded from space Z and $z(\cdot) \in C([0,\infty); Z)$, we may take the limit as $t \to 0+$. Now (11) follows since $z_0 \in Z$ was arbitrary. ■

VI. Reflecting mirror

This example is classical, and a more general version has been treated in terms of "thin air" systems in [23, Section 7]; a construction that bears some resemblance to feedback techniques appearing in [22]. Our approach resembles the techniques of [8].

Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded set with C^2-boundary $\partial\Omega$. We assume that $\partial\Omega$ is the union of two sets Γ_0 and Γ_1 with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ [4]. System S is described by the exterior problem

$$\begin{cases} z_{tt}(t,\xi) = \Delta z(t,\xi) \quad \text{for } \xi \in \Omega \text{ and } t \geq 0, \\ -z_t(t,\xi) - \frac{\partial z}{\partial \nu}(t,\xi) = \sqrt{2}\, u(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ \sqrt{2}\, y(t,\xi) = -z_t(t,\xi) + \frac{\partial z}{\partial \nu}(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ z(t,\xi) = 0 \quad \text{for } \xi \in \Gamma_0 \text{ and } t \geq 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in \Omega. \end{cases} \tag{12}$$

We obtain equations of the form (1) by using the rule

$$z_{tt} = \Delta z \quad \hat{=} \quad \frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\Delta & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}.$$

The spaces Z, X and and operator L are defined by

$$\begin{gathered} L := \left[\begin{smallmatrix} 0 & -1 \\ -\Delta & 0 \end{smallmatrix}\right] : Z \to X \text{ with} \\ Z := Z_0 \times H^1_{\Gamma_0}(\Omega) \text{ and } X := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \end{gathered}$$

where $Z_0 := \left\{ z \in H^1_{\Gamma_0}(\Omega) \cap H^{3/2}(\Omega) : \Delta z \in L^2(\Omega) \right\}$. The norm of Z_0 is given by

$$\|z_0\|_{Z_0}^2 := \|z_0\|_{H^1(\Omega)}^2 + \|z_0\|_{H^{3/2}(\Omega)}^2 + \|\Delta z_0\|_{L^2(\Omega)}^2.$$

For space X, we use the energy norm

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|_X^2 := \||\nabla z_0|\|_{L^2(\Omega)}^2 + \|w_0\|_{L^2(\Omega)}^2. \tag{13}$$

As is well known, it follows from Poincaré inequality $\|z_0\|_{L^2(\Omega)} \leq K\||\nabla z_0|\|_{L^2(\Omega)}$ for $z_0 \in H^1_{\Gamma_0}(\Omega)$ that this norm is equivalent to the direct sum norm of X, see e.g. [8, p. 168]. Thus $Z \subset X$ with a bounded inclusion and $L \in \mathcal{L}(Z; X)$.

Define the input and output spaces by $U = Y := L^2(\Gamma_1)$, together with

$$\begin{aligned} G \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] &:= \frac{1}{\sqrt{2}} \left(-\frac{\partial z_0}{\partial \nu}|\Gamma_1 + w_0|\Gamma_1 \right) \text{ and} \\ K \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] &:= \frac{1}{\sqrt{2}} \left(\frac{\partial z_0}{\partial \nu}|\Gamma_1 + w_0|\Gamma_1 \right). \end{aligned}$$

It follows from the basic theory of boundary traces for Sobolev spaces (see e.g. [3]) that $G \in \mathcal{L}(Z; U)$ and $K \in \mathcal{L}(Z; Y)$. For details, see the discussion in [10, page 29]. Clearly, the wave equation (12) is translated (at least formally) to the form of (1), by using L, G and K.

Proposition 4: Let the operators L, G, K and spaces Z, X be defined as above. Then $\Xi = (G, L, K)$ is doubly boundary node.

Proof: This is [10, Proposition 18], and its proof requires several (but well-known) facts from the elliptic regularity theory. ■

It is now almost trivial to check (using Theorem 5) that $\Xi = (G, L, K)$ defines a conservative linear system S. Indeed, for an arbitrary $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$, the Green's formula [3, Lemma 1.5.3.8] implies

$$\begin{aligned} &-2\Re \langle \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \rangle_X = 2\Re \langle \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_0 \\ \Delta z_0 \end{smallmatrix}\right] \rangle_X \\ &= 2\Re \left(\langle \Delta \overline{z_0}, w_0 \rangle_{L^2(\Omega)} + \int_\Omega \nabla \overline{z_0} \cdot \nabla w_0 \, d\Omega \right) \\ &= 2\Re \left(\int_{\Gamma_0 \cup \Gamma_1} \frac{\partial \overline{z_0}}{\partial \nu} w_0 \, d\omega \right) = 2\|w_0|\Gamma_1\|_{L^2(\Gamma_1)}^2 \end{aligned}$$

because $\frac{\partial z_0}{\partial \nu}|\Gamma_1 = w_0|\Gamma_1$. Clearly $K \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = \sqrt{2} w_0|\Gamma_1$ for all $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$, and condition (i) of Theorem 5 holds. Similarly,

$$\begin{aligned} &\langle \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \rangle_X + \langle L \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \rangle_X \\ &= -\int_{\Gamma_1} \frac{\partial \overline{z_0}}{\partial \nu} y_0 \, d\omega - \int_{\Gamma_1} \overline{w_0} \frac{\partial x_0}{\partial \nu} \, d\omega \end{aligned} \tag{14}$$

for any $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in \operatorname{Ker} K$. On the other hand,

$$\begin{aligned} &\langle G \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], G \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \rangle_{L^2(\Gamma_1)} = -\frac{1}{\sqrt{2}} \left\langle \frac{\partial z_0}{\partial \nu}|\Gamma_1, G \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \right\rangle_{L^2(\Gamma_1)} \\ &+ \frac{1}{\sqrt{2}} \langle w_0|\Gamma_1, G \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \rangle_{L^2(\Gamma_1)}. \end{aligned} \tag{15}$$

Since $G \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] = \sqrt{2} y_0|\Gamma_1 = -\sqrt{2} \frac{\partial x_0}{\partial \nu}|\Gamma_1$ for any $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in \operatorname{Ker} K$, condition (ii) of Theorem 5 follows from (14) and (15).[5] We have obtained [10, claim (i) of Lemma 3], namely:

[4] The sets Γ_1 and Γ_0 are allowed to have zero distance in [23]. This is possible because stronger "background results" from [13] are used there.

[5] We leave it to the reader to carry out the similar computation leading to the Green – Lagrange identity (11).

Lemma 3: Let the operators L, G, K and spaces Z, X be defined as above. Use the energy norm (13) for X. Then the boundary node $\Xi = (G, L, K)$ associated to wave equation (12) describes a (tory) conservative system $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ through Theorem 1 and Lemma 1.

REFERENCES

[1] D. Z. Arov and M. A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral equations and operator theory*, 24:1–45, 1996.

[2] H. O. Fattorini. Boundary control systems. *SIAM J. Control*, 6(3), 1968.

[3] P. Grisvard. *Elliptic problems in non-smooth domains*. Pitman, 1985.

[4] M. S. Brodskiĭ. On operator colligations and their characteristic functions. *Soviet Mat. Dokl.*, 12:696–700, 1971.

[5] M. S. Brodskiĭ. *Triangular and Jordan representations of linear operators*, volume 32. American Mathematical Society, Providence, Rhode Island, 1971.

[6] M. S. Brodskiĭ. Unitary operator colligations and their characteristic functions. *Russian Math. Surveys*, 33(4):159–191, 1978.

[7] V. Havu and J. Malinen. Laplace and Cayley transforms – an approximation point of view, *Proc. of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05)*, 2005.

[8] J. Lagnese. Decay of solutions of wave equations in a bounded region with boundary dissipation. *Journal of Differential equations*, 50:163–182, 1983.

[9] J. L. Lions and E. Magenes. *Non-homogenous boundary value problems and applications I*, volume 181 of *Die Grundlehren der mathematischen Wissenchaften*. Springer Verlag, 1972.

[10] J. Malinen. Conservativity of time-flow invertible and boundary control systems. *Helsinki Univ. of Tech. Inst. of Math. Research Reports A479*, 2004.

[11] J. Malinen and O. J. Staffans. Conservative boundary control systems. *Submitted*, 2005.

[12] J. Malinen, O. Staffans, and G. Weiss. When is a linear system conservative? To appear in: *Quarterly of Applied Mathematics*, 2005.

[13] A. Rodrígues-Bernal and E. Zuazua. Parabolic singular limit of a wave equation with localized boundary damping. *Discrete Contin. Dynam. Systems*, 1(3):303–346, 1995.

[14] D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of American Mathematical Society*, 300:383–431, 1987.

[15] D. Salamon. Realization theory in Hilbert spaces. *Math. Systems Theory*, 21:147–164, 1989.

[16] M. S. Livšic and A. A. Yantsevich. *Operator colligations in Hilbert space*. John Wiley & sons, Inc., 1977.

[17] Yu. L. Smuljan. Invariant subspaces of semigroups and Lax–Phillips scheme. *Dep. in VINITI, No. 8009-B86, Odessa (A private translation by Daniela Toshkova, 2001)*, 1986.

[18] O. J. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, 2004.

[19] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems, Part II: The system operator and the Lax – Phillips semigroup. *Transactions of the American Mathematical Society*, 354(8):3229–3262, 2002.

[20] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems, Part III: Inversions and duality. *Integral Equations Operator Theory*, 49(4):517–558, 2004.

[21] B. Sz.-Nagy and C. Foias. *Harmonic Analysis of Operators on Hilbert space*. North-Holland Publishing Company, Amsterdam, London, 1970.

[22] R. Triggiani. Wave equation on a bounded domain with boundary dissipation: An operator approach. *Journal of Mathematical Analysis and applications*, 137:438–461, 1989.

[23] G. Weiss and M. Tucsnak. How to get a conservative well-posed linear system out of thin air. I. Well-posedness and energy balance. *ESAIM Control Optim. Calc. Var.*, 9:247–274, 2003.