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Invariant Equilibria of Polytopic Games via Optimized Robust Control Invariance

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Abstract— **In the context of non-cooperative linear discrete-time games, invariant equilibria are introduced as an extension to many players of the corresponding concept for two-player games (*doubly invariant equilibria*). It is established that the set of equilibrium control amplitudes is convex. An efficient computational procedure for the computation of piecewise affine equilibrium strategies based on the recent concept of optimized robust control invariance is proposed. The procedure makes use of a simple linear programming formulation when constraints are polytopic, thus avoiding recursive projection of convex sets. A numerical example for three player game is presented.**

I. Introduction

Recently, the theory of robust control invariant (RCI) sets has been used to define a dynamic game in which two players, each facing own constraints on state and strategy spaces, tries to make a constraint-feasible subset of state space robust control invariant with respect to the opponent's strategies. Since control amplitude of one player is perceived as disturbance by the other, it is possible to define a game-theoretic equilibrium in control amplitude space. Such an equilibrium, termed *doubly invariant* in two-person games, was studied in the context of discrete-time linear time-invariant (DLTI) systems subject to polytopic constraints on state and control variables [1]–[3]. It was noted in [3] that the notion is related to the Nash equilibrium concept in game theory [4] and to viability theory in control [5], [6]. In applications, the concept has potential relevance for diverse fields such as, for example, fault-tolerant control systems, communication networks and economics. When extending the notion to many person non-cooperative games or dealing with relatively high dimensional state space, the issue of computational efficiency stands up prominently. Recent developments reported in [7]–[9] introduced an interesting procedure to characterize RCI sets. The procedure allows to optimize size and shape of RCI sets via linear programming (LP), when constraints are polytopic, while yielding easily computable state feedback control laws that turn out to be piecewise-affine. In this paper, using the above developments, we generalize *doubly invariant equilibria* to several players and devise a procedure based on a single LP to compute equilibrium strategies.

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This paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 discusses robust control invariance issue for the i^{th} player. Section 4 provides a computationally tractable characterization of robust control invariant sets for the i^{th} player. Section 5 provides a tractable optimization problem whose feasibility reveals the existence of a q-person invariant equilibrium. Section 6 gives an appropriate example – three player game case. Finally, Section 7 presents conclusions and indicates possible extensions. A more detailed exposition and all proofs for the results stated in this paper can be found in [10].

Notation and Basic Definitions: Let $\mathbb{R}^n_+$ be the set of non-negative n-tuples of $\mathbb{R}^n$ (i.e. $\mathbb{R}^n_+ \triangleq \{x \in \mathbb{R}^n \mid x \geq 0\}$), $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_q \triangleq \{0, 1, \ldots, q\}$, $\mathbb{N}^+_q \triangleq \{1, \ldots, q\}$. Given any $j \in \mathbb{N}^+_q$ let $\mathbb{N}_{q,j} \triangleq \mathbb{N}^+_q \setminus \{j\}$. Let $\mathbf{1}_t$ denote the vector $(1, 1, \ldots, 1)' \in \mathbb{R}^t$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u+v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^b$, we denote $\bigoplus_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$ and $\bigotimes_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \times \cdots \times \mathcal{U}_b$. Given a set $\mathcal{U} \subseteq \mathbb{R}^n$, $\text{interior}(\mathcal{U})$ denotes its interior. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces. A *polytope* is a compact (i.e. closed and bounded) polyhedron. A collection of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}, i \in \mathbb{N}^+_q$ is a C (CP)-collection if for every $i \in \mathbb{N}^+_q$ the set $\mathcal{U}_i$ is a convex, compact set, (polytope) with non–empty interior containing the origin in its interior.

II. Preliminaries

Consider the following discrete-time linear time-invariant q-person game:

$$x^+ = Ax + \sum_{i=1}^{q} B^i u^i, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state, $u^i \in \mathbb{R}^{m_i}$ is the current control action of the i^{th} player, $q \in \mathbb{N},\ q > 0$ is a finite integer and $(A, B^1, B^2, \ldots, B^q) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m_1} \times \mathbb{R}^{n\times m_2} \times \ldots \times \mathbb{R}^{n\times m_q}$.

We make the following standing assumption:

Assumption 1: The couples (A, B^i) are controllable for all $i \in \mathbb{N}^+_q$.

Game (1) is subject to the following set of hard state and control constraints with respect to player i:

$$(x, u^i) \in \mathbb{X}^i \times \beta_i \mathbb{U}^i,\ i \in \mathbb{N}^+_q \tag{2}$$

where, for any $\beta = (\beta_1, \beta_2, \ldots, \beta_q)' \in \mathbb{R}^q_+,\ \beta > 0$, sets:

$$\mathcal{C}(\beta) \triangleq \{(\mathbb{X}^i, \beta_i \mathbb{U}^i),\ i \in \mathbb{N}^+_q\} \tag{3}$$

form a C-collection. We call $\mathcal{C}(\mathbf{1})$ normalized constraint set collection for game (1). In this paper we are interested in a computationally tractable procedure that enables verifying existence of a properly defined q-person invariant equilibrium for game (1).

Definition 1: A set of vectors $\beta = (\beta_1, \beta_2, \dots, \beta_q) \in \mathbb{R}^q_+,\ \beta > 0$ is called a non-trivial q–person invariant equlibrium of game (1) if there exists a collection of compact, non–empty, sets $\{\Omega^i \subset \mathbb{R}^n\},\ i \in \mathbb{N}^+_q$ such that Ω^i for all $i \in \mathbb{N}^+_q$ satisfy $\Omega^i \subseteq \mathbb{X}^i$ and for all $x \in \Omega^i$ there exists a $u^i \in \beta_i \mathbb{U}^i$ such that $x^+ = Ax + B^i u^i + \sum_{k \in \mathbb{N}_{q,i}} B^k u^k \in \Omega^i$ for all $u^k \in \beta_k \mathbb{U}^k$ and all $k \in \mathbb{N}_{q,i}$.

Any collection of sets $\{\Omega^i \subset \mathbb{R}^n\}, i \in \mathbb{N}^+_q$ in definition 1 is termed a *q–person invariant equilibrium collection*. A trivial q–person invariant equilibrium exists for a vector $\beta = \mathbf{0}$ – the corresponding q–person invariant equilibrium collection is trivially $\{\{\mathbf{0}\}, \{\mathbf{0}\}, \dots, \{\mathbf{0}\}\}$. We are interested in geometric properties of the set of β's for which a non–trivial q–person invariant equilibrium exists. Definition 1 , for $q = 2$, reduces to the *doubly invariant equilibrium* definition introduced in [1].

For a discrete time-invariant system $x^+ = f(x, u, w),\ f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ under constraints $(x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}$ we recall standard definitions in set invariance theory [6].

Definition 2: A set Ω is *robust control invariant* (RCI) for system $x^+ = f(x, u, w)$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ there exists a $u \in \mathbb{U}$ such that $f(x, u, w) \in \Omega$ for all $w \in \mathbb{W}$.

Given a control law $\nu :\ \mathbb{R}^n \to \mathbb{R}^m$, let:

$$\mathbb{X}_\nu \triangleq \mathbb{X} \cap \{x \mid \nu(x) \in \mathbb{U}\} \tag{4}$$

Definition 3: A set $\Omega \subset \mathbb{R}^n$ is *robust positively invariant* (RPI) for system $x^+ = f(x, \nu(x), w)$ and constraint set $(\mathbb{X}_\nu, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}_\nu$ and $f(x, \nu(x), w) \in \Omega,\ \forall w \in \mathbb{W}$, $\forall x \in \Omega$.

We perform first analysis for a fixed value of the vector β, i.e. $\beta = \hat{\beta} \in \mathbb{R}^q_+,\ \hat{\beta} > 0$. Initially, we assume that, without loss of generality, $\mathcal{C}(\hat{\beta})$ (3) has been normalized so that we can consider the case when the constraint set collection is given by $\mathcal{C}(\mathbf{1})$.

III. Robust Control Invariance Issue for the i^{th} player

From the perspective of player i game (1) is regarded as an additively disturbed DLTI system:

$$x^+ = Ax + B^i u^i + \sum_{j \in \mathbb{N}_{q,i}} w^j \tag{5}$$

where for any $j \in \mathbb{N}_{q,i}$:

$$w^j \in \mathbb{W}^j \triangleq B^j \mathbb{U}^j, \tag{6}$$

is considered as an external disturbance corresponding to the possible actions of player $j,\ j \in \mathbb{N}_{q,i}$. Equation (5) can be rewritten as:

$$x^+ = Ax + B^i u^i + w, \tag{7}$$

with

$$w \triangleq \sum_{j \in \mathbb{N}_{q,i}} B^j u^j, \quad w \in \mathbb{W} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} B^j \mathbb{U}^j \tag{8}$$

By definition 2, a set Ω^i is RCI for system $x^+ = Ax + B^i u^i + w$ with $w \in \mathbb{W} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} B^j \mathbb{U}^j$ and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W})$ if $\Omega^i \subseteq \mathbb{X}^i$ and for all $x \in \Omega^i$ there exists a $u \in \mathbb{U}^i$ such that $x^+ = Ax + B^i u^i + w \in \Omega^i$ for all $w \in \mathbb{W}$.

We also need to define pairwise robust control invariance:

Definition 4: A set $\Omega^{(i,j)},\ j \neq i$ is *robust control invariant* for $x^+ = Ax + B^i u^i + w^j,\ w^j \triangleq B^j u^j$ and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W}^j)$ if $\Omega^{(i,j)} \subseteq \mathbb{X}^i$ and for all $x \in \Omega^{(i,j)}$ there exists a $u \in \mathbb{U}^i$ such that $x^+ = Ax + B^i u^i + w^j \in \Omega^{(i,j)}$ for all $w^j \in \mathbb{W}^j \triangleq B^j \mathbb{U}^j$.

We first establish an interesting and relevant result that exploits linearity of game (1) and basic properties of Minkowski set addition. We assume that:

Assumption 2: There exists a collection of sets $\{\Omega^{(i,j)}\},\ j \in \mathbb{N}_{q,i}$ such that $\Omega^{(i,j)}$ is RCI for system $x^+ = Ax + B^i u^i + w^j,\ w^j \triangleq B^j u^j$ and constraint set $(\alpha^{(i,j)} \mathbb{X}^i, \mu^{(i,j)} \mathbb{U}^i, \mathbb{W}^j)$ with $\mathbb{W}^j \triangleq B^j \mathbb{U}^j$ and $(\alpha^{(i,j)}, \mu^{(i,j)}) \in \mathbb{R}_+ \times \mathbb{R}_+$.

If assumption 2 holds then there exists a collection of control laws $\{\nu^{(i,j)} :\ \Omega^{(i,j)} \to \mu^{(i,j)} \mathbb{U}^i,\ j \in \mathbb{N}_{q,i}\}$ such that sets $\{\Omega^{(i,j)}\},\ j \in \mathbb{N}_{q,i}$ are RPI for system $x^+ = Ax + B^i \nu^{(i,j)}(x) + w^j$ and constraint set $(\mathbb{X}^{(i,j)}_\nu, \mathbb{W}^j)$ with $w^j \triangleq B^j u^j,\ w^j \in \mathbb{W}^j \triangleq B^j \mathbb{U}^j$ and $\mathbb{X}^{(i,j)}_\nu \triangleq \alpha^{(i,j)} \mathbb{X}^i \cap \{x \mid \nu^{(i,j)}(x) \in \mu^{(i,j)} \mathbb{U}^i\}$ for $j \in \mathbb{N}_{q,i}$. We can now formally state the following results:

Theorem 1: Suppose Assumption 2 holds. Then the set $\Omega^i \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} \Omega^{(i,j)}$ is RCI for system $x^+ = Ax + B^i u^i + w$ with $w \in \mathbb{W} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} B^j \mathbb{U}^j$ and constraint set $(\alpha^i \mathbb{X}^i, \mu^i \mathbb{U}^i, \mathbb{W})$ where $\alpha^i \triangleq \sum_{j \in \mathbb{N}_{q,i}} \alpha^{(i,j)}$ and $\mu^i \triangleq \sum_{j \in \mathbb{N}_{q,i}} \mu^{(i,j)}$.

Corollary 1: If assumption 2 holds and $(\alpha^i, \mu^i) \in [0,1] \times [0,1]$, the set $\Omega^i \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} \Omega^{(i,j)}$ is RCI for system $x^+ = Ax + B^i u^i + w$ with $w \in \mathbb{W} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} B^j \mathbb{U}^j$ and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W})$.

It follows from discussion above that there exists a control law $\nu^i : \Omega^i \to \mu^i \mathbb{U}^i$ such that the set Ω^i is RPI for system $x^+ = Ax + B^i \nu^i(x) + w$ with $w \in \mathbb{W} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} B^j \mathbb{U}^j$ and constraint set $(\mathbb{X}^i_\nu, \mathbb{W})$, where $\mathbb{X}^i_\nu \triangleq \alpha^i \mathbb{X}^i \cap \{x \mid \nu^i(x) \in \mu^i \mathbb{U}^i\}$.

Theorem 1 states that the i^{th} player can construct a feedback strategy robust to all other players in game (1) by exploiting pairwise feedback control strategies robust to the individual players in the game. This fact motivates our investigation into computationally tractable procedures for checking existence of an RCI set for the i^{th} player with respect to all other players in non–trivial cases.

IV. A family of RCI sets for the i^{th} player

An appropriate characterization of a family of RCI sets for constrained linear discrete time systems was recently presented in [7]–[9]. We exploit these results to characterize a family of pairwise RCI sets for player i (with respect to player j) and then exploit Theorem 1 to obtain a characterization of a family of RCI sets for player i with respect to all other players.

A. Two families of RCI sets for constrained linear discrete time systems

Here we consider a standard linear time invariant discrete time system:

$$x^+ = Ax + Bu + w \tag{9}$$

subject to constraints:

$$(x, u, w) \in \mathbb{X} \times \mathbb{U} \times \mathbb{W}. \tag{10}$$

where the couple (A, B) is assumed controllable and the set collection $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ is a C-collection.

Let $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}$ and for each $k \in \mathbb{N}$ let $\mathbf{M}_k \triangleq (M_0, M_1, \ldots, M_{k-2}, M_{k-1})$. An appropriate characterization of a family of RCI sets for (9) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$, is given by the following sets for $k \geq n$

$$R_k(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} D_i(\mathbf{M}_k)\mathbb{W} \tag{11}$$

where the matrices $D_i(\mathbf{M}_k)$ are defined by:

$$D_0(\mathbf{M}_k) \triangleq I,$$
$$D_i(\mathbf{M}_k) \triangleq A^i + \sum_{j=0}^{i-1} A^{i-1-j}BM_j, i \geq 1 \tag{12}$$

and $\mathbf{M}_k$ satisfies:

$$D_k(\mathbf{M}_k) = \mathbf{0}. \tag{13}$$

Since the couple (A, B) is controllable such a choice exists for all $k \geq n$. Let $\mathbb{M}_k$ denote the set of all matrices $\mathbf{M}_k$ satisfying condition (13):

$$\mathbb{M}_k \triangleq \{\mathbf{M}_k \mid D_k(\mathbf{M}_k) = \mathbf{0}\} \tag{14}$$

Theorem 2: [7], [9] Given any $\mathbf{M}_k \in \mathbb{M}_k$, $k \geq n$ and the corresponding set $R_k(\mathbf{M}_k)$ there exists a control law $\nu: R_k(\mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\nu(x) \oplus \mathbb{W} \subseteq R_k(\mathbf{M}_k)$, $\forall x \in R_k(\mathbf{M}_k)$, i.e. the set $R_k(\mathbf{M}_k)$ is RCI for the system (9) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$.
The feedback control law $\nu: R_k(\mathbf{M}_k) \to \mathbb{R}^m$ in Theorem 2 is a selection from the set valued map:

$$\mathcal{U}(x) \triangleq \mathbf{M}_k\mathbf{W}(x) \tag{15}$$

where $\mathbf{M}_k \in \mathbb{M}_k$ and the set of *disturbance sequences* $\mathbf{W}(x)$ is defined for each $x \in R_k(\mathbf{M}_k)$ by

$$\mathbf{W}(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in \mathbf{W}^k,\ D\mathbf{w} = x\}, \tag{16}$$

where $\mathbf{w} = \{w_0, \ldots, w_{k-1}\}$, $\mathbf{W}^k \triangleq \mathbb{W} \times \ldots \times \mathbb{W}$ and $D = [D_{k-1}(\mathbf{M}_k)\ \ldots\ D_0(\mathbf{M}_k)]$. A $\nu(\cdot)$ satisfying Theorem 2 can be defined, for instance, as follows:

$$\nu(x) \triangleq \mathbf{M}_k\mathbf{w}^0(x) \tag{17a}$$
$$\mathbf{w}^0(x) \triangleq \arg\min_{\mathbf{w}}\{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}(x)\} \tag{17b}$$

The feedback control law $\nu: R_k(\mathbf{M}_k) \to \mathbb{R}^m$ is a piecewise affine function of $x \in R_k(\mathbf{M}_k)$ when constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ is *a CP collection* [7]. Let

$$U(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} M_i\mathbb{W} \tag{18}$$

The set $R_k(\mathbf{M}_k)$ and the feedback control law $\nu(\cdot)$ are parametrized by the matrix $\mathbf{M}_k$. It should be observed that since each $D_i(\mathbf{M}_k)$ is affine in $\mathbf{M}_k$ and when constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ is *a CP collection* it follows, by standard duality arguments used in linear programming, that we can pose a linear programming problem, whose feasibility yields the RCI set $R_k(\mathbf{M}_k)$ for system (9) and constraint set (10). However, a family of sets $R_k(\mathbf{M}_k)$ (11) is merely a subset of the second and richer family of RCI sets defined below.

Let $\mathcal{F}$ denote the set of equilibrium points for the nominal part of (9) ($x^+ = Ax + Bu$)

$$\mathcal{F} \triangleq \{(\bar{x}, \bar{u}) \mid (A - I)\bar{x} + B\bar{u} = \mathbf{0}\} \tag{19}$$

If $A - I$ is invertible than $\bar{x}(\bar{u}) \triangleq -(A - I)^{-1}B\bar{u}$ is a singleton for any $\bar{u} \in \mathbb{R}^m$. If $\mathcal{F} \cap \text{interior}(\mathbb{X}) \times \text{interior}(\mathbb{U}) \neq \emptyset$ it is easy to show that the family of RCI sets (11) for the system (9) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$ is merely a subset of a richer family of RCI sets for system (9) defined for $k \geq n$ by:

$$S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \triangleq \bar{x} \oplus R_k(\mathbf{M}_k) \tag{20}$$

and for a triple $(\bar{x}, \bar{u}, \mathbf{M}_k) \in \mathcal{F} \times \mathbb{M}_k$.

Theorem 3: [7], [8] Given any triple $(\bar{x}, \bar{u}, \mathbf{M}_k) \in \mathcal{F} \times \mathbb{M}_k$, $k \geq n$ and the corresponding set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ there exists a control law $\theta: S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\theta(x) \oplus \mathbb{W} \subseteq S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$, $\forall x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$, i.e. the set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ is RCI for the system (9) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$.
Analysis in the sequel of this paper is developed for sets $R_k(\mathbf{M}_k)$ (11). However, it is straightforward to extend it with a set of minor but appropriate modifications to sets $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ (20).

The condition (13) can be relaxed as shown in [7], [9]; also the sets $R_k(\mathbf{M}_k)$ (11) and $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ (20) can be characterized without having to explicitly compute the Minkowski set additions involved in their definition. For instance, it follows from (11) and (16) that for a given $\mathbf{M}_k$ the set $R_k(\mathbf{M}_k)$ is also characterized by:

$$\begin{aligned} R_k(\mathbf{M}_k) &= \{x \mid \mathbf{W}(x) \neq \emptyset\} \\ &= \{x \mid \exists \mathbf{w} \in \mathbf{W}^k \text{ s.t. } D\mathbf{w} = x\} \end{aligned}$$

When constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ is a CP collection, an equivalent representation of $R_k(\mathbf{M}_k)$ is given by a polytopic set $\{(x, \mathbf{w}) \mid \mathbf{w} \in \mathbf{W}^k,\ D\mathbf{w} = x\}$ in x–$\mathbf{w}$ space so that checking whether $x \in R_k(\mathbf{M}_k)$ can be verified by solving a single LP.

B. Pairwise RCI problem for player couple $(i,j),\ i \neq j$

Here we consider the $(i,j)^{th}$ system in game (1):

$$x^+ = Ax + B^i u^i + w^j,\ i \neq j \tag{21}$$

where $w^j \triangleq B^j u^j$ and $w^j \in \mathbb{W}^j \triangleq B^j \mathbb{U}^j$. System (21) is subject to constraints

$$(x, u^i, w^j) \in \mathbb{X}^i \times \mathbb{U}^i \times \mathbb{W}^j. \tag{22}$$

Exploiting this fact and the discussion in [7], [9] it can be shown that a RCI set (a member of a family of RCI sets (11)) for system (21) and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W}^j)$ can be obtained as follows . Let $\delta^{(i,j)} \triangleq \{\mathbf{M}^{(i,j)}, \alpha^{(i,j)}, \mu^{(i,j)}, \gamma^{(i,j)}\}$, *(the index k appearing in the equations in subsection IV-A is omitted in order to simplify notation)*, and let

$$\begin{aligned}\Delta^{(i,j)} \triangleq \{\delta^{(i,j)} \mid\ & \mathbf{M}^{(i,j)} \in \mathbb{M}^{(i,j)},\\ & R(\mathbf{M}^{(i,j)}) \subseteq \alpha^{(i,j)}\mathbb{X}^i,\\ & U(\mathbf{M}^{(i,j)}) \subseteq \mu^{(i,j)}\mathbb{U}^i,\\ & (\alpha^{(i,j)}, \mu^{(i,j)}, \gamma^{(i,j)}) \in [0,1]^3,\\ & \alpha^{(i,j)} \leq \gamma^{(i,j)},\ \mu^{(i,j)} \leq \gamma^{(i,j)}\}\end{aligned} \tag{23}$$

with $R(\cdot)$ defined by (11), $U(\cdot)$ defined by (18) with respect to the constraint set (22). The constraint $\gamma^{(i,j)} \leq 1$ ensures that, if $\Delta^{(i,j)} \neq \emptyset$, the set $R^{(i,j)} = R(\mathbf{M}^{(i,j)})$ (11) is RCI for system (21) and constraint set (22). We remark that a suitable $\mathbf{M}^{(i,j)}$ that yields a RCI set for the system (21) and constraint set (22) can be obtained by solving the following convex optimization problem:

$$\mathbb{P}^{(i,j)}:\ \delta^{(i,j)^0} = \arg\min_{\delta^{(i,j)}}\{\gamma^{(i,j)} \mid \delta^{(i,j)} \in \Delta^{(i,j)}\} \tag{24}$$

Note that if $\Delta^{(i,j)} \neq \emptyset$ the solution to problem $\mathbb{P}^{(i,j)}$ yields a set $R^{(i,j)^0} \triangleq R(\mathbf{M}^{(i,j)^0})$ and feedback control law $\nu^{(i,j)^0}(x) = \mathbf{M}^{(i,j)^0}\mathbf{w}^0(x)$ satisfying

$$R^{(i,j)^0} \subseteq \alpha^{(i,j)^0}\mathbb{X}^i,\ \nu^{(i,j)^0}(x) \in U(\mathbf{M}^{(i,j)^0}) \subseteq \mu^{(i,j)^0}\mathbb{U}^i, \tag{25}$$

for all $x \in R^{(i,j)^0}$. We now proceed to exploit the structure of the sets $\Delta^{(i,j)}$ and to provide an efficient procedure for checking existence of the RCI set for the i^{th} player with respect to all other players for *the relevant case when $\mathcal{C}(\mathbf{1})$ is a CP collection*.

C. RCI problem for player i with respect to all others

Discussion in Subsection IV-B and Theorem 1 suggest that existence of an RCI set characterized by Theorem 1 and the sets $\{R(\mathbf{M}^{(i,j)})\},\ j \in \mathbb{N}_{q,i}$ for system (7) and constraint set (8) can be verified by checking feasibility of a set of linear inequalities (and equalities). Let $\delta^i \triangleq (\delta^{(i,j)}, \gamma^i)$ where $j \in \mathbb{N}_{q,i}$ and define:

$$\Delta^i \triangleq \bigotimes_{j \in \mathbb{N}_{q,i}} \Delta^{(i,j)} \times \mathbb{R}_+ \tag{26}$$

so that $\delta^i \in \Delta^i$, where the sets $\Delta^{(i,j)}$ are defined by (23). Let the set $\bar{\Delta}^i \subseteq \Delta^i$ be defined as follows:

$$\bar{\Delta}^i \triangleq \{\delta^i \in \Delta^i \mid \sum_{j \in \mathbb{N}_{q,i}} \gamma^{(i,j)} \leq \gamma^i,\ \gamma^i \in [0,1]\} \tag{27}$$

The following optimization problem

$$\mathbb{P}^i:\ \delta^{i^0} = \arg\min_{\delta^i}\{\gamma^i \mid \delta^i \in \bar{\Delta}^i\} \tag{28}$$

is a linear programming problem. If $\bar{\Delta}^i \neq \emptyset$, the solution to problem $\mathbb{P}^i$ yields a collection of sets $R^{(i,j)^0} \triangleq R(\mathbf{M}^{(i,j)^0})$ and a collection of feedback control laws $\nu^{(i,j)^0}(x) = \mathbf{M}^{(i,j)^0}\mathbf{w}^0(x)$ for $j \in \mathbb{N}_{q,i}$. By Theorem 1, a collection of sets $\{R^{(i,j)^0}\},\ j \in \mathbb{N}_{q,i}$ yields a set

$$R^{i^0} \triangleq \bigoplus_{j \in \mathbb{N}_{q,i}} R^{(i,j)^0} \tag{29}$$

Note that if $\bar{\Delta}^i \neq \emptyset$, we have $\gamma^{i^0} \leq 1$ so that the set $R^{i^0}$is RCI for the system (7) and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W})$ where $\mathbb{W}$ is given by (8). The corresponding feedback control law $\nu^{i^0}:\ R^{i^0} \to \sum_{j \in \mathbb{N}_{q,i}} \mu^{(i,j)^0}\mathbb{U}^i$ can be defined by:

$$\nu^{i^0}(x) \triangleq \sum_{j \in \mathbb{N}_{q,i}} \nu^{(i,j)^0}(x^{(i,j)}) \tag{30}$$

where the vectors $\{x^{(i,j)} \in R^{(i,j)^0}\},\ j \in \mathbb{N}_{q,i}$ are a decomposition of the actual state $x \in R^{i^0}$ satisfying:

$$x = \sum_{j \in \mathbb{N}_{q,i}} x^{(i,j)} \tag{31}$$

An appropriate selection of decomposition (31) can be obtained as follows. Let $\mathbf{x} \triangleq \{x^{(i,j)}\},\ j \in \mathbb{N}_{q,i}$ and define the selection of the decomposition (31) by:

$$\mathbf{x}^0(x) \triangleq \arg\min_{\mathbf{x}}\{|\mathbf{x}|^2 \mid \mathbf{x} \in \mathcal{X}(x)\} \tag{32a}$$

$$\mathcal{X}(x) \triangleq \{\mathbf{x} \mid x^{(i,j)} \in R^{(i,j)^0},\ x = \sum_{j \in \mathbb{N}_{q,i}} x^{(i,j)}\} \tag{32b}$$

We now proceed, in similar fashion, to exploit the structure of sets $\bar{\Delta}^i,\ i \in \mathbb{N}_q^+$ to propose a computationally efficient procedure for checking existence of a q-person invariant equilibrium collection for game (1) and (CP) constraint set collection $\mathcal{C}(\mathbf{1})$ – see (3).

V. q-PERSON INVARIANT EQUILIBRIA

A. Checking existence of a q-person Invariant Equilibria for game (1) and constraint (CP) set collection $\mathcal{C}(\mathbf{1})$ given by (3)

Since existence of an RCI set for the i^{th} player, i.e. system (7) and constraint set $(\mathbb{X}^i, \mathbb{U}^i, \mathbb{W})$ where $\mathbb{W}$ is given by (8) can be verified by solving a single linear programming problem (when constraints are polytopic), we proceed to exploit the structure in order to pose a single linear programming problem to verify existence of an invariant equilibrium set collection for system (1) and constraint set collection $\mathcal{C}(\mathbf{1})$ given by (3).

Let $\delta \triangleq \{\delta^i, \gamma\}$ where $i \in \mathbb{N}_q^+$ and define:

$$\Delta \triangleq \bigotimes_{i \in \mathbb{N}_q^+} \bar{\Delta}^i \times \mathbb{R}_+ \tag{33}$$

so that $\delta \in \Delta$, where sets $\bar{\Delta}^i$ are defined by (27). Let the set $\bar{\Delta} \subseteq \Delta$ be defined as follows:

$$\bar{\Delta} \triangleq \{\delta \in \Delta \mid \gamma^i \leq \gamma,\ \forall i \in \mathbb{N}_q^+, \gamma \in [0,1]\} \tag{34}$$

and consider the following linear programming problem:

$$\mathbb{P}:\ \delta^0 = \arg\min_{\delta}\{\gamma \mid \delta \in \bar{\Delta}\} \tag{35}$$

If $\bar{\Delta} \neq \emptyset$ the solution to problem $\mathbb{P}$ yields a collection of sets $\{R^{i^0}\}$ and a collection of feedback control laws $\{\nu^{i^0}(\cdot)\}$ for $i \in \mathbb{N}_q^+$ as already discussed in Subsection IV-C. The optimal value $\gamma^0 \leq 1$ (if $\bar{\Delta} \neq \emptyset$) ensures that the collection of sets $\{R^{i^0}\}$, by Definition 1, constitute an invariant equilibrium collection for game (1) and (CP) constraint set collection $\mathcal{C}(\mathbf{1})$ given by (3).

Note that if $\bar{\Delta} \neq \emptyset$, any element of $\bar{\Delta}$ yields a q–person invariant equilibrium collection.

B. *q-person Invariant Equilibria for constraint set collection $\mathcal{C}(\beta)$ with $\beta \in \mathbb{R}_+^q$*

The above analysis is performed for the case of fixed vector of control amplitudes, i.e.$\beta = \hat{\beta}$. Allowing control amplitudes β to vary it is, in principle, possible to pose a single bi–linear programming problem (when constraints are polytopic), whose feasibility would reveal a q-person invariant equilibrium and a corresponding q-person invariant equilibrium set collection with respect to the particular parametrization of RCI sets, used in previous sections.

As already discussed in previous sections, for a fixed value of $\hat{\beta} \in \mathbb{R}_+^q,\ \hat{\beta} > 0$ and with the considered parametrization of RCI sets the corresponding problem is a convex problem (35). We proceed to check whether the convexity result of a *doubly invariant equilibrium* established in [3] can be established in many player game case. This fact is formally given by the following result established for *the general case of q-person invariant equilibrium set collection.*

Theorem 4: The set of non–trivial q-person invariant equilibria for system (1) is either an empty or a convex set.

The result of Theorem 4 suggests that the set of non–trivial q–person equilibria is also a convex set (if it is not an empty set) for the class of RCI sets considered in this paper. Furthermore, it is possible to show that the set of non–trivial q-person invariant equilibria β is bounded, since $\mathbb{X}^i,\ i \in \mathbb{N}_q^+$ are compact sets (otherwise $\mathbb{X}^i$ would contain a RCI set for player i with respect to unbounded control actions of some other player/s). Hence a notion of extremal q-person invariant equilibrium, generalizing to many players *doubly invariant Nash equilibrium* defined in [3] can be defined as follows:

Definition 5: A vector $\hat{\beta} \in \mathbb{R}_+^q,\ \hat{\beta} > 0$ is an extremal non-trivial q-person invariant equilibrium if $\hat{\beta}$ is such that there does not exists a q–person invariant equilibrium β satisfying that $\beta \neq \hat{\beta}$ and $\beta \geq \hat{\beta}$.

An iterative procedure for the computation of the extremal equilibrium can be obtained as follows. Let, for any $i \in \mathbb{N}_q^+$, $f_i:\ \mathbb{R}_+^q \to \mathbb{R}_+$ be defined by:

$$f_i(\beta) \triangleq \sup_{\beta}\{\beta_i \mid \beta \text{ is a } q \text{ person invariant equilibrium}\} \tag{36}$$

where β_i is the i^{th} coordinate of a vector β. We observe that $f_i(\beta)$ maps the i^{th} coordinate β_i of a vector β into β_i^+ such that $\beta_i^+ \geq \beta_i$. The functions $f_i:\ \mathbb{R}_+^q \to \mathbb{R}_+$ are non–decreasing (for all $i \in \mathbb{N}_q^+$). An algorithmic procedure for the determination of the extremal q–person invariant equilibrium is formulated in Algorithm 1.

Algorithm 1 Computation of the extremal q–person invariant equilibrium

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Require: β^0 – an initial q–person invariant equilibrium, $k_{max} \in \mathbb{N}$
1: Set $k = 0$, where k is iteration index.
2: **repeat**
3: Increment k by one.
4: Set $j = 0$ and $\hat{\beta}^{(k,j)} = \beta^{k-1}$, where j is player index.
5: **repeat**
6: Increment j by one.
7: Set $\hat{\beta}_j^{(k,j)} = f_j(\hat{\beta}^{(k,j-1)})$ where $f_j(\cdot)$ is defined in in (36), where $\hat{\beta}_j^{(k,j)}$ is the j^{th} coordinate of $\hat{\beta}^{(k,j)}$,
8: Set $\hat{\beta}_i^{(k,j)} = \hat{\beta}_i^{(k,j-1)}$ for all $i \in \mathbb{N}_{q,j}$,
9: **until** $j \leq q$.
10: Set $\beta^k = \hat{\beta}^{(k,q)}$.
11: If $\beta^k = \beta^{k-1}$ terminate the algorithm.
12: **until** $k \leq k_{max}$.

Since the functions $f_i:\ \mathbb{R}_+^q \to \mathbb{R}_+$ are non–decreasing for all $i \in \mathbb{N}_q^+$, each $\hat{\beta}^{(k,j)} \geq \hat{\beta}^{(k,j-1)}$ implying that the Algorithm 1 generates a non–decreasing sequence $\{\beta^k\}$ of q–person invariant equilibria, since $\beta^{k-1} = \hat{\beta}^{(k,1)} \leq \hat{\beta}^{(k,2)} \leq \ldots \leq \hat{\beta}^{(k,q)} = \beta^k$ by construction. Since the set of equilibrium points is bounded and convex by Theorem 4 it follows that β^k as $k_{max} \to \infty$ approaches the extremal q person invariant equilibrium $\hat{\beta}$. If the stopping criteria in step 11 of algorithm 1 is not satisfied, k_{max} should be increased or otherwise $\beta^{k_{max}}$ is a q–person invariant equilibrium that is an approximation of the extremal q–person invariant equilibrium. The selection of β^0 can be done by a grid search or a generalization of the tatonnement process described in [3] although further research is needed to handle the case in which the set of invariant equilibria has empty interior. An improved algorithmic procedure for the computation of β^0 is under current investigation.

VI. Illustrative Example

To illustrate our procedure we consider a three person dynamic game:

$$x^+ = \begin{bmatrix} 0.2 & 0.9 \\ -0.9 & 0.1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u^1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u^2 + \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} u^3 \tag{37}$$

which is subject to the control constraints

$$u^i \in \mathbb{U}^i \triangleq \{\ u^i \in \mathbb{R} \mid -\beta_i \leq u^i \leq \beta_i\},\ i = 1,2,3$$
$$(\beta_1, \beta_2, \beta_3) = (0.0508, 0.0461, 0.0485) \tag{38}$$

and respective state constraints for the first, the second and the third player are

$$\begin{aligned}\mathbb{X}^1 &\triangleq \{\ x \in \mathbb{X} \mid -6 \leq 2x^1 + 3x^2 \leq 6\},\\ \mathbb{X}^2 &\triangleq \{\ x \in \mathbb{X} \mid -4 \leq -x^1 + 2x^2 \leq 4\},\\ \mathbb{X}^3 &\triangleq \{\ x \in \mathbb{X} \mid -2 \leq x^2 \leq 2\},\\ \mathbb{X} &\triangleq \{(x^1, x^2) \in \mathbb{R}^2 \mid |x|_\infty \leq 4\} \end{aligned} \tag{39}$$

In Figure 1 we show the 3 person invariant equilibrium collection constructed by our procedure and computed by solving a single linear programming problem with parameters characterizing the sets R^1, R^2 and R^3 (see subsection IV-A) $(k_{(1,2)}, k_{(1,3)}) = (k_{(2,1)}, k_{(2,3)}) = (k_{(3,1)}, k_{(3,2)}) = (11, 11)$. The sets R^1, R^2 and R^2 are shaded with different levels of gray shading and in this case they satisfy $R^1 \subseteq R^3 \subseteq R^2$ so that $\cap_{i=1,2,3} R^i = R^1$. An improved the 3–person invariant

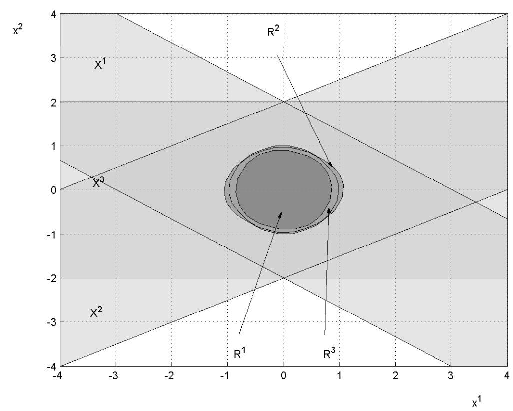

Fig. 1. A 3–person invariant equilibrium collection for three player game

equilibrium collection $\{\Omega^1, \Omega^2, \Omega^3\}$, in sense that $R^i \subseteq \Omega^i,\ i = 1, 2, 3$ together with the set $\Omega^{(1,2,3)} = \cap_{i=1,2,3} \Omega^i$ is shown in Figure 2. This improved 3–person invariant equilibrium collection was computed by exploiting the standard set recursions [11]–[14] for computation of the RCI sets for robust time optimal control problem for constrained linear time invariant discrete time systems, where the appropriate target set collection was chosen to be the initial 3–person invariant equilibrium collection $\{R^1, R^2, R^3\}$.

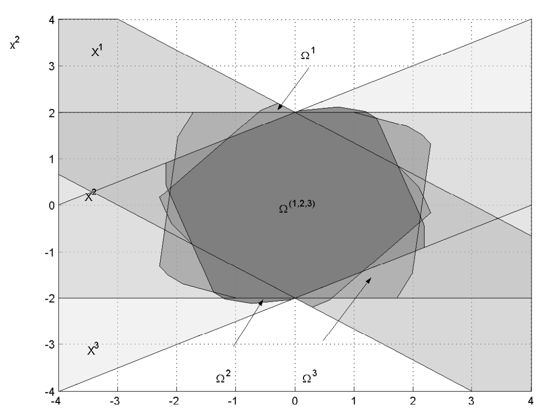

Fig. 2. A 3 person invariant equilibrium collection for three player game

VII. Conclusions

This paper introduced a computationally tractable procedure, requiring solution to a number of linear programming problems when constraints are polytopic, for verifying existence of a q–person invariant equilibrium and the corresponding q–person invariant equilibrium collection within a particular family of RCI sets for game (1). A relevant consequence of Theorem 1 is the fact that each player in the game can construct the feedback strategies robust with respect to all other players by exploiting the pairwise feedback strategies robust with respect to each individual player in game.

A set of results has been obtained for the case of a fixed vector of control amplitudes requiring merely solution of a single linear programming problem when constraints are polytopic in contrast to the standard recursive set computations employed in set invariance theory. These results are then exploited in an iterative algorithmic procedure for the computation of the extremal q–person invariant equilibrium and the corresponding q–person invariant equilibrium collection. The proposed procedure was illustrated by a numerical example.

References

[1] P. Caravani, “Double invariance: A new equilibrium concept for two person dynamic games,” *International Game Theory Review*, vol. 2, pp. 193–207, 2000.

[2] P. Caravani and E. De Santis, “A polytopic game,” *Automatica*, vol. 36, pp. 973–981, 2000.

[3] ——, “Doubly invariant equilibria of linear discrete–time games,” *Automatica*, vol. 38, pp. 1531–1538, 2002.

[4] T. Basar and J. Olsder, *Dynamic noncooperative game theory*. New York: Academic Press, 1995.

[5] J. P. Aubin, *Viability theory*, ser. Systems & Control: Foundations & Applications. Boston, Basel, Berlin: Birkhauser, 1991.

[6] F. Blanchini, “Set invariance in control,” *Automatica*, vol. 35, pp. 1747–1767, 1999, survey paper.

[7] S. V. Raković, “Robust Control of Constrained Discrete Time Systems: Characterization and Implementation,” Ph.D. dissertation, Imperial College London, London, United Kingdom, 2005.

[8] S. V. Raković and D. Q. Mayne, “Regulation of discrete-time linear systems with positive state and control constraints and bounded disturbances,” in *Proceedings of the 16th IFAC World Congress IFAC 2005*, Praha, Czech Republic, July 2005.

[9] S. V. Raković, D. Q. Mayne, E. C. Kerrigan, and K. I. Kouramas, “Optimized robust control invariant sets for constrained linear discrete – time systems,” in *Proceedings of the 16th IFAC World Congress IFAC 2005*, Praha, Czech Republic, July 2005.

[10] S. V. Raković, E. De Santis, and P. Caravani, “Invariant Equilibria of Polytopic Games via Optimized Robust Control Invariance,” Imperial College London, Downloadable from http://www2.ee.ic.ac.uk/cap/cappp/ projects/11/reports.htm, Tech. Rep. EEE/C&P/SVR/9-c/2005, 2005.

[11] D. P. Bertsekas and I. B. Rhodes, “On the minimax reachability of target sets and target tubes,” *Automatica*, vol. 7, pp. 233–247, 1971.

[12] F. Blanchini, “Minimum-time control for uncertain discrete-time linear systems,” in *Proc. 31st IEEE Conference on Decision and Control*, vol. 3, Tuczon AZ, USA, December 1992, pp. 2629–34.

[13] D. Q. Mayne and W. R. Schroeder, “Robust time-optimal control of constrained linear systems,” *Automatica*, vol. 33, pp. 2103–2118, 1997.

[14] D. Q. Mayne, “Control of constrained dynamic systems,” *European Journal of Control*, vol. 7, pp. 87–99, 2001, survey paper.