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Adaptive Compensation of Control Dependent Modeling Uncertainties using Time-Scale Separation

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Abstract— **For nonlinear systems with uncertain nonaffine-in-control dynamics, a direct adaptive model reference control design is proposed that augments a given baseline/nominal dynamic inversion controller. The proposed adaptive augmentation is computed online as a solution of a fast dynamical equation. Such a solution is shown to compensate for control dependent modeling uncertainties via time-scale separation. A simulation example which is motivated by aerospace applications illustrates the theoretical results.**

I. Introduction

In [1], an approximate Dynamic Inversion (DI) methodology is developed for nonaffine-in-control systems using time-scale separation. The methodology invokes fast dynamics to invert the system, and hence relies on time-scale separation property between the system dynamics and the dynamics of the inverting controller. In [2], the methodology is extended to uncertain systems, by developing a direct adaptive counterpart of the method. In this paper, we present an alternative design approach and consider an adaptive augmentation of a fixed gain linear tracking controller for compensation of *control dependent* modeling uncertainties.

In order to motivate further discussion, we consider an unknown scalar system:

$$\dot{x} = f(x, u), \qquad x(0) = x_0, \quad t \geq 0\,, \tag{1}$$

where f is an unknown function of the system state x and the control input u. The control objective is to design $u(t)$ to ensure that $x(t)$ can track a continuously differentiable bounded $\mathtt{r}(t)$. We assume that $f(x, u)$ is invertible with respect to u. In addition, we assume that the inversion of $f(x, u)$ can not be obtained analytically even in the case when f is a known function. Let $\hat{f}(x, u)$ denote an approximation of the unknown system dynamics. Rewrite the system in (1) as:

$$\dot{x} = \hat{f}(x, u) + \Delta(x, u), \qquad x(0) = x_0, \quad t \geq 0\,, \tag{2}$$

where $\Delta(x, u) = f(x, u) - \hat{f}(x, u)$ is commonly referred to as the modeling error. Assuming that the approximator $\hat{f}(x, u)$ was chosen to be invertible with respect to u, the inversion is written in the form:

$$u = \hat{f}^{-1}(x, \nu)\,, \tag{3}$$

where $\nu = \hat{f}(x, u)$ is the so-called pseudo-control. We choose

$$\nu(t) = -k(x(t) - \mathtt{r}(t)) + \dot{\mathtt{r}}(t) - \nu_{ad}(t)\,, \tag{4}$$

where $\nu_{ad}(t)$ is the adaptive augmentation term. Let $e(t) = x(t) - \mathtt{r}(t)$ represent the tracking error signal. Substituting (4) into (2) leads to the following tracking error dynamics:

$$\dot{e}(t) = -ke(t) - \nu_{ad}(t) + \Delta(x(t), u(t)), \quad e(0) = e_0\,. \tag{5}$$

Thus, the adaptive signal ν_{ad} is to be designed to compensate for modeling error $\Delta(x, u)$, which in turn depends upon the control signal u. Notice that u also depends upon ν_{ad} via the relationships in (3), (4). To resolve this algebraic loop, the control design methodologies in [6]–[9] introduced a fixed-point assumption for the map $\nu_{ad} \to \Delta(x, u(\nu_{ad}))$, which is almost impossible to verify numerically in real-life applications. The methodology in [3], [4] takes advantage of the mean value theorem to modify the modeling error to render it control independent at the price of introducing an assumption on boundedness of the time-derivative of control effectiveness, which is also hard to ensure apriori.

Using the results of [1], [2], in this paper we present a direct adaptive model following control design methodology for constructing an on-line approximation of an unknown/ideal DI based tracking controller. Towards this end, we utilize the well-known universal approximation property of Radial Basis Functions (RBF) and parameterize the unknown function $f(x, u)$ on a compact set $\Omega_x \times \Omega_u$:

$$f(x, u) = W^\top \Phi(x, u) + \varepsilon(x, u). \tag{6}$$

In (6), Φ is a vector of Gaussians, $\varepsilon(x, u)$ is a uniformly bounded approximation error, $|\varepsilon(x, u)| < \varepsilon^*$, while W is a vector of unknown constants (ideal RBF weights). This implies that the modeling error can be written as $\Delta(x, u) = W^\top \Phi(x, u) + \varepsilon(x, u) - \hat{f}(x, u)$. Consider the following one-step-ahead tracking error predictor dynamics:

$$\dot{\hat{e}}(t) = -k\hat{e}(t) - \nu_{ad}(t) + \hat{W}^\top(t)\Phi(x, u) - \hat{f}(x, u) \tag{7}$$

with $\hat{e}(0) = \hat{e}_0$, where $\hat{W}(t)$ is the estimate of W. Subtracting (5) from (7), prediction error dynamics is written:

$$\dot{\tilde{e}}(t) = -k\tilde{e}(t) + \tilde{W}^\top(t)\Phi(x, u) - \varepsilon(x, u) \tag{8}$$

with $\tilde{e}(0) = \tilde{e}_0$, where $\tilde{e}(t) = \hat{e}(t) - e(t)$, $\tilde{W}(t) = \hat{W}(t) - W$. The projection based adaptive law

$$\dot{\hat{W}}(t) = \Gamma \mathrm{Proj}(\hat{W}(t), -\tilde{e}(t)\Phi(x(t), u(t))) \tag{9}$$

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with $\hat{W}(0) = W_0$ leads to ultimate boundedness of the prediction error $\tilde{e}(t)$ and the parameter estimation error $\tilde{W}(t)$. Thus, the adaptive control signal ν_{ad} is sought to be the solution, with respect to ν_{ad} for $t \geq 0$, of the following relationship:

$$\nu_{ad} = \hat{W}^\top(t)\Phi(x, \hat{f}^{-1}(x(t), \nu(t, \nu_{ad}))) - \hat{f}(x(t), u(t)) \quad (10)$$

Using eqs. (3), (4), yields the following implicit relationship for ν_{ad} in time:

$$\begin{aligned} &\hat{W}^\top(t)\Phi(x(t), \hat{f}^{-1}(x(t), \nu(t, \nu_{ad}))) \\ + \quad & k(x(t) - \mathtt{r}(t)) - \dot{\mathtt{r}}(t) = 0\,. \end{aligned} \quad (11)$$

Assuming that (11) has an isolated root with respect to ν_{ad}, rewrite equation (7) viewing $\tilde{e}(t) + \mathtt{r}(t)$ and $\dot{\mathtt{r}}(t)$ as time-varying signals:

$$\dot{\hat{e}}(t) = -k\hat{e}(t) + \mathtt{f}(t, \hat{e}(t), \nu_{ad}(t))\,, \quad (12)$$

where $\mathtt{f}(t, \hat{e}, \nu_{ad}) = \hat{W}^\top(t)\Phi\Big(\hat{e}+\tilde{e}(t)+\mathtt{r}(t), \hat{f}^{-1}(\hat{e}+\tilde{e}(t)+\mathtt{r}(t), -k(\hat{e}+\tilde{e}(t)) + \dot{\mathtt{r}}(t) - \nu_{ad})\Big) + k(x(t) - \mathtt{r}(t)) - \dot{\mathtt{r}}(t))$. Consider the following fast dynamics:

$$\epsilon\dot{\nu}_{ad} = -\mathrm{sgn}\left(\frac{\partial \mathtt{f}}{\partial \nu_{ad}}\right)\mathtt{f}(t, \hat{e}, \nu_{ad})\,, \quad \epsilon \ll 1. \quad (13)$$

We argue that under a set of mild assumptions the solution of (13) ensures that $\hat{e}(t) = \hat{e}_r(t) + O(\epsilon)$, where $\hat{e}_r(t)$ is the solution of the exponentially stable system $\dot{\hat{e}}_r(t) = -k\hat{e}_r(t)$. Since $e(t) = \hat{e}(t) + \tilde{e}(t)$, and $\tilde{e}(t)$ is bounded, then $e(t)$ is bounded. If RBF approximation error is zero, i.e. $\varepsilon(x, u) = 0$ in (6), then $e(t) \approx O(\epsilon)$, where ϵ is introduced in (13). In summary, the control design proposed in this paper for the system in (1) is given by (3), (4), with the adaptive signal $\nu_{ad}(t)$ defined as the solution of (13), and with the adaptive weights $\hat{W}(t)$ propagating according to (9).

The paper is organized as follows. In Section II, we recall Tikhonov's theorem from singular perturbation theory, which is the key result used in proving our main theorem. We give our main result on tracking a given reference signal for single input systems in Section III. A relevant simulation example is given in Section IV.

II. Preliminaries on singular perturbations

For proving our main result we will use Tikhonov's theorem on singular perturbations, which we recall below (see for instance Theorem 11.2 on page 439 of [10]).

Consider the problem of solving the system

$$\Sigma_0: \left\{ \begin{array}{ll} \dot{x}(t) = f(t, x(t), u(t), \epsilon), & x(0) = \xi(\epsilon) \\ \epsilon\dot{u}(t) = g(t, x(t), u(t), \epsilon), & u(0) = \eta(\epsilon) \end{array} \right\}, \quad (14)$$

where $\xi : \epsilon \mapsto \xi(\epsilon)$ and $\eta : \epsilon \mapsto \eta(\epsilon)$ are smooth. Assume that f and g are continuously differentiable in their arguments for $(t, x, u, \epsilon) \in [0, \infty] \times D_x \times D_u \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_u \subset \mathbb{R}^m$ are domains, $\epsilon_0 > 0$. In addition, let Σ_0 be in *standard form*, that is,

$$0 = g(t, x, u, 0) \quad (15)$$

has $k \geq 1$ isolated real roots $u = h_i(t, x)$, $i \in \{1, \ldots, k\}$ for each $(t, x) \in [0, \infty] \times D_x$. We choose one particular i, which is fixed. We drop the subscript i henceforth. Let $v(t, x) = u - h(t, x)$. In singular perturbation theory, the system given by

$$\Sigma_{00}: \quad \dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \quad x(0) = \xi(0) \quad (16)$$

is called the *reduced system*, and the system given by

$$\begin{aligned} \Sigma_b: \frac{dv}{d\tau} &= g(t, x, v+h(t, x), 0) \\ v(0) &= \eta_0 - h(0, \xi_0) \end{aligned} \quad (17)$$

is called the *boundary layer system*, where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$, $(t, x) \in [0, \infty) \times D_x$ are treated as fixed parameters. The new time scale τ is related to the original time t via the relationship $\tau = \dfrac{t}{\epsilon}$. The following result is due to Tikhonov.

Theorem 1: Consider the singular perturbation system Σ_0 given in (14) and let $u = h(t, x)$ be an isolated root of (15). Assume that the following conditions are satisfied for all $[t, x, u - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_v \times [0, \epsilon_0]$ for some domains $D_x \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}^m$, which contain their respective origins:

A1. On any compact subset of $D_x \times D_v$, the functions f, g, their first partial derivatives with respect to (x, u, ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t, x)$ and $\left[\frac{\partial g}{\partial u}(t, x, u, 0)\right]$ have bounded first derivatives with respect to their arguments, $\left[\frac{\partial f}{\partial x}(t, x, h(t, x))\right]$ is Lipschitz in x, uniformly in t, and the initial data given by ξ and η are smooth functions of ϵ.

A2. The origin is an exponentially stable equilibrium point of the reduced system Σ_{00} given by equation (16). There exists a Lyapunov function $V : [0, \infty) \times D_x \to [0, \infty)$ that satisfies

$$\begin{gathered} W_1(x) \leq V(t, x) \leq W_2(x) \\ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, h(t, x), 0) \leq -W_3(x) \end{gathered}$$

for all $(t, x) \in [0, \infty) \times D_x$, where W_1, W_2, W_3 are continuous positive definite functions on D_x, and let c be a nonnegative number such that $\{x \in D_x \mid W_1(x) \leq c\}$ is a compact subset of D_x.

A3. The origin is an equilibrium point of the boundary layer system Σ_b given by equation (17) which is exponentially stable uniformly in (t, x).

Let $R_v \subset D_v$ denote the region of attraction of the autonomous system $\frac{dv}{d\tau} = g(0, \xi_0, v + h(0, \xi_0), 0)$, and let Ω_v be a compact subset of R_v. Then for each compact set $\Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c,\ 0 < \rho < 1\}$, there exists a positive constant ϵ_* such that for all $t \geq 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(0, \xi_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, Σ_0 has a unique solution x_ϵ on $[0, \infty)$ and

$$x_\epsilon(t) - x_{00}(t) = O(\epsilon)$$

holds uniformly for $t \in [0, \infty)$, where $x_{00}(t)$ denotes the solution of the reduced system Σ_{00} in (16).

The following Remark will be useful in the sequel.

Remark 1: Assumption A3 can be *locally* verified by linearization. Let φ denote the map $v \mapsto g(t,\xi,v+h(t,\xi),\epsilon)$. It can be shown that if there exists $\omega_0 > 0$ such that the Jacobian matrix $\left[\frac{\partial\varphi}{\partial v}\right]$ satisfies the eigenvalue condition

$$\mathrm{Re}\left(\lambda\left[\frac{\partial\varphi}{\partial v}(t,x,h(t,x),0)\right]\right) \le -\omega_0 < 0$$

for all $(t,x) \in [0,\infty)\times D_x$, then Assumption A3 is satisfied.

III. Tracking design for single input systems

Consider the following nonaffine-in-control single-input system in Brunovsky normal form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bf(x(t),z(t),u(t)) \\ \dot{z}(t) &= \zeta(x(t),z(t),u(t)) \end{aligned} \tag{18}$$

with $x(0)=x_0$, $z(0)=z_0$, for $(x,z,u)\in D_x\times D_z\times D_u$, where $D_x\subset\mathbb{R}^r$, $D_z\subset\mathbb{R}^{n-r}$ and $D_u\subset\mathbb{R}$ are domains containing their respective origins, while A and B correspond to the controllable canonical normal form representation of the nonlinear system dynamics, i.e.

$$A=\begin{bmatrix} 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \dots & 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}.$$

Here $[x^\top(t)\ z^\top(t)]^\top$ denotes the state vector of the system, $x(t)=[x_1(t)\ \cdots\ x_r(t)]^\top$, $u(t)$ is the control input, r is the relative degree of the system, and $f: D_x\times D_z\times D_u\to\mathbb{R}$, $\zeta: D_x\times D_z\times D_u\to\mathbb{R}^{n-r}$ are continuously differentiable *unknown* functions of their arguments. Furthermore, assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for $(x,z,u)\in\Omega_{x,z,u}\subset D_x\times D_z\times D_u$, where $\Omega_{x,z,u}$ is a compact set of possible initial conditions; i.e. there exists $b_0>0$ such that $\left|\frac{\partial f}{\partial u}\right| > b_0$. This assumption guarantees existence of an ideal DI tracking solution. In addition, assume that a *known* model $\hat{f}(x,z,u)$ of the unknown function $f(x,z,u)$ is available over the compact set $\Omega_{x,z,u}\in D_x\times D_z\times D_u$, which is *invertible with respect to* u, and, in addition, it permits a representation of the unknown nonlinearity $f(x,z,u)=\hat{f}(x,z,u)+\Delta(x,z,u)$, where $\Delta(x,z,u)=f(x,z,u)-\hat{f}(x,z,u)$ is referred to as the modeling error. Let $\mathtt{r}(t)$ be an r-times continuously differentiable reference input signal of interest to track. The control *objective* is to design a tracking control law to ensure that $x(t)\to\mathbf{r}(t)$ as $t\to\infty$, where $\mathbf{r}(t)\triangleq[\mathtt{r}(t),\ \dot{\mathtt{r}}(t),\cdots,\mathtt{r}^{(r-1)}(t)]^\top$, while all other error signals in the closed-loop system remain bounded.

If $\Delta(x,z,u)=0$, then the dynamic inversion controller is defined as the solution of the following algebraic equation

$$\hat{f}(x,z,u)=\nu_0\,, \tag{19}$$

where ν_0 is commonly referred to as the *nominal pseudo-control* and it is designed to achieve exponential stability of the error dynamics via the following relationship:

$$\begin{aligned}\nu_0(t) &= -k_1(x_1(t)-\mathtt{r}(t))-k_2(x_2(t)-\dot{\mathtt{r}}(t))-\cdots\\ &- k_r(x_r(t)-\mathtt{r}^{(r-1)}(t))+\mathtt{r}^{(r)}(t),\ k_i>0\,.\end{aligned} \tag{20}$$

Indeed, in that case the system dynamics in (18) is reduced to

$$\dot{x}(t)=Ax(t)+B\hat{f}(x(t),z(t),u(t)). \tag{21}$$

Since $\hat{f}(x,z,u)$ is invertible with respect to u, then upon substitution of (19) and (20) into (21), one obtains an asymptotically stable error dynamics $\dot{e}(t)=A_k e(t)$, where $e(t)=[x_1(t)-\mathtt{r}(t),\ x_2(t)-\dot{\mathtt{r}}(t),\ \cdots,\ x_r(t)-\mathtt{r}^{(r-1)}(t)]^\top$ is the tracking error vector, while A_k is the Hurwitz matrix of the coefficients $k_i, i=1,\cdots,r$, positioned on its last row.

For a nonzero $\Delta(x,z,u)\neq 0$, consider an RBF approximation of $f(x,z,u)$ over the compact set $\Omega_{x,z,u}\in D_x\times D_z\times D_u$:

$$f(x,z,u)=W^\top\Phi(x,z,u)+\varepsilon(x,z,u)\,, \tag{22}$$

where W is a vector of unknown constants, while Φ is a vector of known basis functions (Gaussians), and $\varepsilon(x,z,u)$ is a uniformly bounded approximation error, $|\varepsilon(x,z,u)|<\varepsilon^*$. Then $\Delta(x,z,u)=W^\top\Phi(x,z,u)+\varepsilon(x,z,u)-\hat{f}(x,z,u)$.

Rewrite the system dynamics in (18) in the following form:

$$\begin{aligned}\dot{x}(t) &= Ax(t)+B\Big(\hat{f}(x(t),z(t),u(t))\\ &+ \Delta(x(t),z(t),u(t))\Big)\\ \dot{z}(t) &= \zeta(x(t),z(t),u(t))\,.\end{aligned} \tag{23}$$

Augment the pseudo-control in (20) with an additional adaptive signal for compensation of the control dependent modeling error $\Delta(x,z,u)$:

$$\nu(t) = \nu_0(t)-\nu_{ad}(t)\,. \tag{24}$$

Thus the control signal u is defined as the solution of

$$\hat{f}(x,z,u)=\nu\,, \tag{25}$$

where ν is given by (24). This leads to the following closed-loop error dynamics of the form:

$$\begin{aligned}\dot{e}(t) &= A_k e(t)-B\Big(\nu_{ad}(t)-\Delta(t,e(t),z(t),u(t))\Big)\\ \dot{z}(t) &= \zeta(x(t),z(t),u(t)),\end{aligned} \tag{26}$$

which can be otherwise presented as:

$$\begin{aligned}\dot{e}(t) &= A_k e(t)-B\Big(\nu_{ad}(t)-W^\top\Phi(x(t),z(t),u(t))\\ &- \varepsilon(x(t),z(t),u(t))+\hat{f}(x(t),z(t),u(t))\Big)\\ \dot{z}(t) &= \zeta(x(t),z(t),u(t))\,.\end{aligned} \tag{27}$$

Substituting (24) and (25) yields:

$$\begin{aligned}\dot{e}(t) &= A_k e(t)+B\Big(W^\top\Phi(x(t),z(t),u(t))\\ &+ \varepsilon(x(t),z(t),u(t))-\nu_0(t)\Big)\\ \dot{z}(t) &= \zeta(x(t),z(t),u(t))\,.\end{aligned} \tag{28}$$

Since W are unknown parameters, we consider the following one-step-ahead error predictor using the following series parallel model:

$$\begin{aligned}&\dot{\hat{e}}(t)=A_k\hat{e}(t)+B\Big(\hat{W}^\top(t)\Phi(x(t),z(t),u(t))-\nu_0(t)\Big)\\ &\dot{z}(t)=\zeta(x(t),z(t),u(t))\end{aligned} \tag{29}$$

with $\hat{e}(0)=\hat{e}_0$. In (29), $\hat{W}(t)$ is the adaptive parameter for estimating W, while $\nu_0(t)$ is the same as in (20). Then the prediction error dynamics for the series parallel model in (29) is:

$$\begin{aligned}\dot{\tilde{e}}(t) &= A_k\tilde{e}(t)+B\Big(\tilde{W}^\top(t)\Phi(x(t),z(t),u(t)) \\ &- \varepsilon(x(t),z(t),u(t))\Big)\,, \quad (30)\\ \dot{z}(t) &= \zeta(x(t),z(t),u(t)) \quad (31)\end{aligned}$$

with $\tilde{e}(t)=\hat{e}(t)-e(t)$, $\tilde{e}(0)=e_0-\hat{e}_0$, $z(0)=z_0$, $\tilde{W}(t)=\hat{W}(t)-W$.

Theorem 2: The adaptive law

$$\dot{\hat{W}}(t)=\Gamma\mathrm{Proj}(\hat{W}(t),-\Phi(x(t),z(t),u(t))\,\tilde{e}^\top(t)PB) \quad (32)$$

with $\hat{W}(0)=W_0$, where $\mathrm{Proj}(\cdot,\cdot)$ denotes the Projection operator [5], $P=P^\top>0$ solves the Lyapunov equation $A_k^\top P+PA_k=-Q$ for arbitrary $Q>0$, $\Gamma>0$ is the adaptation gain matrix, ensures that the prediction error dynamics (30), (31) is ultimately bounded with respect to $\tilde{e}(t),\tilde{W}(t)$, uniformly in z_0.

Proof: Consider the following Lyapunov function candidate

$$V(\tilde{e}(t),\tilde{W}(t))=\tilde{e}^\top(t)P\tilde{e}(t)+\tilde{W}^\top(t)\Gamma^{-1}\tilde{W}(t)\,. \quad (33)$$

Its derivative along the trajectories of (30), (32) can be upper bounded $\dot{V}\le-\lambda_{\min}(Q)\|\tilde{e}\|^2+2\|\tilde{e}\|PB\varepsilon^*\le 0$, where the following property of the Projection operator is used $\tilde{W}^\top(\mathrm{Proj}(\hat{W},y)-y)\le 0$, which is true for all vectors y [5], while $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of Q. Hence $\dot{V}\le 0$ outside the compact set

$$\Big\{\|\tilde{e}\|\le\frac{2\varepsilon^*\|PB\|}{\lambda_{\min}(Q)}\Big\}\bigcap\Big\{\|W\|\le W^*\Big\}, \quad (34)$$

where W^* is the maximum allowable norm upper bound selected for the Projection operator, $\|\cdot\|$ denotes the 2-norm. Following standard invariant set arguments one can conclude that the prediction error dynamics (30), (31) is ultimately bounded with respect to $\tilde{e}(t),\tilde{W}(t)$, uniformly in z_0. ■

Remark 2: If $\varepsilon^*=0$, then the adaptive law

$$\dot{\hat{W}}(t)=-\Gamma\Phi(x(t),z(t),u(t))\tilde{e}^\top(t)PB\,,\quad \hat{W}(0)=W_0$$

yields asymptotic prediction, i.e. $\tilde{e}(t)\to 0$ as $t\to\infty$. Indeed, in that case, $\dot{V}=-\tilde{e}^\top Q\tilde{e}\le 0$, and application of Barbalat's lemma further implies that $\lim_{t\to\infty}\tilde{e}(t)=0$ uniformly in z_0.

From (24) and (25) it follows that

$$u=\hat{f}^{-1}(x,z,\nu_0-\nu_{ad})\,,$$

where the inversion is performed with respect to the last argument. Then the forcing term $\hat{W}^\top(t)\Phi(x(t),z(t),u(t))-\nu_0(t)$ in (29) can be represented as: $\hat{W}^\top(t)\Phi(x(t),z(t),u(t))-\nu_0(t)=\hat{W}^\top(t)\Phi(x(t),z(t),\hat{f}^{-1}(x(t),z(t),\nu_0(t)-\nu_{ad}(t)))-\nu_0(t)$. Thus, the adaptive controller $\nu_{ad}(t)$ is defined implicitly as the solution of the following equation:

$$\hat{W}^\top(t)\Phi(x(t),z(t),\hat{f}^{-1}(x(t),z(t),\nu_0(t)-\nu_{ad}))=\nu_0(t) \quad (35)$$

resulting in asymptotically stable estimator dynamics $\dot{\hat{e}}(t)=A_k\hat{e}(t)$.

Assumption 3: Using Projection Operator and a proper choice of the regressor vector Φ (other than RBFs), the adaptive process in (32) can be constructed such that the control effectiveness of the estimator is bounded away from zero for all $t>0$:

$$\Big|\hat{W}^\top(t)\frac{\partial\Phi(x,z,u(t,\nu_{ad}))}{\partial\nu_{ad}}\Big|>a_0>0\,. \quad (36)$$

Notice that using the definition of the pseudo-control from (24) yields

$$\begin{aligned}\frac{\partial\Phi(x,z,u(t,\nu_{ad}))}{\partial\nu_{ad}} &= \frac{\partial\Phi(x,z,u)}{\partial u}\frac{\partial u}{\partial\nu}\frac{\partial\nu}{\partial\nu_{ad}}\\ &= -\frac{\partial\Phi(x,z,u)}{\partial u}\frac{\partial\hat{f}^{-1}(x,\nu)}{\partial\nu}\,.\end{aligned}$$

If the inversion model is chosen to be linear $\hat{f}(x,u)=ku$, then

$$\frac{\partial\Phi(x,z,u(t,\nu_{ad}))}{\partial\nu_{ad}}=-\frac{1}{k}\frac{\partial\Phi(x,z,u)}{\partial u}\,,$$

which is just a scaled version of the bounded derivative of the regressor.

Remark 3: Assumption 3 is required to ensure exponential stability of the boundary layer system in application of Tikhonov's theorem as discussed below. One way to satisfy this assumption is to redefine the regressor vector, include the control signal as its first component, $bu+W^\top\Phi(x,z,u)$, and define the estimator to be $\hat{b}(t)u(t)+\hat{W}(t)\Phi(x,z,u)$. The requirement

$$b_0>W^*\phi^*\,,$$

where W^* is a norm bound imposed by the Projection operator $\|\hat{W}(t)\|\le W^*$, while $\phi^*\ge\|\frac{\partial\Phi(x,z,u)}{\partial u}\|$, will ensure that the control effectiveness of this estimator $\hat{b}(t)u+\hat{W}(t)\Phi(x,z,u)$ is bounded away from zero:

$$\Big|\hat{b}(t)+\hat{W}^\top(t)\frac{\partial\Phi(x,z,u)}{\partial u}\Big|>a_0>0\,, \quad (37)$$

where $a_0=b_0-W^*\phi^*$.

Subject to Assumption 3, we consider the following fast dynamics:

$$\epsilon\dot{\nu}_{ad}=-\mathrm{sign}\left(\frac{\partial\mathbf{f}}{\partial\nu_{ad}}\right)\mathbf{f}(t,\hat{e},z,\nu_{ad}),\quad \nu_{ad}(0)=\nu_{ad_0}, \quad (38)$$

where

$$\begin{aligned}\mathbf{f}(t,\hat{e},z,\nu_{ad}) &= \hat{W}^\top(t)\Phi\Big(\hat{e}+\mathbf{r}(t)+\tilde{e}(t),z,\\ &\hat{f}^{-1}(\hat{e}+\mathbf{r}(t)+\tilde{e}(t),z,\nu_0(t)-\nu_{ad})\Big)-\nu_0(t).\end{aligned}$$

Let $\nu_{ad}=h(t,\hat{e},z)$ be an isolated root of $\mathbf{f}(t,\hat{e},z,\nu_{ad})=0$. The reduced system for the dynamics in (29), (38) is given by:

$$\dot{\hat{e}}(t)=A_k\hat{e}(t) \quad (39)$$

$$\dot{z}(t)=\boldsymbol{\zeta}(t,\hat{e}(t),z(t),h(t,\hat{e}(t),z(t))) \quad (40)$$

with $\hat{e}(0) = \hat{e}_0, z(0) = z_0$, where $\boldsymbol{\zeta}(t, \hat{e}, z, h) = \zeta\big(\hat{e} + \mathbf{r}(t) + \tilde{e}(t), z, f^{-1}\big(\hat{e} + \mathbf{r}(t) + \tilde{e}(t), z, \nu_0(t) - h(t, \hat{e}, z)\big)\big)$. The boundary layer system is given by:

$$\frac{dv}{d\tau} = -\text{sign}\left(\frac{\partial \mathbf{f}}{\partial \nu_{ad}}\right)\mathbf{f}(t, \hat{e}, z, v + h(t, \hat{e}, z)). \tag{41}$$

Applying Theorem 1, we now get the following result for single input systems:

Theorem 4: Assume that the adaptive process is such that the following conditions are satisfied for all $[t, \hat{e}, z, \nu_{ad} - h(t, \hat{e}, z), \epsilon] \in [0, \infty) \times D_{\hat{e},z} \times D_v \times [0, \epsilon_0]$ for some domains $D_{\hat{e},z} \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}$, which contain their respective origins:

B1. On any compact subset of $D_{\hat{e},z} \times D_v$, the functions $\mathbf{f}$, ζ, and their first partial derivatives with respect to $(\hat{e}, z, \nu_{ad})$, and the first partial derivative of $\mathbf{f}$ with respect to t are continuous and bounded, $h(t, \hat{e}, z)$ and $\frac{\partial \mathbf{f}}{\partial \nu_{ad}}(t, \hat{e}, z, \nu_{ad})$ have bounded first derivatives with respect to their arguments, $\frac{\partial \mathbf{f}}{\partial \hat{e}}$, $\frac{\partial \mathbf{f}}{\partial z}$ as functions of $(t, \hat{e}, z, h(t, \hat{e}, z))$ are Lipschitz in $\hat{e}, z$, uniformly in t.

B2. The origin is an exponentially stable equilibrium point of the system

$$\dot{z} = \zeta(\mathbf{r}(t) + \tilde{e}(t), z, f^{-1}(\mathbf{r}(t) + \tilde{e}(t), z, \nu_0(t) - h(t, 0, z)))$$

The map $(\hat{e}, z) \mapsto \zeta(\hat{e} + \mathbf{r}(t) + \tilde{e}(t), z, f^{-1}(\hat{e} + \mathbf{r}(t) + \tilde{e}(t), z, \nu_0(t) - h(t, \hat{e}, z)))$ is continuously differentiable and Lipschitz in $(\hat{e}, z)$, uniformly in t.

B3. The adaptive process is such that $(t, \hat{e}, z, v) \mapsto \frac{\partial \mathbf{f}}{\partial \nu_{ad}}(t, \hat{e}, z, v + h(t, \hat{e}, z))$ is bounded away from zero for all $(t, \hat{e}, z) \in [0, \infty) \times D_{\hat{e},z}$.

Then the origin of (41) is exponentially stable. Moreover, let Ω_v be a compact subset of R_v, where $R_v \subset D_v$ denotes the region of attraction of the autonomous system

$$\frac{dv}{d\tau} = -\text{sign}\left(\frac{\partial \mathbf{f}}{\partial \nu_{ad}}\right)\mathbf{f}(0, \hat{e}_0, z_0, v + h(0, \hat{e}_0, z_0)).$$

Then for each compact subset $\Omega_{z,\hat{e}} \subset D_{z,\hat{e}}$ there exist a positive constant ϵ_* and a $T > 0$ such that for all $t \geq 0$, $(\hat{e}_0, z_0) \in \Omega_{\hat{e},z}$, $\nu_{ad_0} - h(0, \hat{e}_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations (29), (38) has a unique solution $\hat{e}_\epsilon(t)$ on $[0, \infty)$, and $\hat{e}_\epsilon(t) = \hat{e}_{\mathtt{r}}(t) + O(\epsilon)$ holds uniformly for $t \in [T, \infty)$, where $\hat{e}_{\mathtt{r}}(t)$ denotes the solution of the reduced system (39).

Proof: We need to verify that Assumptions A1, A2, A3 in Theorem 1 are satisfied. Assumption B1 clearly implies that A1 holds.

We now show that Assumption A2 holds. Assumption B2 implies (see Lemma 4.6, page 176 of [10]), that the system (40) (with $\hat{e}$ viewed as the input) is input to state stable. Thus there exists class $\mathcal{K}$ and class $\mathcal{KL}$ functions γ and β, respectively, such that $\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma\big(\sup_{t_0 \leq \tau \leq t} \|\hat{e}(\tau)\|\big)$ for all $t \geq t_0$, $t_0 \in [0, \infty)$. Furthermore from the proof of Lemma 4.6 of [10], it follows that $\gamma(\rho) = c\rho$, for some constant $c > 0$. Using the fact that the unforced system has 0 as an exponentially stable equilibrium point, it can be seen from the proof of Lemma 4.6 of [10] that $\beta(\rho, t) = k\rho\exp(-\omega t)$ for some positive constants k and ω. Thus the solution to the reduced system (39)-(40) satisfies $\|\hat{e}(t)\| \leq \|\hat{e}_0\| c_1 \exp(-\omega_0 t)$ and $\|z(t)\| \leq (\|x_0\| + \|z_0\|) c_2 \exp(-\omega_0 t)$ for all $t \geq 0$ and for some $\omega_0 > 0$. Hence, the origin $(0, 0)$ is an exponentially stable equilibrium point of (39)-(40). From a converse Lyapunov theorem (Theorem 4.14 on pages 162-163 of [10]), it follows that there exists a Lyapunov function $V : [0, \infty) \times D_{\hat{e},z} \to \mathbb{R}$ such that $w_1\|(\hat{e}, z)\|^2 \leq V(t, \hat{e}, z) \leq w_2\|(\hat{e}, z)\|^2$ and $\frac{\partial V}{\partial t}(t, \hat{e}, z) + \nabla_{\hat{e},z} V \cdot \mathbf{F}(t, \hat{e}, z) \leq -w_3\|(\hat{e}, z)\|^2$, where $\mathbf{F}(t, \hat{e}, z) = \big[(A_k \hat{e})^\top \;\; \boldsymbol{\zeta}(t, \hat{e}, z, h)^\top\big]^\top$. We note that any positive c can be chosen in A2 of Theorem 1, and so a compact $\Omega_{\hat{e},z} \subset \{(\hat{e}, z) \in D_{\hat{e},z} \mid W_2(\hat{e}, z) \leq \rho c,\ 0 < \rho < 1\}$ can be chosen to be any subset of $D_{\hat{e},z}$.

In light of the Remark 2.1, it is easy to see that with the definition of the boundary layer system given by (41), subject to Assumption 3, its exponential stability can be verified locally by linearization with respect to v.

Hence Theorem 1 applies and so it follows that for each compact set $\Omega_{\hat{e},z} \subset D_{\hat{e},z}$ there exist a positive constant $\epsilon_* > 0$ and $T > 0$, such that for all $(\hat{e}_0, z_0) \in \Omega_{\hat{e},z}$, $\nu_{ad_0} - h(0, \hat{e}_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations given by (29), (38) has a unique solution $\hat{e}_\epsilon$, z_ϵ on $[0, \infty)$ and

$$\hat{e}_\epsilon(t) \;\; = \;\; \hat{e}_{\mathtt{r}}(t) + O(\epsilon), \quad z_\epsilon(t) = z_{\mathtt{r}}(t) + O(\epsilon) \tag{42}$$

hold uniformly for $t \in [T, \infty)$, while $\hat{e}_{\mathtt{r}}, z_{\mathtt{r}}$ denote the solution of the reduced system (39)-(40). ■

Corollary 5: From Theorems 2 and 4, it follows that $x(t)$ tracks $\mathbf{r}(t)$ with bounded errors.

IV. Simulations

Consider tracking problem for the scalar nonlinear system given by

$$\dot{x}(t) = 0.5x(t) + \tanh(u(t) + 3) + \tanh(u(t) - 3) + 0.01u(t) \tag{43}$$

with $x(0) = 0$. It is easy to see that the system dynamics is invertible, but not in terms of elementary functions. This system is motivated by aircraft applications, in which control effectiveness $\dfrac{\partial f}{\partial u}$ is small for both small and large control inputs u.

Dynamic inversion is performed using the following model $\hat{f}(x, u) = 0.5u$, implying that $u = 2\nu$, with the following choice of pseudo-control $\nu(t) = -10(x(t) - \mathtt{r}(t)) + \dot{\mathtt{r}}(t) - \nu_{ad}(t)$. Thus, the modeling error can be written as: $\Delta(x, u) = 0.5x + \tanh(u + 3) + \tanh(u - 3) - 0.49u$. The error dynamics are $\dot{e}(t) = -10e(t) + \Delta(x(t), u(t)) - \nu_{ad}(t)$, with $e(t) = x(t) - \mathtt{r}(t)$. The series parallel model is designed with the use of 25 RBFs, distributed over the grid $x \in [-2, 2]$, $u \in [-2, 2]$ with the step size equal to 1 in both dimensions: $\Phi_i(x, u) = \exp(-3((x - x_i)^2 + (u - u_i)^2)/2)$, where the point (x_i, u_i) represents the center of the i^{th} RBF. The one-step-ahead error predictor is designed as:

$$\dot{\hat{e}}(t) = -20\hat{e}(t) + \hat{W}^\top(t)\Phi(x(t), u(t)) + 10e(t) - \dot{\mathtt{r}}(t)\,.$$

The norm upper bound for the projection operator is set to $W^* = 10$, adaptation gain is set to $\Gamma = 25$. Simulation

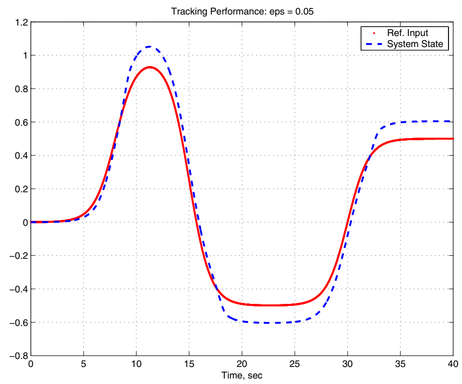

Fig. 1. Tracking without adaptive augmentation

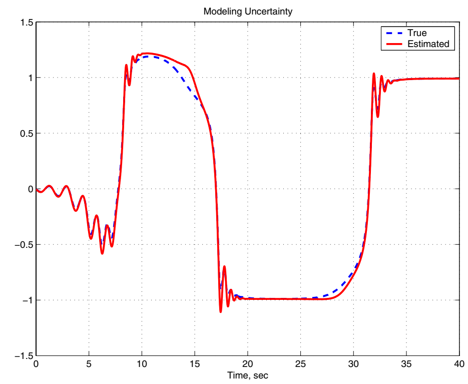

Fig. 3. Estimation of nonlinearity over time

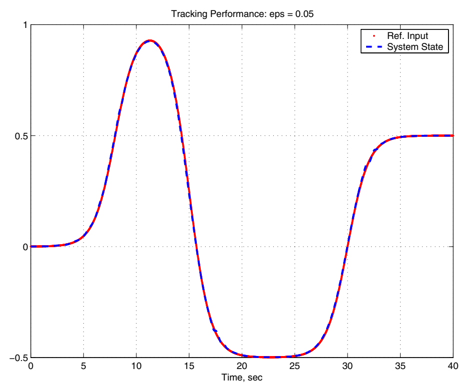

Fig. 2. Tracking performance with adaptive augmentation

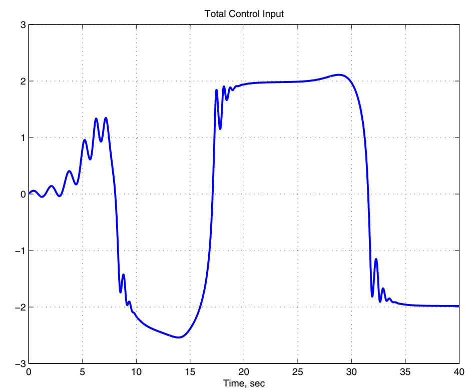

Fig. 4. Control history

is performed using the following reference input $\mathtt{r}(t) = \frac{1}{1+\exp(t-8)} - \frac{1.5}{1+\exp(t-15)} + \frac{1}{1+\exp(t-30)}$. The fast dynamics are designed as:

$$\begin{aligned} 0.05\dot{\nu}_{ad} &= -(\hat{W}^\top(t)\Phi(x(t), 2(-10e(t) + \dot{\mathtt{r}}(t) - \nu_{ad}(t))) \\ &+ \quad 10e(t) - \dot{\mathtt{r}}(t)). \end{aligned}$$

Figure 1 demonstrates closed-loop performance of the dynamic inversion controller, when applied to the nonlinear system without adaptive augmentation. Turning adaptive augmentation on, Figure 2 shows significant improvement of the tracking performance. Finally, Figure 3 shows estimation of the nonlinearity via the adaptive signal over time, and Figure 4 demonstrates total control effort.

V. Conclusions

For nonlinear systems with uncertain nonaffine-in-control dynamics, a direct adaptive model reference control design is presented that augments an approximate dynamic inversion controller. The proposed adaptive augmentation is computed online as a solution of a fast dynamical equation. Such a solution is shown to compensate for control dependent modeling uncertainties via time-scale separation. The open problem in this context is the choice of adaptive law and a regressor vector to ensure exponential stability of the boundary layer system associated with the fast dynamical equation. One way to ensure this is discussed in Remark 3. Alternate ways are currently under investigation.

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