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The Infinite-Dimensional Continuous Time Kalman–Yakubovich–Popov Inequality (for Scattering Supply Rate)

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***Abstract*— We study the set M_Σ of all generalized solutions (that may be unbounded and have an unbounded inverse) of the KYP (Kalman–Yakubovich–Popov) inequality for a infinite-dimensional linear time-invariant system Σ in continuous time with scattering supply rate. It is shown that if M_Σ is nonempty, then the transfer function of Σ coincides with a Schur class function in some right half-plane. For a minimal system Σ the converse is also true. In this case the set of all $H \in M_\Sigma$ with the property that the system is still minimal when the original norm in the state space is replaced by the norm induced by H is shown to have a minimal and a maximal solution, which correspond to the available storage and the required supply, respectively. We show by an example that the stability of the system with respect to the norm induced by some $H \in M_\Sigma$ depends crucially on the particular choice of H. In this example, depending on the choice of the original realization, some or all $H \in M_\Sigma$ will be unbounded and/or have an unbounded inverse.**

I. Introduction

Linear finite-dimensional time-invariant systems in continuous time are typically modeled by the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \ge 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

on a triple of finite-dimensional vector spaces, namely, the *input* space $\mathcal{U}$, the *state* space $\mathcal{X}$, and the *output* space $\mathcal{Y}$. We have $u(t) \in \mathcal{U}$, $x(t) \in \mathcal{X}$ and $y(t) \in \mathcal{Y}$. We are interested in the case where, in addition to the dynamics described by (1), the components of the system satisfy an energy inequality. In this paper we shall use the *scattering supply rate*

$$j(u,y) = \|u\|^2 - \|y\|^2 = \left\langle \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} 1_\mathcal{U} & 0 \\ 0 & -1_\mathcal{Y} \end{smallmatrix}\right] \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \right\rangle \tag{2}$$

and the *storage (or Lyapunov) function*

$$E_H(x) = \langle x, Hx \rangle, \quad H > 0. \tag{3}$$

A system is *scattering H-passive* (or simply scattering passive if $H = 1_\mathcal{X}$) if for any admissible data $(x_0, u(\cdot))$ the solution of the system (1) satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}t} E_H(x(t)) \le j(u(t), y(t)) \text{ a.e. on } (0,\infty). \tag{4}$$

This inequality is often written in integrated form

$$E_H(x(t)) - E_H(x(s)) \le \int_s^t j(u(v), y(v))\, dv, \quad 0 \le s \le t. \tag{5}$$

It is not difficult to see that the inequality (4) with supply rate (2) is equivalent to the inequality

$$\begin{aligned} 2\Re\langle Ax + Bu, Hx\rangle &+ \|Cx + Du\|^2 \\ &\le \|u\|^2, \; x \in \mathcal{X}, u \in \mathcal{U}, \end{aligned} \tag{6}$$

which is usually rewritten in the form

$$\begin{bmatrix} HA + A^*H + C^*C & HB + C^*D \\ B^*H + D^*C & D^*D - 1_\mathcal{U} \end{bmatrix} \le 0. \tag{7}$$

This is the standard KYP (Kalman–Yakubovich–Popov) inequality for continuous time and scattering supply rate.

In the development of the theory of absolute stability (or hyperstability) of systems which involve nonlinear feedback those linear systems which are H-passive with respect to a scattering supply rate are of special interest, especially in H^∞ control. One of the main problems is to find conditions on the coefficients A, B, C, and D under which the KYP inequality has at least one solution $H > 0$.

To formulate a classical result about the solution of this problem we introduce the main frequency characteristic of the system (1), namely its *transfer function* defined by

$$\mathfrak{D}(z) = D + C(z - A)^{-1}B, \qquad z \in \rho(A). \tag{8}$$

We also introduce the *Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{C}^+)$ of *holomorphic contractive* functions $\mathfrak{D}$ defined on $\mathbb{C}^+$ with values in $\mathcal{B}(\mathcal{U}, \mathcal{Y})$. Here $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \Re z > 0\}$. If $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are finite-dimensional, then the transfer function is rational and $\dim \mathcal{X} \ge \deg \mathfrak{D}$, where $\deg \mathfrak{D}$ is the MacMillan degree of $\mathfrak{D}$. A finite-dimensional system is *minimal* if $\dim X = \deg \mathfrak{D}$. The state space of a minimal system has the smallest dimension among all systems with the same transfer function $\mathfrak{D}$.

The (finite-dimensional) system (1) is *controllable* if, given any $z_0 \in \mathcal{X}$ and $T > 0$, there exists some continuous function u on $[0, T]$ such that the solution of (1) with $x(0) = 0$ satisfies $x(T) = z_0$. It is *observable* if it has the following property: if both the input function u and the

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output function y vanish on some interval $[0, T]$ with $T > 0$, then necessarily the initial state x_0 is zero.

Theorem 1.1 (Kalman): A finite-dimensional system is minimal if and only if it is controllable and observable.

Theorem 1.2 (Kalman–Yakubovich–Popov): Let $\Sigma = ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a finite-dimensional system with transfer function $\mathfrak{D}$.

(i) If the KYP inequality (7) has a solution $H > 0$ then $\mathbb{C}^+ \subset \rho(A)$ and $\mathfrak{D}|_{\mathbb{C}^+} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{C}^+)$.

(ii) If Σ is minimal and $\mathfrak{D}|_{\mathbb{C}^+} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{C}^+)$, then the KYP inequality (7) has a solution H, i.e., Σ is scattering H-passive for some $H > 0$.

Here $\mathfrak{D}|_\Omega$ is the restriction of $\mathfrak{D}$ to $\Omega \subset \rho(A)$.

It is can be shown that $H > 0$ is a solution of (7) if and only if $\tilde{H} = H^{-1}$ is a solution of the the dual KYP inequality

$$\begin{bmatrix} \tilde{H}A^* + A\tilde{H} + BB^* & \tilde{H}C^* + BD^* \\ C\tilde{H} + DB^* & DD^* - 1_\mathcal{Y} \end{bmatrix} \leq 0. \tag{9}$$

The *discrete time* scattering KYP inequality is given by

$$\begin{bmatrix} A^*HA + C^*C - H & A^*HB + C^*D \\ B^*HA + D^*C & D^*D + B^*HB - 1_\mathcal{U} \end{bmatrix} \leq 0. \tag{10}$$

The corresponding Kalman–Yakubovich–Popov theorem is still valid with $\mathbb{C}^+$ replaced by $\mathbb{D}^+ = \{z \in \mathbb{C} \mid |z| > 1\}$ and with the transfer function defined by the same formula (8).[1]

In the seventies the classical results on the KYP inequalities were extended to systems with $\dim \mathcal{X} = \infty$ by V. A. Yakubovich and his students and collaborators (see [22], [23], [8] and the references listed there). There is now a rich literature on this subject; see, e.g., the discussion in [10] and the references cited there. However, as far as we know, in these and all later generalizations it was assumed (until [2]) that *either H itself is bounded or H^{-1} is bounded*.[2] This is not always a realistic assumption. The operator H is very sensitive to the choice of the state space $\mathcal{X}$ and its norm, and the boundedness of H and H^{-1} depends entirely on this choice. By allowing both H and H^{-1} to be unbounded we can use an analogue of the standard finite-dimensional procedure to determine whether a given transfer function θ is a Schur function or not, namely to *choose an arbitrary minimal realization of θ, and then check whether the KYP inequality* (7) *has a positive (generalized) solution*. This procedure would not work if we require H or H^{-1} to be bounded, because our first main theorem (Theorem 3.3) is not true in that setting. We shall discuss this further in Section V by means of an example.

A generalized solution of the discrete time KYP inequality (10) that permits both H and H^{-1} to be unbounded was developed by Arov, Kaashoek and Pik in [2]. There it was required that

$$A\mathcal{D}(\sqrt{H}) \subset \mathcal{D}(\sqrt{H}) \text{ and } \mathcal{R}(B) \subset \mathcal{D}(\sqrt{H}), \tag{11}$$

and (10) was rewritten using the corresponding quadratic form defined on $\mathcal{D}(\sqrt{H}) \oplus \mathcal{U}$. *Here we extend this approach to continuous time.*

II. Continuous Time System Nodes

In discrete time one always assumes that A, B, C, and D are bounded operators. In continuous time this assumption is not reasonable. Below we will use a natural continuous time setting, earlier used in, e.g., [3], [9], [12], [13], and [14] (in slightly different forms).

In the sequel, we think about the block matrix $S = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ as *one single closed (possibly unbounded) linear operator* from $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]$ (the cross product of $\mathcal{X}$ and $\mathcal{U}$) to $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}]$ with dense domain $\mathcal{D}(S) \subset [\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]$, and write (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{12}$$

In the infinite-dimensional case such an operator S need not have a four block decomposition corresponding to the decompositions $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]$ and $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}]$ of the domain and range spaces. However, we shall throughout assume that the operator

$$\begin{aligned} Ax &:= P_\mathcal{X} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \\ x &\in \mathcal{D}(A) := \{x \in \mathcal{X} \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S)\}, \end{aligned} \tag{13}$$

is closed and densely defined in $\mathcal{X}$ (here $P_\mathcal{X}$ is the orthogonal projection onto $\mathcal{X}$). We define $\mathcal{X}^1 := \mathcal{D}(A)$ with the graph norm of A, $\mathcal{X}^1_* := \mathcal{D}(A^*)$ with the graph norm of A^*, and let $\mathcal{X}^{-1}$ to be the dual of $\mathcal{X}^1_*$ when we identify the dual of $\mathcal{X}$ with itself. Then $\mathcal{X}^1 \subset \mathcal{X} \subset \mathcal{X}^{-1}$ with continuous and dense embeddings, and the operator A has a unique extension to an operator $\widehat{A} = (A^*)^* \in \mathcal{B}(\mathcal{X}; \mathcal{X}^{-1})$, where we interpret A^* as an operator in $\mathcal{B}(\mathcal{X}^1_*; \mathcal{X})$.[3] Additional assumptions on A will be added in Definition 2.1 below.

The remaining blocks of S will be only partially defined. The 'block' B will be an operator in $\mathcal{B}(\mathcal{U}; \mathcal{X}^{-1})$. In particular, it may happen that $\mathcal{R}(B) \cap \mathcal{X} = \{0\}$. The 'block' C will be an operator in $\mathcal{B}(\mathcal{X}^1; \mathcal{Y})$. We shall make no attempt to define the 'block' D in general since this can be done only under additional assumptions (see, e.g., [14, Chapter 5] or [17], [18]). Nevertheless, we still use a modified block notation $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where $A\&B = P_\mathcal{X} S$ and $C\&D = P_\mathcal{Y} S$.

Definition 2.1: By a *system node* we mean a colligation $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and the *system operator* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a (possibly unbounded) linear operator from $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}]$ to $[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}]$ with the following properties:

[1] This is the standard "engineering" version of the transfer function. In the mathematical literature one usually replace z by $1/z$ and $\mathbb{D}^+$ by the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

[2] Results where H^{-1} is bounded are typically proved by replacing the primal KYP inequality by the dual KYP inequality (9).

[3] This construction is found in most of the papers listed in the bibliography (in slightly different but equivalent forms), including [3], [9], and [11]–[19].

(i) S is closed.

(ii) The operator A defined in (13) is the generator of a C_0 semigroup $t \mapsto T^t$, $t \geq 0$, on $\mathcal{X}$.

(iii) $A\&B$ has an extension $\begin{bmatrix} \widehat{A} & B \end{bmatrix} \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \mathcal{X}^{-1})$ (where $B \in \mathcal{B}(\mathcal{U}; \mathcal{X}^{-1})$).

(iv) $\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \;\middle|\; \widehat{A}x + Bu \in \mathcal{X} \right\}$, and $A\&B = \begin{bmatrix} \widehat{A} & B \end{bmatrix}|_{\mathcal{D}(S)}$;

It can be shown that (ii)–(iv) imply that the domain of S is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. It is also true that if (ii)–(iv) holds, then (i) is equivalent to the following condition:

(v) $C\&D \in \mathcal{B}(\mathcal{D}(S); \mathcal{Y})$, where we use the graph norm of $A\&B$ on $\mathcal{D}(S)$.

We call $A \in \mathcal{B}(\mathcal{X}^1; \mathcal{X})$ the *main operator* of Σ, $t \mapsto T^t$, $t \geq 0$, is the *evolution semigroup*, $B \in \mathcal{B}(\mathcal{U}; \mathcal{X}^{-1})$ is the *control operator*, and $C\&D \in \mathcal{B}(V; \mathcal{Y})$ is the *combined observation/feedthrough operator*. From the last operator we can extract $C \in \mathcal{B}(\mathcal{X}^1; \mathcal{Y})$, the *observation operator* of Σ, defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in \mathcal{X}^1. \tag{14}$$

It can be proved that

$$\begin{bmatrix} (z - \widehat{A})^{-1} Bu \\ u \end{bmatrix} \in \mathcal{D}(S)$$

for all $z \in \rho(A)$ and $u \in \mathcal{U}$. We can therefore define the *transfer function* $\mathfrak{D}$ of Σ by

$$\mathfrak{D}(z) = C\&D \left[\begin{smallmatrix} (z-\widehat{A})^{-1} B \\ 1_{\mathcal{U}} \end{smallmatrix}\right], \quad z \in \rho(A). \tag{15}$$

In the case where the 'block' D is well-defined, e.g., in the case where $\mathcal{R}(B) \subset \mathcal{X}$, the formula (15) can be written in the standard form (8).

If $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a system node, then (12) has (smooth) trajectories of the following type.

Lemma 2.2: Let $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system node. Then for each $x_0 \in \mathcal{X}$ and $u \in W^{2,2}_{\mathrm{loc}}([0,\infty); \mathcal{U})$ such that $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, there is a unique function $x \in C^1([0,\infty); \mathcal{X})$ (called a *state trajectory*) satisfying $x(0) = x_0$, $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$, $t \geq 0$, and $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, $t \geq 0$. If we define the output by $y(t) = C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, $t \geq 0$, then $y \in C([0,\infty); \mathcal{Y})$, and the three functions u, x, and y satisfy (12).

The lemma is contained in [14, Lemmas 4.7.7–4.7.8].

By the *system* induced by a system node $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean the node itself together with all the trajectories of Σ. We use the same notation Σ for the system as for the node.

A system Σ is (approximately) *controllable* if the set of all possible states $x(t)$ in Lemma 2.2 with $x_0 = 0$ and $u(0) = 0$ is dense in $\mathcal{X}$ (i.e., we let u vary over all functions in $u \in W^{2,2}_{\mathrm{loc}}([0,\infty); \mathcal{U})$ with $u(0) = 0$, and let t vary over $[0,\infty)$). It is (approximately) *observable* if the only trajectory $x(\cdot)$ with $x(0) \in \mathcal{D}(A)$ for which both the input function $u(\cdot)$ and output function $y(\cdot)$ vanish identically is the zero trajectory $x(\cdot) = 0$. Finally, we define Σ to be minimal if it is both controllable and observable. It can be shown that Σ is controllable if and only if

$$\bigvee_{\lambda \in \rho^+_\infty(A)} \mathcal{R}((\lambda - \widehat{A})^{-1} B) = \mathcal{X}, \tag{16}$$

where $\rho^+_\infty(A)$ is the connected component of A which contains a right half-plane. Similarly, Σ is observable if and only if

$$\bigcap_{\lambda \in \rho^+_\infty(A)} \mathcal{N}(C(\lambda - A)^{-1}) = 0. \tag{17}$$

Finally, it is minimal if and only if both (16) and (17) hold.

III. The Generalized KYP Inequality

In our study of the KYP inequality we do not only allow the operators A, B, and C to be unbounded (as explained above), but we allow both the *storage operator* $H > 0$ and its inverse H^{-1} to be *unbounded* as well. This means that one must be very careful about the domain on which the different operators act.

In the case of an unbounded operator H we rewrite the storage function E_H in (3) in the form

$$E_H(x) = \|\sqrt{H}x\|^2, \quad x \in \mathcal{D}(\sqrt{H}). \tag{18}$$

This is equivalent to replacing the operator $H > 0$ by the corresponding (closed) quadratic form induced by H. In addition we shall require $\mathcal{D}(\sqrt{H})$ to be invariant under trajectories of Σ.

Definition 3.1: A system node $\Sigma := (S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is (scattering) *H-passive* (or simply *passive* if $H = 1_{\mathcal{X}}$) if the following conditions hold:

(i) H is a positive (injective, possibly unbounded) self-adjoint operator on $\mathcal{X}$. We denote the positive self-adjoint square root of H by $Q := \sqrt{H}$.

(ii) If $u \in W^{2,2}_{\mathrm{loc}}([0,\infty); \mathcal{U})$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$ with $x_0 \in \mathcal{D}(Q)$ and $A\&B \left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(Q)$, then the solution x in Lemma 2.2 satisfies $x(t), \dot{x}(t) \in \mathcal{D}(Q)$ for all $t \geq 0$, and both Qx and $Q\dot{x}$ are continuous in $\mathcal{X}$ on $[0,\infty)$.

(iii) Each solution of the type described in (ii) satisfies

$$\begin{aligned} &\langle Qx(t), Qx(t) \rangle_{\mathcal{X}} + \int_s^t \|y(v)\|^2_{\mathcal{Y}}\, dv \\ &\qquad \leq \langle Qx(s), Qx(s) \rangle_{\mathcal{X}} + \int_s^t \|u(v)\|^2_{\mathcal{U}}\, dv, \\ &\qquad 0 \leq s \leq t. \end{aligned} \tag{19}$$

In the present infinite-dimensional case the connection between the H-passivity of a system node and the generalized KYP inequality is more subtle than in the finite-dimensional case. In particular, solutions of the generalized KYP inequality must satisfy a certain invariance condition.

The following is our first main result.

Theorem 3.2: Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system node with main operator A and control operator B, and let $\rho_\infty^+(A)$ be the connected component of $\rho(A) \cap \mathbb{C}^+$ which contains some right half-plane. Then Σ is H-passive if and only if the following conditions hold:

(i) H is a positive (injective, possibly unbounded) self-adjoint operator on $\mathcal{X}$. We denote the positive self-adjoint square root of H by $Q := \sqrt{H}$.

(ii) $(\lambda - A)^{-1}\mathcal{D}(Q) \subset \mathcal{D}(Q)$ for some $\lambda \in \rho_\infty^+(A)$.

(iii) $(\lambda - \widehat{A})^{-1}B\mathcal{U} \subset \mathcal{D}(Q)$ for some $\lambda \in \rho_\infty^+(A)$.

(iv) The operator QAQ^{-1}, defined on its natural domain consisting of those $x \in \mathcal{R}(Q)$ for which $Q^{-1}x \in \mathcal{D}(A)$ and $AQ^{-1}x \in \mathcal{D}(Q)$, is closable.

(v) For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$ with $x_0 \in \mathcal{D}(Q)$ and $A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(Q)$ we have

$$\begin{aligned} 2\Re\langle Q[A\&B]\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], Qx_0\rangle_{\mathcal{X}} \\ + \|C\&D\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|_{\mathcal{Y}}^2 \le \|u_0\|_{\mathcal{U}}^2. \end{aligned} \tag{20}$$

Here conditions (ii) and (iv) can alternatively be replaced by the condition

(ii') $T^t\mathcal{D}(Q) \subset \mathcal{D}(Q)$ for all $t \ge 0$, and the function $t \mapsto QT^tx_0$ is continuous on $[0, \infty)$ (with values in $\mathcal{X}$) for all $x_0 \in \mathcal{D}(Q)$,

where $t \mapsto T^t$ is the evolution semigroup of Σ.

One half of the proof of Theorem 3.2 is easy, namely the claim that (i)–(iii) in Definition 3.1 imply (i)–(v) in Theorem 3.2. The most difficult part of the opposite direction of the proof is to show that (i)–(v) in Theorem 3.2 imply condition (ii) in Definition 3.1.

We shall call (20) the *generalized (continuous time scattering) KYP inequality*, and we call H a solution of this inequality if and only if (i)–(v) in Theorem 3.2 hold. Thus, by Theorem 3.2, H is a solution of the generalized KYP inequality if and only if Σ is H-passive. If all the operators in (20) are bounded together with H^{-1}, then (20) reduces to the standard KYP inequality (7).

For the formulation of our next main theorem we recall the definition of the restricted Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$, where Ω is an open connected subset of $\mathbb{C}^+$: $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$ means that θ is the restriction to Ω of a function in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y}, \mathbb{C}^+)$. It is known that $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$ if and only if θ is a $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued holomorphic function on Ω and the kernel

$$K_\theta(z, \omega) = \frac{1_{\mathcal{Y}} - \theta(z)\theta(\omega)^*}{z + \overline{\omega}}, \quad z,\ \omega \in \Omega,$$

is positive definite on $\Omega \times \Omega$ (or more generally, on $\Omega_0 \times \Omega_0$ where $\Omega_0 \subset \Omega$ contains some interior cluster point; see [1]).

Theorem 3.3: Let $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a system node with main operator A and transfer function $\mathfrak{D}$. Let $\rho_\infty^+(A)$ be the connected component of $\rho(A) \cap \mathbb{C}^+$ which contains some right half-plane.

(i) If the generalized KYP inequality (20) has a solution H, i.e., if Σ is H-passive, then $\mathfrak{D}|_{\rho_\infty^+(A)} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \rho_\infty^+(A))$.

(ii) Conversely, suppose that Σ is minimal and that $\mathfrak{D}|_{\rho_\infty^+(A)} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \rho_\infty^+(A))$. Then the generalized KYP inequality (20) has a solution H satisfying the two additional minimality conditions

$$\begin{aligned} \bigvee_{\lambda \in \rho_\infty^+(A)} \mathcal{R}(\sqrt{H}(\lambda - \widehat{A})^{-1}B) = \mathcal{X}, \\ \bigcap_{\lambda \in \rho_\infty^+(A)} \mathcal{N}(C(\lambda - A)^{-1}|_{\mathcal{D}(\sqrt{H})}) = 0. \end{aligned} \tag{21}$$

These minimality conditions mean that if we replace the original norm in the state space by the norm obtained from the storage function E_H (and then complete the space with respect to the new norm), then the resulting system Σ_H is still minimal. The KYP inequality says that this new system is scattering passive. If both H and H^{-1} are bounded, then the conditions (21) hold if and only if the original system Σ is minimal.

In our third main theorem we compare solutions of the generalized KYP inequality to each other by using the partial ordering of nonnegative self-adjoint operators on $\mathcal{X}$: if H_1 and H_2 are two nonnegative self-adjoint operators on the Hilbert space $\mathcal{X}$, then we write $H_1 \preceq H_2$ whenever $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$ and $\|H_1^{1/2}x\| \le \|H_2^{1/2}x\|$ for all $x \in \mathcal{D}(H_2^{1/2})$. For *bounded* nonnegative operators H_1 and H_2 with $\mathcal{D}(H_2) = \mathcal{D}(H_1) = \mathcal{X}$ this ordering coincides with the standard ordering of bounded self-adjoint operators.

We denote the set of all solutions H of the generalized KYP inequality (20) satisfying the additional minimality conditions (21) by $M_\Sigma^{\min}$.

Theorem 3.4: Let $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system node with transfer function $\mathfrak{D}$ satisfying the condition $\mathfrak{D}|_{\rho_\infty^+(A)} \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \rho_\infty^+(A))$ (this notation is explained before and in Theorem 3.3). Then $M_\Sigma^{\min}$ is nonempty, and it contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e.,

$$H_\circ \preceq H \preceq H_\bullet, \quad H \in M_\Sigma^{\min}.$$

The two extremal storage functions $E_{H_\circ}$ and $E_{H_\bullet}$ correspond to Willems' [20], [21] *available storage* and *required supply*, respectively. See [14, Remark 11.8.11] for details. We define $H_\odot \in M_\Sigma^{\min}$ to be the *balanced* solution of the generalized KYP inequality (20), i.e., the solution $H_\odot$ for which the system $\Sigma_{H_\odot}$ is the passive balanced realization constructed in [14, Theorem 11.8.14].[4]

IV. H-Stability

The possible unboundedness of H and and H^{-1} where H is a solution of the generalized KYP inequality (20) has important consequences for the stability analysis of Σ. Indeed, in the finite-dimensional setting it is sufficient to prove stability with respect to the storage function E_H defined in (3) in order to get stability with respect to the original norm in the state space, since all norms in

[4] $H_\odot$ can in a certain sense be interpreted as a geometric mean of $H_\circ$ and $H_\bullet$.

a finite-dimensional space are equivalent. This is not true in the infinite-dimensional setting unless H and H^{-1} are bounded. Stability with respect to one storage function E_{H_1} is not equivalent to stability with respect to another storage function E_{H_2}. Moreover, the natural norm to use for the adjoint system is the one obtained from $E_{H^{-1}}$ instead of E_H, taking into account that H is a solution of the generalized KYP inequality (20) if and only if $\tilde{H} = H^{-1}$ is a solution of the adjoint generalized KYP inequality.

Definition 4.1: Let H be a solution of the generalized KYP inequality (20). Then the evolution semigroup $t \mapsto T^t$, $t \geq 0$, is

(i) strongly H-stable, if

$$\lim_{t\to\infty} \|H^{1/2}T^t x\| \to 0 \text{ for all } x \in \mathcal{D}(H^{1/2}),$$

(ii) strongly H-$*$-stable, if

$$\lim_{t\to\infty} \|H^{-1/2}(T^t)^* x_*\| \to 0 \text{ for all } x_* \in \mathcal{R}(H^{1/2}),$$

(iii) strongly H-bistable if both (i) and (ii) above hold.

Theorem 4.2: Let $\Sigma := (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a minimal system node with transfer function $\mathfrak{D}$ satisfying the condition $\mathfrak{D}|_{\rho^+_\infty(A)} = \theta|_{\rho^+_\infty(A)}$ for some $\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{C}^+)$. Let $H_\circ$, $H_\bullet$, and $H_\odot$ be the special solutions in $M^{\min}_\Sigma$ defined in and after Theorem 3.4. Let $t \mapsto T^t$, $t \geq 0$, be the evolution semigroup of Σ. Then the following claims are true:

(i) $t \mapsto T^t$ is strongly $H_\circ$-stable if and only if the factorization problem

$$\varphi(z)^*\varphi(z) = 1_{\mathcal{U}} - \theta(z)^*\theta(z) \text{ a.e. on } i\mathbb{R}$$

has a solution $\varphi \in \mathcal{S}(\mathcal{U}, \mathcal{Y}_\varphi; \mathbb{C}^+)$ for some Hilbert space $\mathcal{Y}_\varphi$.

(ii) $t \mapsto T^t$ is strongly $H_\bullet$-$*$-stable if and only if the factorization problem

$$\psi(z)\psi(z)^* = 1_{\mathcal{Y}} - \theta(z)\theta(z)^* \text{ a.e. on } i\mathbb{R}$$

has a solution $\psi \in \mathcal{S}(\mathcal{U}_\psi, \mathcal{Y}; \mathbb{C}^+)$ for some Hilbert space $\mathcal{U}_\psi$.

(iii) $t \mapsto T^t$ is strongly $H_\odot$-bistable if and only if both the factorization problems in (i) and (ii) are solvable.

V. An Example

In this section we present an example where all the solutions H of the generalized KYP inequality (20) are unbounded and have an unbounded inverse. This example is a continuous time analogue of the discrete time examples given in [7, p. 267] and [2]. The same example will be used to illustrate the conclusion of Theorem 4.2.

The impulse response of a suitably normalized damped heat equation on $[0, \infty)$ with Neumann control and Dirichlet observation at the origin is given by $b(t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-2t}$, $t \geq 0$, with transfer function $\theta(z) = 1/\sqrt{z+2}$, $z \in \mathbb{C}^+$. This is a Schur function on $\mathbb{C}^+$, and it is possible to realize this function with the help of the damped heat equation. However, instead we choose another realization, namely an exponentially weighted version of one of the standard Hankel realizations. We begin by first replacing θ by the shifted function $\theta_0(z) := 1/\sqrt{z+3}$, $z \in \mathbb{C}^+$. The corresponding impulse response is $b_0(t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-3t}$, $t \geq 0$. We realize θ_0 by means of the standard time domain output normalized Hankel realization described in, e.g., [14, Example 2.6.5(ii)], and we denote this realization by $\Sigma_0 := (S_0; \mathcal{X}, \mathbb{C}, \mathbb{C})$. The state space of this realization is $\mathcal{X} = L^2(0, \infty)$ and the system operator $S_0 = \left[\begin{smallmatrix} [A\&B]_0 \\ [C\&D]_0 \end{smallmatrix}\right]$ is defined as follows. We take the main operator to be $(A_0 x)(\xi) = x'(\xi)$ for $x \in \mathcal{D}(A_0) := W^{2,1}(0, \infty)$. Then $\mathcal{X}^{-1} = W^{-1,2}(0, \infty)$, and $\hat{A}_0 x$ is the distribution derivative of $x \in L^2(0, \infty)$. We take the control operator to be $(B_0 c)(\xi) = b_0(\xi) c$ for $c \in \mathbb{C}$. We define $\mathcal{D}(S_0)$ to consist of those $\left[\begin{smallmatrix} x \\ c \end{smallmatrix}\right]$ for which $x \in L^2(0, \infty)$ is of the form $x(\xi) = x(0) + \int_0^\xi h(\nu)\, d\nu - c \int_0^\xi b_0(\nu)\, d\nu$ for some $h \in L^2(0, \infty)$, and define $[A\&B]_0 \left[\begin{smallmatrix} x \\ c \end{smallmatrix}\right] = h$ and $[C\&D]_0 \left[\begin{smallmatrix} x \\ c \end{smallmatrix}\right] = x(0)$. This realization is output normalized in the sense that the observability Gramian is the identity, and it is minimal because the range of the Hankel operator induced by b_0 is dense in $L^2(0, \infty)$ (see [7, Theorem 3-5, p. 254]). The evolution semigroup $t \mapsto T_0^t$ is the left-shift semigroup on $L^2(0, \infty)$, i.e., $(T_0^t x)(\xi) = x(t + \xi)$ for t, $\xi \geq 0$, and the spectrum of A_0 is the closed left half-plane $\{\Re z \leq 0\}$. From this realization we get a minimal realization $\Sigma := (S; \mathcal{X}, \mathbb{C}, \mathbb{C})$ of the original transfer function θ by taking $S = S_0 + \left[\begin{smallmatrix} 1_\mathcal{X} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Clearly the spectrum of the main operator $A := A_0 + 1_\mathcal{X}$ is the closed half-plane $\{\Re z \leq 1\}$, the evolution semigroup $t \mapsto T^t$, given by $(T^t x)(\xi) = e^t x(t + \xi)$ for $t, \xi \geq 0$, is unbounded, and the transfer function $\mathfrak{D}$ is the restriction of θ to the half-plane $\Re z > 1$.

Since θ is a Schur function, it follows from Theorem 3.3 that the generalized KYP inequality (20) has a solution H. Suppose that both H and H^{-1} are bounded. Then our original realization becomes passive if we replace the original norm by the norm induced by the storage function E_H. In particular, with respect to this norm the evolution semigroup is contractive. However, this is impossible since we known that the semigroup is unbounded with respect to the original norm, and the two norms are equivalent. This contradiction shows that H or H^{-1} is unbounded. In this particular case it follows from [14, Theorems 9.4.7 and 9.5.2] that H^{-1} is bounded, hence H itself must be unbounded.

From the above example we can get another one where both H and H^{-1} must be unbounded as follows. We take two independent copies of the transfer function θ considered above, i.e, we look at the matrix-valued transfer function $\left[\begin{smallmatrix} \theta(z) & 0 \\ 0 & \theta(z) \end{smallmatrix}\right]$. We realize this transfer function by taking two independent realizations of the two blocks, so that we realize one of them with the exponentially weighted output normalized shift realization described above, and the other block with the adjoint of this realization. This will force both H and H^{-1} to be unbounded for every solution H of the generalized KYP inequality (20) for the combined system.

The above example illustrates our earlier claim that it is possible that all the solutions of the generalized KYP inequality (20) are unbounded and have an unbounded inverse. However, taking a closer look at the situation we find that there is another even more severe problem. Through a careful choice of the original realization one can always assure that the identity is a solution of (20) (in particular, H and H^{-1} are bounded) whenever the function θ that we want to realize is a Schur function (one way to do this is to start with an arbitrary minimal realization, find an arbitrary solution of the generalized KYP inequality (20), replace the norm in the state space by the norm induced by the storage function E_H, and finally complete the space with respect to this norm). However, it will still be true in many cases that (20) also has other solutions H for which H or H^{-1} is unbounded. In particular, if we choose the original realization to be the passive balanced realization constructed in [14, Theorem 11.8.14] (which corresponds to the balanced solution $H_\odot$ of the generalized KYP inequality (20)), then it is more a rule than an exception that the maximal solution $H_\bullet$ in Theorem 3.4 is unbounded, and that the minimal solution $H_\circ$ in Theorem 3.4 has an unbounded inverse. The only case where this is *not* true is where the norms induced by the two storage functions $E_{H_\circ}$ and $E_{H_\bullet}$ are equivalent.[5] In the example discussed above the balanced realization can be identified with the standard realization based on the damped heat equation, and its spectrum is the half-line $(-\infty, -2]$. If we want to study this example by starting from the damped heat equation realization, then $H_\bullet$ is unbounded and $H_\circ$ has an unbounded inverse.

To illustrate Theorem 4.2 we observe that in the example studied above with $\theta(z) = 1/\sqrt{z+2}$ both factorization problems (i) and (ii) in that theorem coincide, and they are solvable. Consequently, the evolution semigroup $t \mapsto T^t$ is strongly $H_\circ$-stable, strongly $H_\bullet$-$*$-stable, and strongly $H_\odot$-bistable (and even exponentially $H_\odot$-stable in this case). Nevertheless, $t \mapsto T^t$ is *not* strongly $H_\circ$-$*$-stable or strongly $H_\bullet$-stable. This follows from the fact that θ does not have a meromorphic pseudo-continuation into the left half-plane (see [5] and [6] for details).

The proofs of all the results mentioned above are given in [6]. They are based on the corresponding results for the discrete time case proved in [2], some new results on the pseudo-similarity of continuous time system nodes obtained in [6], and the connection between discrete and continuous time-invariant systems via the Cayley transform, considered in [3] (this transform is described in detail in [14]).

[5]Necessary and sufficient condition on the transfer function for these two norms to be the same or equivalent can be derived from [4, Theorems 2 and 3].

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