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Uncertain Variables as a Tool for Design of Uncertain Control Systems

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***Abstract*—The paper is concerned with closed-loop control systems with unknown parameters which are assumed to be values of uncertain variables described by certainty distributions given by an expert. The non-parametric optimization problem and three versions of the parametric optimization problem based on the uncertain variables are described. Analogies with optimization problems based on random variables are indicated. Simple examples illustrate the presented approach.**

I. Introduction

There exists a great variety of formal descriptions of uncertainties and uncertain systems (see e.g. [1]–[6]). The idea of uncertain variables, introduced and developed in recent years, is specially oriented for analysis and decision problems in a class of uncertain systems described by classical models or relational knowledge representations with unknown parameters characterized by an expert [7]–[10]. The uncertain variable is defined by a set of values and a certainty distribution characterizing the expert's knowledge on approximate values of the variable. The uncertain variables are related to random and fuzzy variables but there are also essential differences explained in [10]. Examples of practical applications have been described in the literature cited above (an example of the application to a class of transportation systems is presented in [11]). This paper is concerned with the application of the uncertain variables for dynamical, closed-loop control systems. In the works [12]–[14] it has been shown how to use the uncertain variables to stability analysis and stabilization of uncertain control systems, based on a general approach to the stability of uncertain systems [15], [16]. The purpose of this paper is to present new results concerning the non-parametric and parametric optimization of the systems with uncertain parameters in the models of the control plants (Sections III and IV). In particular, in Section IV a new approach consisting in the formulation and solution of the optimization problem for the given certainty threshold is proposed. Section V presents simple examples illustrating the parametric problem and in Section VI analogies with a similar approach based on random variables are indicated. We shall start with a very short presentation of the uncertain variables in Section II. Details may be found in the books [9], [10].

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II. Uncertain Variables and Basic Decision Problem

In the definition of the uncertain variable $\bar{x}$ we consider two soft properties (i.e. such properties $\varphi(x)$ that for the fixed x the logic value $v[\varphi(x)] \in [0,1]$): "$\bar{x} \cong x$" which means "$\bar{x}$ is approximately equal to x" or "x is the approximate value of $\bar{x}$," and "$\bar{x} \,\tilde{\in}\, D_x$" which means "$\bar{x}$ approximately belongs to the set D_x" or "the approximate value of $\bar{x}$ belongs to D_x". The *uncertain variable* $\bar{x}$ is defined by a set of values X (real number vector space), the function $h(x) = v(\bar{x} \cong x)$ (i.e. the certainty index that $\bar{x} \cong x$, given by an expert) and the following definitions for $D_x, D_1, D_2 \subseteq X$:

$$v(\bar{x} \,\tilde{\in}\, D_x) = \begin{cases} \max\limits_{x \in D_x} h(x) & \text{for} \quad D_x \neq \varnothing \\ 0 & \text{for} \quad D_x = \varnothing, \ \text{(empty set)} \end{cases}$$

$$v(\bar{x} \,\tilde{\notin}\, D_x) = 1 - v(\bar{x} \,\tilde{\in}\, D_x),$$

$$v(\bar{x} \,\tilde{\in}\, D_1 \vee \bar{x} \,\tilde{\in}\, D_2) = \max\{v(\bar{x} \,\tilde{\in}\, D_1), v(\bar{x} \,\tilde{\in}\, D_2)\},$$

$$v(\bar{x} \,\tilde{\in}\, D_1 \wedge \bar{x} \,\tilde{\in}\, D_2) = \begin{cases} \min\{v(\bar{x} \,\tilde{\in}\, D_1), \ v(\bar{x} \,\tilde{\in}\, D_2)\} & \text{for} \quad D_1 \cap D_2 \neq \varnothing \\ 0 & \text{for} \quad D_1 \cap D_2 = \varnothing. \end{cases}$$

The function $h(x)$ is called a *certainty distribution*.

C-uncertain variable $\bar{x}$ is defined by the set of values X, the function $h(x) = v(\bar{x} \cong x)$ given by an expert, and the following definitions:

$$v_c(\bar{x} \,\tilde{\in}\, D_x) = \frac{1}{2}[v(\bar{x} \,\tilde{\in}\, D_x) + 1 - v(\bar{x} \,\tilde{\in}\, \overline{D}_x)] \tag{1}$$

where $\overline{D}_x = X - D_x$,

$$v_c(\bar{x} \,\tilde{\notin}\, D_x) = 1 - v_c(\bar{x} \,\tilde{\in}\, D_x),$$

$$v_c(\bar{x} \tilde{\in} D_1 \vee \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cup D_2),$$

$$v_c(\bar{x} \tilde{\in} D_1 \wedge \bar{x} \tilde{\in} D_2) = v_c(\bar{x} \tilde{\in} D_1 \cap D_2).$$

The application of *C*-uncertain variable means better using of the expert's knowledge, but may be more complicated. The mean value of $\bar{x}$ is defined as follows:

$$\mathrm{M}(\bar{x}) = \int_X x h(x)\,dx \cdot [\int_X h(x)\,dx]^{-1}. \qquad (2)$$

For C-uncertain variable the mean value $\mathrm{M}_c(\bar{x})$ has the same form with $h_c(x) = v_c(\bar{x} \cong x)$ instead of $h(x)$. Let us consider a static plant with the input vector $u \in U$ and the output vector $y \in Y$, described by a relation $R(u, y; x) \subset U \times Y$ where the vector of unknown parameters $x \in X$ is assumed to be a value of an uncertain variable described by the certainty distribution $h(x)$ given by an expert. If the relation R is not a function then the value u determines a set of possible outputs

$$D_y(u; x) = \{y \in Y : (u, y) \in R(u, y; x)\}$$

For the requirement $y \in D_y \subset Y$ given by a user, we can formulate the following **decision problem**: For the given $R(u, y; x)$, $h(x)$ and D_y one should find the decision u^* maximizing the certainty index that the set of possible outputs approximately belongs to D_y (i.e. belongs to D_y for an approximate value of $\bar{x}$). Then

$$u^* = \arg\max_{u \in U} v[D_y(u; \bar{x}) \tilde{\subseteq} D_y] = \arg\max_{u \in U} \max_{x \in D_x(u)} h(x) \qquad (3)$$

where $D_x(u) = \{x \in X : D_y(u; x) \subseteq D_y\}$.

III. Non-parametric Optimization of a Closed-loop System

Let us consider a dynamical control plant described by the equation

$$\dot{s}(t) = g[s(t), u(t); x]$$

where *s* is the state vector, *u* is the input vector and *x* is an unknown vector parameter which is assumed to be a value of an uncertain variable $\bar{x}$ described by the certainty distribution $h_x(x)$ given by an expert. Consequently, for the given *T* and the function φ, the performance index

$$Q = \int_0^T \varphi(s, u)\,dt$$

is a function of *x*. Assume that the state *s*(*t*) is put at the input of the controller. Then the *uncertain control algorithm* (or shortly speaking – the *uncertain controller*) has a form $u = \Psi(s, x)$ which may be obtained as a result of non-parametric optimization, i.e. Ψ is the optimal control algorithm obtained by the minimization of Q for the given model of the plant with the fixed *x* and for the given form of a performance index. The deterministic control algorithm $u_d = \Psi_d(s)$ may be obtained by so called *determinization* of the uncertain controller [10], consisting in replacing the uncertain variable $\bar{u} = \Psi(s, \bar{x})$ by its mean value. Then

$$u_d = \mathrm{M}(\bar{u}; s) \triangleq \Psi_d(s)$$

where the mean value $\mathrm{M}(\bar{u}; s)$ for $\bar{u}$ is determined according to (2) by using the certainty distribution for $\bar{u}$:

$$h_u(u; s) = v[\bar{x} \tilde{\in} D_x(u; s)] = \max_{x \in D_x(u; s)} h_x(x)$$

where $D_x(u; s) = \{x \in X :\ u = \Psi(s, x)\}$.

A similar approach may be applied in the discrete-time case with the plant described by the equation

$$s_{n+1} = g(s_n, u_n; x)$$

and the performance index

$$Q = \sum_{n=1}^{N} \varphi(s_n, u_{n-1}).$$

Example 1

Let us consider the time-optimal control of the plant with the transfer function $K_{\mathrm{P}}(p; x) = xp^{-2}$ (Fig. 1), subject to constraint $|u(t)| \le M$. It is well known that the optimal control algorithm $u = \Psi(s, x)$ is the following:

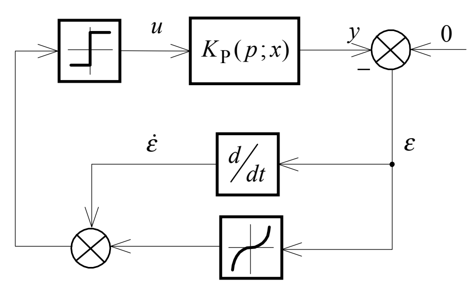

Fig. 1. Example of control system.

$$u(t) = M\,\mathrm{sgn}(\varepsilon + \frac{|\dot{\varepsilon}|\dot{\varepsilon}}{2\,x\,M})$$

where $s = [\varepsilon, \dot{\varepsilon}]$ and $\varepsilon = -y$ is the control error. For the given $h_x(x)$ we can determine $h_u(u;\varepsilon,\dot{\varepsilon})$, which is reduced to three values $v_1 = v(\bar{u} \cong M)$, $v_2 = v(\bar{u} \cong -M)$, $v_3 = v(\bar{u} \cong 0)$. Then

$$u_d(t) = \mathrm{M}(\bar{u}) = M(v_1 - v_2)(v_1 + v_2 + v_3)^{-1}.$$

It is easy to see that

$$v_1 = \max_{x \in D_{x1}} h_x(x), \qquad v_2 = \max_{x \in D_{x2}} h_x(x)$$

where

$$D_{x1} = \{x : x \operatorname{sgn} \varepsilon > -|\dot{\varepsilon}|\dot{\varepsilon}(2M|\varepsilon|)^{-1}\},$$

$$D_{x2} = \{x : x \operatorname{sgn} \varepsilon < -|\dot{\varepsilon}|\dot{\varepsilon}(2M|\varepsilon|)^{-1}\}$$

and

$$v_3 = h_x\left(\frac{-|\dot{\varepsilon}|\dot{\varepsilon}}{2M\varepsilon}\right).$$

Assume that $\bar{x}$ has a triangular certainty distribution presented in Fig. 2. For $\varepsilon > 0$, $\dot{\varepsilon} < 0$ and $x_g < a$ it is easy to obtain the following control algorithm

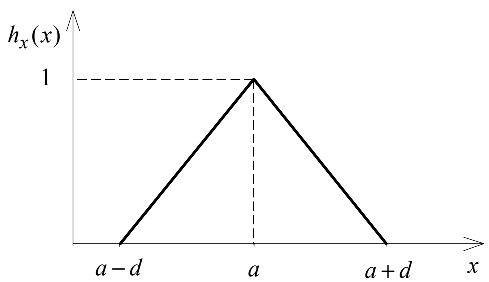

Fig. 2. Example of certainty distribution.

$$u_d = \mathrm{M}(\bar{u}) = \begin{cases} M & \text{for } d \le a - x_g \\ M\dfrac{a - x_g}{3d - 2(a - x_g)} & \text{for } d \ge a - x_g \end{cases}$$

where $x_g = (\dot{\varepsilon})^2(2M\varepsilon)^{-1}$. For example, for $M = 0.5$, $\dot{\varepsilon} = -3$, $\varepsilon = 1$, $a = 16$ and $d = 10$ we obtain $u_d = 0.2$.

IV. Parametric Optimization of a Closed-loop System

The control plant is now described by the equations

$$\dot{s}(t) = g[s(t), u(t); x], \qquad y(t) = \eta[s(t)]$$

or by the transfer function $K_P(p;x)$ in a linear case; x is an uncertain vector parameter described by $h_x(x)$. The controller with input y (or control error ε) is described by the analogous model with a vector of parameters b which is to be determined. The optimization problem consists then in the determination of the parameter b in the given assumed form of the control algorithm. Consequently, for the given T and φ, the performance index

$$Q = \int_0^T \varphi(y,u)\,dt \triangleq \boldsymbol{\Phi}(b,x)$$

is a function of b and x. In particular, for a one-dimensional plant

$$Q = \int_0^\infty \varepsilon^2(t)dt = \boldsymbol{\Phi}(b,x).$$

The closed-loop control system is then considered as a static plant with the input b, the output Q and an unknown parameter x, for which we can formulate and solve the decision problem described for static (memory less) uncertain plants in [10]. We can consider three versions of the control problem.

Version 1. For the given models of the plant and the controller find the value b^* minimizing $\mathrm{M}(\bar{Q})$, i.e. the mean value of the performance index. The procedure for solving the problem is then the following:

1. To determine the function $Q = \boldsymbol{\Phi}(b,x)$.
2. To determine the certainty distribution $h_q(q;b)$ for $\bar{Q}$ using the function $\boldsymbol{\Phi}$ and the distribution $h_x(x)$.
3. To determine the mean value $\mathrm{M}(\bar{Q};b)$.
4. To find b^* minimizing $\mathrm{M}(\bar{Q};b)$.

Version 2. By the minimization of $Q = \boldsymbol{\Phi}(b,x)$ with respect to b for the fixed x we obtain the value $b(x)$. The controller with the uncertain parameter $b(x)$ is an *uncertain controller* in our case. To obtain the deterministic control algorithm, one should substitute $\mathrm{M}(\bar{b})$ in place of $b(x)$ in the uncertain control algorithm, where the mean value $\mathrm{M}(\bar{b})$ should be determined by using the function $b(x)$ and the certainty distribution $h_x(x)$.

Version 3. This is a new approach based on the general formulation of the decision problem for the uncertain static plant, presented in Section II. For the plant $y = \boldsymbol{\Phi}(u,x)$ with one-dimensional nonnegative output y one can formulate the decision problem with the requirement determined by

$D_y = [0, \alpha]$, i.e. $y \le \alpha$ where $\alpha > 0$ is a given number. Consequently, the decision problem consists in finding the decision u^* maximizing the certainty index

$$v[\Phi(u, \bar{x}) \tilde{\le} \alpha],$$

i.e. the certainty index that the property $\Phi(u, \bar{x}) \le \alpha$ is "approximately satisfied" (is satisfied for an approximate value of $\bar{x}$). For the closed-loop dynamical control system under consideration $u = b$ and $y = Q$. According to (3)

$$v[\Phi(b, \bar{x}) \tilde{\le} \alpha] = v\{\Phi(b, \bar{x}) \tilde{\in} [0, \alpha]\} \triangleq v(b, \alpha) = \max_{x \in D_x(b)} h_x(x) \quad (4)$$

where

$$D_x(b) = \{x \in X : \Phi(b, x) \le \alpha\}. \quad (5)$$

In the case of *C*-uncertain variables, according to (1)

$$v_c[\Phi(b, \bar{x}) \tilde{\le} \alpha] = v_c\{\Phi(b, \bar{x}) \tilde{\in} [0, \alpha]\} \triangleq v_c(b, \alpha)$$

$$= \frac{1}{2}[\max_{x \in D_x(b)} h_x(x) + 1 - \max_{x \in X - D_x(b)} h_x(x)]. \quad (6)$$

The **parametric optimization problem for the given α** is formulated as follows: For the given function $\Phi(b, x)$, the number α and the certainty distribution $h_x(x)$ one should find the optimal parameter of the controller b maximizing the certainty index that the requirement concerning the performance index Q is satisfied for an approximate value of $\bar{x}$, i.e. one should find the value $b^*(\alpha)$ maximizing the certainty index (4) or the value $b_c^*(\alpha)$ maximizing the certainty index (6):

$$b^*(\alpha) = \arg\max_b v(b, \alpha), \quad (7)$$

$$b_c^*(\alpha) = \arg\max_b v_c(b, \alpha). \quad (8)$$

It is easy to note that for the given b, $v(b, \alpha)$ and $v_c(b, \alpha)$ are increasing (in general, non-decreasing) functions of α. Consequently, for the given desirable value $\bar{v}$ (or $\bar{v}_c$) which may be called a *certainty threshold*, it is possible to determine the strongest requirement, i.e. the minimum possible value of $\alpha \triangleq \bar{\alpha}$ (or $\alpha \triangleq \bar{\alpha}_c$), which should be determined by solving the equation

$$v[b^*(\alpha), \alpha] = \bar{v} \quad (9)$$

or

$$v_c[b_c^*(\alpha), \alpha] = \bar{v}_c. \quad (10)$$

Finally, a designer should determine the optimal value

$$\bar{b} = b^*(\bar{\alpha}) \quad \text{or} \quad \bar{b}_c = b_c^*(\bar{\alpha}_c). \quad (11)$$

Thus, we have the following procedure of solving the **parametric optimization problem for the given certainty threshold:**

1. The determination of v or v_c according to (4) or (6), respectively.
2. The maximization (7) or (8).
3. The solution of the equation (9) or (10).
4. The determination of the optimal value $\bar{b}$ or $\bar{b}_c$ according to (11).

It is worth noting that similar three versions of the parametric optimization may be presented for the discrete control system, with the plant described by the equations

$$s_{n+1} = g(s_n, u_n; x), \quad y_n = \eta(s_n)$$

and the performance index

$$Q = \sum_{n=1}^{N} \varphi(y_n, u_{n-1}) \triangleq \Phi(b, x).$$

Remark. The described method of the parametric optimization in **version 3** may be extended for the uncertain plant with two uncertain parameters x_1 and x_2 in two equations describing the continuous plant:

$$\dot{s}(t) = g[s(t), u(t); x_1], \quad y(t) = \eta[s(t); x_2],$$

and in the analogous equations for the discrete case. Then $Q = \Phi(b, x_1, x_2)$ and we use the joint certainty distribution $h(x_1, x_2)$ in the place of $h_x(x)$ [10]. If the uncertain variables $\bar{x}_1$ and $\bar{x}_2$ are independent, described by the certainty distributions $h_1(x_1)$ and $h_2(x_2)$, respectively, then $h(x_1, x_2) = \max\{h_1(x_1), h_2(x_2)\}$.

V. EXAMPLES

Example 2

To illustrate **version 2** presented in Section IV, let us consider the linear control system (Fig. 3) with the following data:

$$K_P(p; x) = \frac{x}{(pT_1 + 1)(pT_2 + 1)}, \quad K_C(p; b) = \frac{b}{p}$$

$z(t) = 0$ for $t < 0$, $z(t) = 1$ for $t \geq 0$, $h_x(x)$ has a form presented in Fig. 2. It is easy to determine

$$Q = \int_0^\infty \varepsilon^2(t)\,dt = \frac{x^2(T_1+T_2)}{2xb(T_1+T_2-xbT_1T_2)} = \Phi(b,x)\,. \quad (12)$$

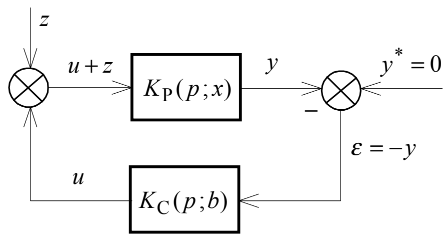

Fig. 3. Closed-loop control system.

The minimization of Q with respect to b gives

$$b(x) = \alpha x^{-1}, \qquad \alpha = (T_1+T_2)(T_1T_2)^{-1},$$

i.e. the uncertain controller is described by

$$K_C(p) = \frac{b(x)}{p} = \frac{\alpha}{xp}. \quad (13)$$

The certainty distribution $h_b(b)$ is as follows:

$$h_b(b) = \begin{cases} 0 & \text{for} \quad 0 < b \leq \dfrac{\alpha}{a+d} \\ \dfrac{ab-\alpha}{db}+1 & \text{for} \quad \dfrac{\alpha}{a+d} \leq b \leq \dfrac{\alpha}{a} \\ \dfrac{-ab+\alpha}{db}+1 & \text{for} \quad \dfrac{\alpha}{a} \leq b \leq \dfrac{\alpha}{a-d} \\ 0 & \text{for} \quad \dfrac{\alpha}{a-d} \leq b < \infty \quad . \end{cases}$$

From the definition of a mean value we obtain

$$\mathrm{M}(\bar{b}) = \frac{\alpha d(d^2+2a^2)}{2a^2 \ln \dfrac{a^2}{a^2-d^2}}\ . \quad (14)$$

Finally, the deterministic controller is described by

$$K_{C,d}(p) = \frac{\mathrm{M}(\bar{b})}{p}.$$

To apply the first version described in the previous section, it is necessary to find the certainty distribution for Q using formula (12) and the distribution $h_x(x)$, then to determine $\mathrm{M}(\bar{Q};b)$ and to find the value b^* minimizing $\mathrm{M}(\bar{Q};b)$. It may be shown that $b^* \neq \mathrm{M}(\bar{b})$ given by the formula (14).

Example 3

To illustrate **version 3** for a discrete case, let us consider very simple example of one-dimensional feed-back control system containing a plant with the input u_n and the output y_n, described by the equation $y_{n+1} - Ty_n = xu_n$, and a controller described by $u_n = b\varepsilon_n$. For $\varepsilon_n = -y_n$ the control system is described by the equation $\varepsilon_{n+1} = (T-xb)\varepsilon_n$. Then, for $|T-xb| < 1$ and $\varepsilon_0 = 1$

$$Q = \sum_{n=0}^{\infty} \varepsilon_n^2 = [1-(T-xb)^2]^{-1} = \Phi(b,x)$$

and the requirement $Q \leq \alpha$ is reduced to

$$(T - \sqrt{1-\alpha^{-1}})b^{-1} \leq x \leq (T + \sqrt{1-\alpha^{-1}})b^{-1}.$$

Assume that $h_x(x)$ has the form presented in Fig. 2, with $a = d = 0.5$. Using (6) we obtain

$$v_c(x,b) = \begin{cases} (T+\sqrt{1-\alpha^{-1}})\,b^{-1} & \text{for } b \geq 2T \\ 0 & \text{for } b \leq 1-(T-\sqrt{1-b^{-1}}) \\ 1-(T-\sqrt{1-\alpha^{-1}})\,b^{-1} & \text{otherwise}\ . \end{cases}$$

It is easy to see that $b_c^*(\alpha) = b_c^* = 2T$ (it does not depend on α) and $v_c(b^*,\alpha) = 0.5\,(1+T^{-1}\sqrt{1-\alpha^{-1}})$. From the equation $v_c(b^*,\alpha) = \bar{v}_c$ we obtain

$$\bar{\alpha}_c = [1-T^2(2\bar{v}_c-1)^2]^{-1}.$$

For the numerical data $T = 1$ and $\bar{v}_c = 0.9$, the results are as follows: $b^* = \bar{b} = 2$, $\bar{\alpha}_c = 2.8$.

VI. Analogies with Approach Based on Random Variables

It is interesting and useful to indicate analogies between the approach used the uncertain variables and a similar consideration based on random variables. Assume that the unknown parameter x is a value of a random variable $\tilde{x}$ described by a probability density $f_x(x)$ and consider the optimization problems analogous to those presented in Section IV.

Version 1. For the given models of the plant and the controller find the value b^* that minimizes $\mathrm{E}(\widetilde{Q};b)$, i.e. the expected value of the performance index. The procedure for solving the problem is as follows:

1. To determine the function $Q = \Phi(b,x)$.
2. To determine the expected value

$$\mathrm{E}(\widetilde{Q};b) = \int_X \Phi(b,x) f_x(x) dx .$$

3. To find b^* minimizing $\mathrm{E}(\widetilde{Q};b)$.

Version 2. One should determine

$$b(x) = \arg\min_b \Phi(b,x) .$$

The control algorithm with the random parameter $b(x)$ may be called a knowledge of the control in our case, and the controller with this parameter is a *random controller* in the closed-loop system. To obtain the deterministic control algorithm, one should substitute

$$\mathrm{E}(\widetilde{b}) = \mathrm{E}[b(\widetilde{x})] = \int_X b(x) f_x(x) dx$$

in place of $b(x)$ in the random algorithm.

Version 3. For the requirement $Q \le \alpha$ one should determine the probability

$$P[\Phi(b,\widetilde{x}) \le \alpha] = \int_{D_x(b)} f_x(x) dx \triangleq p(b,\alpha) \qquad (15)$$

where the set $D_x(b)$ is determined by (5). Let

$$b^*(\alpha) = \arg\max_b p(b,\alpha). \qquad (16)$$

For the given probability threshold $p = \overline{p}$, the value $\alpha = \overline{\alpha}$ is obtained by solving the equation

$$p[b^*(\alpha),\alpha] = \overline{p} . \qquad (17)$$

Finally, a designer should determine the optimal value $\overline{b} = b^*(\overline{\alpha})$ .The formulas (15), (16) and (17) are analogous to (4), (7) and (9), respectively. It is worth noting that in the case of the random description, the non-parametric optimization problem analogous to that described in Section III may be considered. Then $u = \Psi(s,x)$ is the *random control algorithm,* and as a result of the determinization the deterministic control algorithm may be obtained:

$$u_d = \mathrm{E}(\widetilde{u};s) = \int_X \Psi(s,x) f_x(x) dx \triangleq \Psi_d(s) .$$

VII. Conclusions

The uncertain variables have been proved to be a convenient tool for the design of a class of uncertain control systems with uncertain parameters described by an expert. The simulations have been shown that the versions presented in Section IV may give different results depending on the description of the plant and the parameters of the certainty distribution. It is worth noting that **version 3** has better practical interpretation connected with the certainty threshold. It has been shown that the approach presented in the paper is similar to the approach based on random variables. Thus, the concepts of the uncertain and random controllers together with the known idea of a fuzzy controller may be treated as parts of a uniform approach to uncertain control systems with different descriptions of the uncertainty.

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