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On the parametrization of all stabilizing controllers

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Abstract— **In the behavioral approach, the stabilization problem is to find, for a given plant behavior, a controller behavior such that the manifest controlled behavior is stable. In this paper we will establish, for a given plant behavior, a parametrization of all stabilizing controller behaviors.**

I. INTRODUCTION

An important issue in control is the problem of parametrizing, for a given plant, the space of all stabilizing controllers. In the context of feedback control of linear systems, this issue was treated in the seminal paper [8] by Youla, Jabr and Bongiorno, that initiated a complete new line of research. Since then, the well-known 'Youla parametrization' of all stabilizing controllers has become instrumental in feedback control of linear systems.

More recently, in the context of the behavioral approach to linear systems, in [2] the problem of *stabilization by interconnection* has drawn attention. In this context, stabilization is no longer restricted to feedback only, but can take place through general interconnection, with feedback as an important special case. Given is a plant behavior with two types of system variables, the variable w to be controlled and the variable c (the control variable) that we are allowed to put constraints on. A controller for our plant is an additional system behavior, called the controller behavior. Interconnecting the plant with the controller simply means requiring c to be an element of the controller behavior. The space of w trajectories that are possible after interconnecting the plant behavior and the controller behavior forms the so called manifest controlled behavior. The interconnection is called regular if no restrictions on c that were already present in the laws on the plant behavior, are repeated in the controller behavior. The stabilization problem is to find, for the given plant behavior, a controller behavior such that the interconnection is regular and the manifest controlled behavior is stable, in the sense that every w converges to zero as time tends to infinity.

For the special case of full interconnection (i.e. the case that the to be controlled variable w and the control variable c coincide) this problem was studied in [6]. For the general case, in [2] (see also [1]) necessary and sufficient conditions on the plant behavior were found for the existence of a stabilizing controller. These conditions were formulated completely *representation independent*, i.e. they were formulated in terms of intrinsic properties of the plant behavior, and not in terms of properties of the representation in terms of which the plant behavior is given.

In the present paper we will address the problem of finding, for a given plant behavior, a parametrization of *all* stabilizing controller behaviors. This problem was studied before in [3] for the special case of full interconnection. Here we will derive a parametrization for the general, partial interconnection case.

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II. LINEAR DIFFERENTIAL SYSTEMS

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^q$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$ (the space of all locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^q$) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a set of differential equations can be represented by a real polynomial matrix R with q columns, and then $\mathfrak{B} = \{w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. Here $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is understood to hold in the distributional sense. Any such dynamical system Σ is called a *linear differential system*. The set of all linear differential systems with q variables is denoted by $\mathfrak{L}^q$. Since the behavior $\mathfrak{B}$ of the system Σ is the central item, we will mostly speak about the system $\mathfrak{B} \in \mathfrak{L}^q$ (instead of $\Sigma \in \mathfrak{L}^q$).

In the behavioral approach a distinction is made between the behavior as the space of all solutions to a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a *representation* of the behavior. If a behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ then we call this a kernel representation of $\mathfrak{B}$, and we often write $\mathfrak{B} = \ker(R)$.

Whereas a linear differential systems is *defined* as the solution space $\mathfrak{B}$ of a differential equation of the form $R(\frac{d}{dt}) = 0$, such system can have other representations as well. One of these is the image representation. Let M be a real polynomial matrix with q rows and, say, l columns. If

$$\mathfrak{B} = \{w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid \text{ there exists } \ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = M(\tfrac{d}{dt})\ell\}$$

then we call $w = M(\frac{d}{dt})\ell$ an image representation of the system behavior $\mathfrak{B}$ and we often write $\mathfrak{B} = \mathrm{im}(M)$

Not all linear differential systems behaviors have an image representation. In fact, the linear differential system $\mathfrak{B}$ has an image representation if and only if it is controllable. If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is controllable if and only if the rank of the complex matrix $R(\lambda)$ is independent of λ for $\lambda \in \mathbb{C}$. If $\mathfrak{B}$ is a linear differential system, then we denote by $\mathfrak{B}_{\mathrm{cont}}$

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the largest controllable subbehavior of $\mathfrak{B}$. The system $\mathfrak{B}$ is stabilizable (see [7]) if and only if $R(\lambda)$ is independent of λ for $\lambda \in \mathbb{C}^+$

Suppose R has p rows. Then the kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least p rows. A given kernel representation, $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is minimal if and only if the polynomial matrix R has full row rank. The number of rows in any minimal kernel representation of $\mathfrak{B}$ is denoted by $\mathtt{p}(\mathfrak{B})$. This number is called the *output cardinality* of $\mathfrak{B}$. It corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$ (see [7]).

We now recall the concept of *minimal left annihilator*, see also [4]. If M is a polynomial matrix, then the polynomial matrix R is called a minimal left annihilator (MLA) of M if $\mathrm{im}(M) = \ker(R)$. The following useful fact is well known:

Lemma 1: **:** Let R and M be polynomial matrices. R is a MLA of M if and only if $RM = 0$, $R(\lambda)$ has rank independent of λ for $\lambda \in \mathbb{C}$ and $\mathrm{rank}(R) = \mathrm{rowdim}(M) - \mathrm{rank}(M)$.

In this paper, often we will need, for a given M, a MLA of R *with full row rank*. If the given M has full row rank, say q, then for consistency we *define* such full row rank MLA as the 'void' matrix R with 0 rows and q columns.

We will also need some facts on the representation of sums of behaviors. It is well known that the space of all linear differential systems $\mathfrak{L}^{\mathtt{q}}$ is closed under addition. Suppose that $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{q}}$, where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have kernel representations $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, respectively. The problem to find a kernel representation of $\mathfrak{B}_1 + \mathfrak{B}_2$ was solved in [4]:

Proposition 2: **:** Let (S_1, S_2) be a MLA of $\mathrm{col}(R_1, R_2)$. Then the polynomial matrix $S_1R_1 = -S_2R_2$ yields a kernel representation of $\mathfrak{B}_1 + \mathfrak{B}_2$

Next, we will review some facts on observability. Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{q}}$ with system variable $w = (w_1, w_2)$, where w_1 and w_2 take values in $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$, respectively, $q = q_1 + q_2$. Suppose w_2 has the interpretation of variable to be deduced from information on the variable w_1. We call w_2 *observable from* w_1 if $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $w_2 = w_2'$. We call w_2 *detectable from* w_1 if $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $\lim_{t\to\infty}(w_2(t) - w_2'(t)) = 0$. If $\mathfrak{B}$ is represented by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$, then w_2 is observable from w_1 if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Also, w_2 is detectable from w_1 if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$.

We now review some facts on elimination. Again, let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{q}}$ with system variable $w = (w_1, w_2)$. Let P_{w_1} denote the projection onto the w_1-component. Then the set $P_{w_1}\mathfrak{B}$ of all w_1 for which there exists w_2 such that $(w_1, w_2) \in \mathfrak{B}$ almost forms a linear differential system, in the sense that the closure $\overline{P_{w_1}\mathfrak{B}}$ in the topology of $\mathcal{L}_1^{\mathrm{loc}}$ is an element of $\mathfrak{L}^{q_1}$ (see [7]). In this paper we will denote $\overline{P_{w_1}\mathfrak{B}}$ by $\mathfrak{B}_{\mathtt{w}_1}$. We will call $\mathfrak{B}_{\mathtt{w}_1}$ the system obtained by eliminating w_2 from $\mathfrak{B}$.

Sometimes, system behaviors are represented by latent variable representations of the form $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, with *latent variable* ℓ. Of course, this equation represents the full behavior of all (w, ℓ) that satisfy the differential equation. The w-behavior $\mathfrak{B}$ obtained by eliminating ℓ from this full behavior is called the manifest behavior associated with this latent variable representation. On several occasions in this paper we will need to compute the output cardinality $\mathtt{p}(\mathfrak{B})$ of this behavior in terms of the polynomial matrices R and M. It was shown in [2] that $\mathtt{p}(\mathfrak{B}) = \mathrm{rank}(R, M) - \mathrm{rank}(M)$.

Finally, we will recall some facts on autonomous systems. If the behavior $\mathfrak{B}$ has the property that $\mathtt{p}(\mathfrak{B}) = q$, the number of variables (so all variables are output), then we call $\mathfrak{B}$ *autonomous*. An autonomous system is called *stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$.

III. ALL STABILIZING CONTROLLERS

In this section we will introduce the main problem that will be considered in this paper. Assume we have a linear differential plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$, with system variable (w, c), where w takes its values in $\mathbb{R}^q$ and c in $\mathbb{R}^k$. The components of w should be interpreted as the variables to be controlled, the components of c are those through which we can interconnect the plant to a controller and are called *the control variables*. Let $\mathcal{C} \in \mathfrak{L}^k$ be a controller behavior, with system variable c. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is defined as the system behavior $\mathcal{K}_{\text{full}}(\mathcal{C}) \in \mathfrak{L}^{q+k}$, defined as

$$\mathcal{K}_{\text{full}}(\mathcal{C}) = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\},$$

which is called *the full controlled behavior*. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is called regular if $\mathtt{p}(\mathcal{K}_{\text{full}}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C})$. In addition to $\mathcal{K}_{\text{full}}(\mathcal{C})$, we have the behavior $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w \in \mathfrak{L}^{\mathtt{q}}$ that is obtained by eliminating c from $\mathcal{K}_{\text{full}}(\mathcal{C})$. $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ will be called the *manifest controlled behavior*.

Given $\mathcal{P}_{\text{full}}$ as above, a controller $\mathcal{C}$ is said to stabilize $\mathcal{P}_{\text{full}}$ through c if the manifest controlled behavior $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ is stable and the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. The controller $\mathcal{C}$ is then called a stabilizing controller. It was shown in [2] that a stabilizing controller $\mathcal{C}$ exists if and only if $(\mathcal{P}_{\text{full}})_w$ is stabilizable and in $\mathcal{P}_{\text{full}}$ w is detectable from c.

Now, let $\mathcal{P}_{\text{full}}$ be represented minimally by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. The main problem that we will consider in this paper is to find a parametrization, in terms of the polynomial matrices R_1 and R_2, of *all* polynomial matrices C such that the controller $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ is a stabilizing controller.

Example 3: **:** Consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0 \\ w_2 + c_1 + c_2 &= 0 \\ \dot{c}_1 + c_1 + \dot{c}_2 + c_2 &= 0 \end{aligned}$$

A stabilizing controller is given by $\mathcal{C} = \{(c_1, c_2) \mid \dot{c}_2 + 2c_1 + c_2 = 0$. Indeed, by eliminating c from the full controlled behavior $\mathcal{K}_{\text{full}}(\mathcal{C})$ we find that $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, with

$$R(\xi) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\xi & \frac{1}{2}\xi^2 + \frac{1}{2}\xi - 1 \\ 0 & -\xi - 1 \end{pmatrix},$$

which is Hurwitz. Yet another class of stabilizing controllers is represented by $C(\xi) = (\xi(\xi+1) + k, \xi + 1 + k)$, $k \in \mathbb{R}$. We want to find a parametrization of *all* 1×2 polynomial matrices $C(\xi)$ such that $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ is a stabilizing controller.

We note that for the special case of full interconnection (see section IV), the problem of parametrizing all stabilizing controllers was considered before in [3]. Of course, in the context of feedback stabilization this parametrization problem dates back to the famous result of Youla ([8]), see also [5].

IV. All stabilizing controllers: the full interconnection case

In the previous section we reviewed stabilization by partial interconnection. The problem of stabilization by *full* interconnection is formulated as follows. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$ be a given plant behavior. Find a controller behavior $\mathcal{C}$ such that the controlled behavior $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ is autonomous and stable, and the interconnection is regular. It was proven in [6] that there exists such stabilizing controller $\mathcal{C}$ if and only if the plant behavior $\mathcal{P}$ is stabilizable. In this section we will solve the problem of parametrizing *all* stabilizing controller behaviors for $\mathcal{P}$. Our result in this section generalizes the main result from [3] that was obtained under the assumption that $\mathcal{P}$ can be represented by an image representation, equivalently, that $\mathcal{P}$ is controllable. Here, we only assume stabilizability.

Assume that $\mathcal{P}$ is represented by the minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Assume that $\mathcal{P}$ is stabilizable, equivalently that $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$. The following theorem yields a parametrization of all stabilizing controllers for the stabilizable plant $\mathcal{P}$:

Theorem 4: **:** Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$ be stabilizable. Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a minimal kernel representation of the controllable part $\mathcal{P}_{\text{cont}}$. Let C_0 be such that $\mathrm{col}(R_1, C_0)$ is unimodular. Then for any $\mathcal{C} \in \mathfrak{L}^{\mathtt{q}}$ represented by the kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ the following statements are equivalent:

1) $\mathcal{P} \cap \mathcal{C}$ is autonomous and stable, the interconnection is regular and the kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is minimal,
2) there exist a polynomial matrix F and a Hurwitz polynomial matrix D such that $C = FR_1 + DC_0$.

Proof : The proof is omitted here. ☐

If, in the above, we assume that $\mathcal{P}$ is controllable, then we can take $R = R_1$, and we recover the parametrization of all stabilizing controllers that was obtained in [3].

V. All stabilizing controllers: the observable case

In this section we return to the original problem of stabilization by partial interconnection. We will solve the problem of parametrizing, for a given plant $\mathcal{P}_{\text{full}}$, all stabilizing controllers. To start with, in this section we will assume that in the full plant behavior $\mathcal{P}_{\text{full}}$, c is observable from w. Next, in the sections to follow, we will lift the observability assumption and describe a parametrization for the general case.

For the observable case the following lemma is instrumental:

Lemma 5: **:** Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c). Assume that in $\mathcal{P}_{\text{full}}$, c is observable from w. Assume that $(\mathcal{P}_{\text{full}})_w$ is stabilizable and that in $\mathcal{P}_{\text{full}}$, w is detectable from c. Let $\mathcal{C} \in \mathfrak{L}^k$. Then the following two statements are equivalent:

1) $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c,
2) $\mathcal{C}$ stabilizes $(\mathcal{P}_{\text{full}})_c$ by full interconnection.

Proof : The proof is omitted. ☐

The following theorem then gives a parametrization of all stabilizing controllers for the observable case:

Theorem 6: **:** Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ satisfy the assumptions of lemma 5. Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Construct polynomial matrices V, S and C_0 as follows:

1) let V be a full row rank MLA of R_1,
2) factorize $VR_2 = TS$ with T square, nonsingular and $S(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$.
3) let C_0 be such that $\mathrm{col}(S, C_0)$ is unimodular

Then for any $\mathcal{C} \in \mathfrak{L}^k$ represented by the kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ the following statements are equivalent:

1) $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c and the kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ is minimal,
2) there exist a polynomial matrix F and a Hurwitz polynomial matrix D such that $C = FS + DC_0$.

Proof : This is an immediate corollary of theorem 4 and lemma 5. ☐

Thus we have obtained a parametrization of all stabilizing controllers for the observable case.

Example 7: **:** Consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + c_1 &= 0 \\ w_2 + c_2 &= 0 \\ \dot{c}_2 + c_2 &= 0 \end{aligned}$$

We will parametrize all controllers $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ that stabilize $\mathcal{P}_{\text{full}}$ through $c = (c_1, c_2)$. We have

$$R_1 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \xi + 1 \end{pmatrix},$$

It is easily seen that c is observable from w and w is detectable from c. Furthermore, $(\mathcal{P}_{\text{full}})_w$ is represented by

$\dot{w}_2 + w_2 = 0$, so is stabilizable. Performing the steps of theorem 6, we obtain $V = (0,0,1)$, $V\tilde{R}_2 = TS$ with $T(\xi) = \xi + 1$ and $S = (0,1)$ Choose $C_0 = (1,0)$. The required parametrization is then $C(\xi) = (d(\xi), f(\xi))$ with d an arbitrary Hurwitz polynomial, and f an arbitrary polynomial.

VI. All stabilizing controllers: the general case

Again, let $\mathcal{P}_{\rm full}$ be represented minimally by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. In order to arrive at a parametrization for the general case, we will show that the general case can be reduced to the observable case. This reduction requires two steps. First, we will reduce the general case to the case that R_2 has full column rank. Next we will reduce the latter to the case that $R_2(\lambda)$ has full column rank for all λ, i.e. the observable case.

1) *Reduction to the case that R_2 has full column rank.* Let V be a unimodular matrix such that
$$R_2 = \begin{pmatrix} \tilde{R}_2 & 0 \end{pmatrix} V,$$
with $\tilde{R}_2$ full column rank k'. Define the new system $\mathcal{P}'_{\rm full} \in \mathfrak{L}^{q+k'}$ as the system (with control variable c') represented by
$$R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + \tilde{R}_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c' = 0.$$
2) *Reduction to the observable case.* Assume now that in $\mathcal{P}_{\rm full}$ the matrix R_2 has full column rank. Let L be a square, nonsingular polynomial matrix such that $R_2 = \tilde{R}_2 L$, with $\tilde{R}_2(\lambda)$ full column rank for all $\lambda \in \mathbb{C}$. Define the new system $\mathcal{P}'_{\rm full}$ as the system (with control variable c') represented by
$$R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + \tilde{R}_2(\tfrac{\mathrm{d}}{\mathrm{d}t})c' = 0.$$
In the system $\mathcal{P}'_{\rm full}$, c' is observable from w.

It will turn out that every controller that stabilizes $\mathcal{P}'_{\rm full}$ leads to a *set* of controllers that stabilizes $\mathcal{P}_{\rm full}$. In the following two subsections we will treat the two reduction steps seperately.

A. Reduction to the case that R_2 has full column rank

The first step concerns the reduction of a general $\mathcal{P}_{\rm full}$ to a full plant behavior $\mathcal{P}'_{\rm full}$ with R_2-matrix full column rank as described in reduction step 1. above.

Theorem 8: **:** $(\mathcal{P}_{\rm full})_w$ is stabilizable if and only if $(\mathcal{P}'_{\rm full})_w$ is stabilizable, and in $\mathcal{P}_{\rm full}$, w is detectable from c if and only in $\mathcal{P}'_{\rm full}$, w is detectable from c'. Furthermore, if $\mathcal{C} \in \mathfrak{L}^k$ is represented by the minimal kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ then the following two statements are equivalent:

1) the controller $\mathcal{C}$ stabilizes $\mathcal{P}_{\rm full}$ through c,
2) there exist a polynomial matrix C_{11}, polynomial matrices C_{12} and C_{21} of full row rank, and a unimodular matrix U such that
$$C = U \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & 0 \end{pmatrix} V, \tag{1}$$
and such that the controller $\mathcal{C}_{21} \in \mathfrak{L}^{k'}$ represented by $C_{21}(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ stabilizes $\mathcal{P}'_{\rm full}$ through c'.

Proof : The first statement follows from the obvious fact that $\mathcal{P}_{\rm full}$ and $\mathcal{P}'_{\rm full}$ share the same hidden behavior $\mathcal{N}$ and the same manifest plant behavior $(\mathcal{P}'_{\rm full})_w = (\mathcal{P}_{\rm full})_w$.

(1. $\Rightarrow$ 2.) If $\mathcal{C}$ stabilizes $\mathcal{P}_{\rm full}$ then (omitting the $\frac{\mathrm{d}}{\mathrm{d}t}$'s) $(\mathcal{K}_{\rm full}(\mathcal{C}))_w$ with $\mathcal{K}_{\rm full}(\mathcal{C}) = \{(w,c) \mid R_1 w + R_2 c = 0 \text{ and } Cc = 0\}$ is stable and
$$\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix}$$
has full row rank. Partition $CV^{-1} = (C_1, C_2)$ with the number of columns of C_1 equal to $k' = \mathrm{rank}(R_2)$. Choose a unimodular matrix U such that $U^{-1}C_2 = \mathrm{col}(C_{12}, 0)$ with C_{12} full row rank. Partition $U^{-1}C_1 = \mathrm{col}(C_{11}, C_{21})$. Since (C_1, C_2) has full row rank, also C_{21} has full row rank. Moreover, (1) holds. Now we claim that the controller $\mathcal{C}_{21}$ represented by $C_{21}(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ stabilizes $\mathcal{P}'_{\rm full}$. Indeed, denote by
$$\mathcal{K}'_{\rm full}(\mathcal{C}_{21}) := \{(w, c_1) \mid R_1 w + \tilde{R}_2 c_1 = 0 \text{ and } C_{21} c_1 = 0\}.$$
the full controlled behavior of $\mathcal{P}'_{\rm full}$ using the controller $\mathcal{C}_{21}$. Using the fact that C_{12} has full row rank it can be shown that
$$\mathcal{K}'_{\rm full}(\mathcal{C}_{21}) = \{w \mid \text{ there exists } c \text{ s.t. } R_1 w + R_2 c = 0,\ Cc = 0\ \}.$$
Thus we obtain $(\mathcal{K}'_{\rm full}(\mathcal{C}_{21}))_w = (\mathcal{K}_{\rm full}(\mathcal{C}))_w = \mathcal{K}$. Finally, again since C_{12} has full row rank,
$$\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix} \tag{2}$$
has full row rank if and only if
$$\begin{pmatrix} R_1 & \tilde{R}_2 \\ 0 & C_{21} \end{pmatrix} \tag{3}$$
has full row rank. Hence the interconnection of $\mathcal{C}_{12}$ and $\mathcal{P}'_{\rm full}$ is regular if and only if the interconnection of $\mathcal{C}$ and $\mathcal{P}_{\rm full}$ is regular.

(2. $\Rightarrow$ 1.) Conversely, if 1 holds then by reversing the above argument we see that if the controller $C_{21}c' = 0$ stabilizes $\mathcal{P}'_{\rm full}$, then $Cc = 0$ stabilizes $\mathcal{P}_{\rm full}$. Again, (3) has full row rank if and only if (2) has full row rank. ☐

B. Reduction to the observable case

In the previous subsection it was shown that our parametrization problem can be reduced to a problem for a plant behavior with R_2-matrix full column rank. In the present subsection we will reduce the full column rank case to the observable case. Let $\mathcal{P}_{\rm full}$ be represented by the minimal kernel representation $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$, with R_2 full column rank. Let L be square, nonsingular such that $R_2 = \tilde{R}_2 L$, with $\tilde{R}_2(\lambda)$ full column rank for all λ. Let $\mathcal{P}'_{\rm full}$ be the (observable) system represented by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + \tilde{R}_2(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$.

Lemma 9: **:** $(\mathcal{P}_{\text{full}})_w$ is stabilizable if and only if $(\mathcal{P}'_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$, w is detectable from c if and only in $\mathcal{P}'_{\text{full}}$, w is detectable from c'. Furthermore, if $\mathcal{C} \in \mathfrak{L}^k$ is represented by the minimal kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ then the following two statements are equivalent:

1) the controller $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c
2) the controller $\mathcal{C}'$ represented in latent variable representation (with latent variable ℓ) by

$$\begin{pmatrix} I \\ 0 \end{pmatrix} c' = \begin{pmatrix} L(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \ell \tag{4}$$

stabilizes $\mathcal{P}'_{\text{full}}$ through c'.

Proof : Again, the hidden behavior and manifest plant behavior of $\mathcal{P}_{\text{full}}$ and $\mathcal{P}'_{\text{full}}$ coincide. This proves the first statement.

Since L is nonsingular we clearly have

$$\{w \mid \text{ there exists } c \text{ s.t. } R_1 w + R_2 c = 0,\ Cc = 0\} =$$
$$\{w \mid \text{ there exists } c \text{ s.t. } R_1 w + \tilde{R}_2 Lc = 0,\ Cc = 0\} =$$
$$\{w \mid \text{ there exists } c, c' \text{ s.t. } R_1 w + \tilde{R}_2 c' = 0,\ c' = Lc,\ Cc = 0\}.$$

Thus we obtain $(\mathcal{K}'_{\text{full}}(\mathcal{C}'))_w = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$.

Next, we will prove that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular if and only if the interconnection of $\mathcal{P}'_{\text{full}}$ and $\mathcal{C}'$ is regular. Note that $\mathcal{K}'_{\text{full}}(\mathcal{C}')$ has latent variable representation

$$\begin{pmatrix} R_1 & \tilde{R}_2 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ c' \end{pmatrix} = \begin{pmatrix} 0 \\ L \\ C \end{pmatrix} \ell.$$

Hence the output cardinality of $\mathcal{K}'_{\text{full}}(\mathcal{C}')$ equals

$$\mathtt{p}(\mathcal{K}'_{\text{full}}(\mathcal{C}')) = \text{rank} \begin{pmatrix} R_1 & \tilde{R}_2 & 0 \\ 0 & I & L \\ 0 & 0 & C \end{pmatrix} - \text{rank} \begin{pmatrix} 0 \\ L \\ C \end{pmatrix}.$$

Using elementary row and column operations and the fact that L is nonsingular, this can be shown to be equal to

$$\text{rank} \begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix} = \mathtt{p}(\mathcal{K}_{\text{full}}(\mathcal{C})).$$

Also,

$$\mathtt{p}(\mathcal{C}') = \text{rank} \begin{pmatrix} I & L \\ 0 & C \end{pmatrix} - \text{rank} \begin{pmatrix} L \\ C \end{pmatrix} = \text{rank}(C) = \mathtt{p}(\mathcal{C}).$$

Finally, $p(\mathcal{P}_{\text{full}}) = \text{rank}(R_1, R_2) = \text{rank}(R_1, \tilde{R}_2) = \mathtt{p}(\mathcal{P}'_{\text{full}})$. This proves our claim. We conclude that $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c if and only if $\mathcal{C}'$ stabilizes $\mathcal{P}'_{\text{full}}$ through c'. $\square$

According to this theorem, a controller represented by $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ works for $\mathcal{P}_{\text{full}}$ if and only if the controller $c' = L(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, $C(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$ (with control variable c') works for the observable system $\mathcal{P}'_{\text{full}}$. What we are looking for here is a parametrization of *all* such polynomial matrices C. Now, we do already have a parametrization of all controllers $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ that work for $\mathcal{P}'_{\text{full}}$. Indeed, this parametrization was established in theorem 6. Hence the question arises under what conditions the latent variable representation $c' = L(\frac{\mathrm{d}}{\mathrm{d}t})\ell, C(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$ and the kernel representation $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ represent the same behavior $\mathcal{C}'$. The answer to this is given in the following lemma:

Lemma 10: **:** Let L be a $k \times k$, square, nonsingular polynomial matrix. Let C and C' be polynomial matrices with k columns. Then the latent variable representation $c' = L(\frac{\mathrm{d}}{\mathrm{d}t})\ell, C(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$ and the kernel representation $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ represent the same manifest behavior if and only if $\ker(C'L) = \ker(L) + \ker(C)$.

Proof : $(\Rightarrow)$ Let $C'L\ell = 0$. Then $c' := L\ell \in \ker(C')$, so there exists ℓ' such that $c' = L(\frac{\mathrm{d}}{\mathrm{d}t})\ell', C(\frac{\mathrm{d}}{\mathrm{d}t})\ell' = 0$. Define $\ell'' := \ell - \ell'$ Then $\ell = \ell' + \ell'' \in \ker(C) + \ker(L)$. Conversely, let $C\ell = 0$. Define $c' = L\ell$. Then $C'c' = 0$ so $C'L\ell = 0$. $(\Leftarrow)$ Assume $C'c' = 0$. Let ℓ be such that $c' = L\ell$. Then $C'L\ell = 0$ so there exists $\ell_1 \in \ker(L)$ and $\ell_2 \in \ker(C)$ such that $\ell = \ell_1 + \ell_2$. This implies that $c' = L\ell_2$, while $C\ell_2 = 0$. Conversely, assume $c' = L\ell$ with $C\ell = 0$. Then cleary $C'L\ell = 0$ so $C'c' = 0$. $\square$

Corollary 11: **:** The controller represented by $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ stabilizes $\mathcal{P}_{\text{full}}$ through c if and only if there exists a polynomial matrix C' such that $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ stabilizes $\mathcal{P}'_{\text{full}}$ through c', and C satisfies $\ker(C) + \ker(L) = \ker(C'L)$.

Since we already have a parametrization of all polynomial matrices C' such that the controller $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ stabilizes $\mathcal{P}'_{\text{full}}$, a parametrization of all controllers that stabilize $\mathcal{P}_{\text{full}}$ can be obtained by parametrizing *for fixed* C' all polynomial matrices C such that $\ker(C) + \ker(L) = \ker(C'L)$. Such parametrization can be derived from the following theorem:

Theorem 12: **:** Let L be a $k \times k$, square, nonsingular polynomial matrix. Let C and C' be a full row rank polynomial matrices with k columns. Then $\ker(C'L) = \ker(L) + \ker(C)$ if and only if there exists a square, nonsingular polynomial matrix X such that $C'L = XC$ and

$$\begin{pmatrix} -X(\lambda) & C'(\lambda) \end{pmatrix}$$

has full row rank for all $\lambda \in \mathbb{C}$.

Proof : $(\Rightarrow)$ Let $(-X, Y)$ be a full row rank MLA of $\text{col}(C, L)$. Then by proposition 2, $\ker(XC) = \ker(YL) = \ker(C'L)$. We claim that XC has full row rank. Indeed, if p is a polynomial row vector such that $pXC = 0$ then also $pX = 0$. Since $XC = YL$ and L is nonsingular, also $pY = 0$. Since $(-X, Y)$ has full row rank this yields $p = 0$. Thus XC and $C'L$ yield minimal representations of the same behavior so there exists a unimodular U such that $C'L = UXC$. This implies $YL = U^{-1}C'L$ so, by the nonsingularity of L, $Y = U^{-1}C'$. Define $\tilde{X} = UX$. Then $(-\tilde{X}, C')$ is a full row rank MLA of $\text{col}(C, L)$. This implies $C'L = \tilde{X}C$. Also, $(-\tilde{X}(\lambda), C'(\lambda))$ has full row rank for all λ. Finally, $\tilde{X}$ is square and nonsingular. Clearly, X has full row rank. Also,

$$\text{rowdim}(X) = \text{rowdim}(C) + \text{rowdim}(L) - \text{rank}(C, L).$$

Since L is nonsingular, $\text{rank}(C, L) = \text{rank}(L)$, so $\text{rowdim}(X) = \text{rowdim}(C)$. Of course, also $\text{coldim}(X) = \text{rowdim}(C)$, so X is square and nonsingular. The same then holds for $\tilde{X}$.

(⇐) We have $XC - C'L = 0$ and $(-X(\lambda), C'(\lambda))$ has full row rank for all λ. It is easily verified that $\mathrm{rank}(X, C') = \mathrm{coldim}(X, C') - \mathrm{rank}(C, L)$, so (X, C') is a full row rank MLA of $\mathrm{col}(C, L)$. By proposition 2, this implies that $\ker(C) + \ker(L) = \ker(C'L)$ as desired. □

Corollary 13: **:** Let $\mathcal{C} \in \mathfrak{L}^k$ be represented by the minimal kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. Then the following two statements are equivalent:

1) the controller $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c,
2) there exists a square, nonsingular polynomial matrix X and a full row rank polynomial matrix C' such that
$$C = X^{-1}C'L,$$
where $(X(\lambda), C'(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ and the controller $\mathcal{C}'$ represented by $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ stabilizes $\mathcal{P}'_{\text{full}}$ through c'.

Thus, any stabilizing controller $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ for $\mathcal{P}_{\text{full}}$ can be written as $C = X^{-1}C'L$ for some nonsingular polynomial matrix X with the property that $(X(\lambda), C'(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$, and where C' represents a stabilizing controller for $\mathcal{P}'_{\text{full}}$. Conversely, for any such C', any nonsingular X such that $(X(\lambda), C'(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ and such that $X^{-1}C'L$ is a polynomial matrix yields a stabilizng controller $C = X^{-1}C'L$.

Example 14: **:** We reexamine example 2. Again, consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0 \\ w_2 + c_1 + c_2 &= 0 \\ \dot{c}_1 + c_1 + \dot{c}_2 + c_2 &= 0 \end{aligned}$$

We will parametrize all controllers $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ that stabilize $\mathcal{P}_{\text{full}}$ through c. We have

$$R_1 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} \xi & 1 \\ 1 & 1 \\ \xi+1 & \xi+1 \end{pmatrix},$$
$$\tilde{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \xi+1 \end{pmatrix}, L = \begin{pmatrix} \xi & 1 \\ 1 & 1 \end{pmatrix}.$$

In $\mathcal{P}'_{\text{full}}$, represented by $R_1 w + \tilde{R}_2 c', = 0$, c' is observable from w. We first parametrize all controllers $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ that stabilize $\mathcal{P}'_{\text{full}}$. Performing the steps of theorem 6, we obtain $V_2 = (0, 0, 1)$, $V_2\tilde{R}_2 = TS$ with $T(\xi) = \xi + 1$ and $S = (0, 1)$ Choose $C_0 = (1, 0)$. The required parametrization is then $C'(\xi) = (d(\xi), f(\xi))$ with d an arbitrary Hurwitz polynomial, and f an arbitrary polynomial. We compute $C'(\xi)L(\xi) = (\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$. A parametrization for the original plant $\mathcal{P}_{\text{full}}$ is obtained by computing, for any choice of d and f, all nonzero common factors $x(\xi)$ of the polynomials $\xi d(\xi) + f(\xi)$ and $d(\xi) + f(\xi)$ with the property that $(x(\lambda), d(\lambda), f(\lambda)) \neq 0$ for all λ. Let d and f be given, d Hurwitz. We distinguish the following cases:

1) $x(\xi) = c$, constant, unequal to zero. These $x(\xi)$'s satisfy the requirements
2) $x(\xi)$ has at least one zero $\lambda \neq 1$. Then $\lambda d(\lambda) + f(\lambda) = 0$ and $d(\lambda) + f(\lambda) = 0$. If $d(\lambda) = 0$ then also $f(\lambda) = 0$, and this leads to $(x(\lambda), d(\lambda), f(\lambda)) = 0$, violating the rank condition. If $d(\lambda) \neq 0$ then a simple calculation shows that $\lambda = 1$, which contradicts the assumption that $\lambda \neq 1$. Thus this case does not yield required $x(\xi)$'s.
3) $x(\xi)$ has only $\lambda = 1$ as zero, in other words, $x(\xi) = c(\xi - 1)^k$ for some $c \neq 0$ and integer $k \geq 1$. In this case we distinguish further between the following cases:
   a) $k = 1$. We have $d(1) + f(1) = 0$. Since d is Hurwitz, $d(1) \neq 0$, so we have $(x(1), d(1), f(1)) \neq 0$, and the rank condition holds. We conclude that $x(\xi) = c(\xi - 1)$, with $c \neq 0$, satisfies the requirements.
   b) $k > 1$. In this case $\lambda = 1$ is also a common zero of the derivative polynomials $d(\xi) + \xi d'(\xi) + f'(\xi)$ and $d'(\xi) + f'(\xi)$. This implies $d(1) = 0$, which contradicts the fact that d is Hurwitz. We conclude that $x(\xi) = c(\xi - 1)^k$ for $k > 2$ does not satisfy the requirements.

Our conclusion is that a parametrization of all stabilizing controllers for $\mathcal{P}_{\text{full}}$ is given by: $C(\xi) = (\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$, d Hurwitz polynomial, f arbitrary polynomial, or $C(\xi) = \frac{1}{\xi - 1}(\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$, d Hurwitz polynomial and f polynomial such that $d(1) + f(1) = 0$.

REFERENCES

[1] M.N. Belur, *Control in a Behavioral Context*, Doctoral Dissertation, University of Groningen, The Netherlands, 2003.

[2] M.N. Belur and H.L. Trentelman, 'Stabilization, pole placement and regular implementability', *IEEE Transactions on Automatic Control*, Vol. 47, nr. 5, pp. 735 - 744, 2002.

[3] M. Kuijper, 'Why do stabilizing controllers stabilize', *Automatica*, Vol. 31, pp. 621 - 625, 1995.

[4] P. Rocha and J. Wood, 'Trajectory control and interconnection of 1D and nD systems', *SIAM Journal of Control and Optimization*, Vol 40, pp. 107 - 134, 2001.

[5] M. Vidyasagar, *Control System Synthesis, A Factorization Approach*, The MIT Press, Cambridge, Massachusetts, 1985.

[6] J.C. Willems, 'On interconnections, control and feedback', *IEEE Transactions on Automatic Control*, Vol. 42, pp. 326 -339, 1997.

[7] J.W. Polderman & J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer-Verlag, Berlin, 1997.

[8] D.C. Youla, H.A. Jabr & J.J. Bongiorno, 'Modern Winer-Hopf design of optimal controllers, Part 2: The multivariable case', *IEEE Trans. Aut. Control*, Vol. 21, pp. 319 - 338, 1976.