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Semi-global minimal time hybrid robust stabilization of analytic driftless control-affine systems

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Abstract— **We investigate the problem of semi-global minimal time robust stabilization of analytic driftless control-affine systems, by means of a hybrid state feedback law. Our main result is that, in the absence of singular minimal time solutions, the solutions of the closed-loop system converge to the origin in quasi-minimal time (for a given bound on the controller) with a robustness property with respect to small measurement noise and external disturbances.**

I. INTRODUCTION

Let m and n be two positive integers. Consider on $\mathbb{R}^n$ the driftless control-affine system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{1}$$

where $f_1, \ldots, f_m$ are analytic vector fields on $\mathbb{R}^n$, and where the control function $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ satisfies the constraint

$$\sum_{i=1}^{m} u_i(t)^2 \leq 1. \tag{2}$$

The system (1), together with the constraint (2), is said to be *globally asymptotically stabilizable*, if, for each point $x \in \mathbb{R}^n$, there exists a control law satisfying the constraint (2) such that the solution of (1) associated to this control law and starting from x tends to 0 as t tends to $+\infty$, and satisfies a stability property (see [36] e.g. for a precise statement).

This asymptotic stabilization problem has a long history and has been widely investigated. Note that, due to *Brockett's condition* [10, Theorem 1, (iii)], if $m < n$ then there does not exist any continuous stabilizing feedback law for (1). However several control laws have been derived for such control systems (see for instance [24], [21], [5] and references therein).

The *robust asymptotic stabilization problem* is under current and active research. Many notions of controllers have been introduced to treat this problem, such as discontinuous sampling feedbacks [13], [35], time varying control laws [15], [14], [25], [26], patchy feedbacks (as in [4]), SRS feedbacks [34], ..., enjoying different robustness properties depending on the errors under consideration.

We consider here feedback laws having both discrete and continuous components, which generate closed-loop systems with *hybrid* terms (see for instance [39]). Such feedbacks appeared first in [28] to stabilize nonlinear systems having a priori no discrete state. They consist in defining a *switching strategy* between several smooth control laws defined on a partition of the state space.

It can be now seen as a paradigm for the robust stabilization of nonlinear control systems for which we are not able (because of the topological obstructions), or we do not know how to compute a continuous or a discontinuous stabilizing feedback. This strategy appears in [28] and has been used, among others, in [30], [29], [18], [33]. It requires a precise statement of the notion of a solution, and of the notion of asymptotically stable systems, as studied *e.g.* in [29], [18].

Here we use this strategy together with an optimal objective, focusing on the robust stabilization property and on the optimality of the speed of convergence. It requires a precise study of the minimal time control problem of (1) under the constraint (2). We use a hybrid strategy to unit different control laws. This allows to switch between the components of the hybrid feedback law by guaranteeing a robustness property with respect to measurement noises and external disturbances.

More precisely, *in a first step*, we consider the minimal time problem for the system (1) with the constraint (2), of steering a point $x_0 \in \mathbb{R}^n$ to the origin. Note that this problem is solvable as soon as *Hörmander's condition* is satisfied on the m-tuple of vector fields $(f_1, \ldots, f_m)$. Of course, in general it is impossible to compute explicitly the optimal time feedback controllers for this problem. Moreover, Brockett's condition implies that such control laws are not smooth whenever $m < n$ and the vector fields $f_1, \ldots, f_m$ are independent. This raises the problem of the regularity of optimal feedback laws. The literature on this subject is immense. In an analytic setting, the problem of determining the analytic regularity of the value function for a given optimal control problem, has been, among others, investigated by [37]. For systems of the form (1), the minimal time problem under the constraint (2) is equivalent to the *sub-Riemannian problem* associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$. In this framework, the minimal time function to x_0 is equal to the sub-Riemannian distance to x_0. The analytic regularity of the sub-Riemannian distance is related to the existence of singular minimizing solutions, see [1], [2], [40]. More precisely, if there does not exist any nontrivial singular minimizing solution starting from 0, then the sub-Riemannian distance to 0 is subanalytic outside 0 (see [19], [20] for a general definition of subanalytic sets). In particular, this function is analytic outside a stratified submanifold $\mathcal{S}$ of $\mathbb{R}^n$, of codimension greater than or equal

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to 1, see [38]. As a consequence, outside this submanifold it is possible to provide an analytic optimal time feedback controller for the system (1) with the constraint (2).

Note that the analytic context is used so as to ensure stratification properties, which do not hold a priori if the system is smooth only. These properties are related to the notion of *o-minimal category* (see [16]).

Then, *in a second step*, by assuming the *Hörmander's condition*, we have to define a suitable feedback law such that all solutions go out of a given neighborhood of $\mathcal{S}$ within a small fixed time.

Finally, in order to achieve a minimal time robust stabilization procedure, using a hybrid feedback law (more precisely, a hysteresis), we unit the three feedback law-components. This part of our work is not new and analogous to [33], however we can not apply the robustness result of [33] directly, since, here, we are interested in a quasi-optimal property.

We thus give an alternative solution to a conjecture of [9, Conj. 1, p. 101], in which the existence of *patchy feedbacks* is conjectured (this is however a different matter).

In a previous paper [31], this program was achieved *explicitly* on the so-called *Brockett system*, for which $n = 3$, $m = 2$, and, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3},\ f_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}.$$

In this case, there does not exist any nontrivial singular solution, and the manifold $\mathcal{S}$ is the axis $(0x_3)$.

II. Definitions and Main Result

A. The minimal time problem

Consider the minimal time problem for the system (1) with the constraint (2).

Definition 2.1: We say that *Hörmander's condition* holds if the Lie algebra spanned by the vector fields $f_1, \dots, f_m$, is equal to $\mathbb{R}^n$ at every point x of $\mathbb{R}^n$.

It is well-known that under this condition, any two points of $\mathbb{R}^n$ can be joined by a minimal time solution of (1), (2).

We denote by $T(x_0)$ the minimal time needed to steer the system (1) with the constraint (2) from a point $x_0 \in \mathbb{R}^n$ to the origin.

Remark 2.1: Obviously, the control function $u : \mathbb{R}^n \to \mathbb{R}^m$ associated to a minimal time solution of (1), (2), actually satisfies $\sum_{i=1}^m u_i^2 = 1$.

For $T > 0$, let $\mathcal{U}_T$ denote the (open) subset of $u(.)$ in $L^\infty([0, T], \mathbb{R}^m)$ such that the solution of (1), starting from 0 and associated to a control $u(\cdot) \in \mathcal{U}_T$, is well defined on $[0, T]$. The mapping

$$\begin{array}{rccc} E_T: & \mathcal{U}_T & \longrightarrow & \mathbb{R}^n \\ & u(\cdot) & \longmapsto & x(T), \end{array}$$

which to a control $u(\cdot)$ associates the end-point $x(T)$ of the corresponding solution $x(\cdot)$ of (1) starting at 0, is called *end-point mapping* at the origin, in time T; it is a smooth mapping.

Definition 2.2: A solution $x(\cdot)$ of system (1), so that $x(0) = 0$, is said to be *singular* on $[0, T]$ if its associated control $u(\cdot)$ is a singular point of the end-point mapping E_T (*i.e.* if the Fréchet derivative of E_T at $u(\cdot)$ is not onto).

Remark 2.2: If $x(\cdot)$ is singular on $[0, T]$, then it is singular on $[0, t]$, for every $t \in (0, T)$.

B. Class of controllers and notion of hybrid solution

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be defined by $f(x, u) = \sum_{i=1}^m u_i f_i(x)$. The system (1) writes

$$\dot{x}(t) = f(x(t), u(t)). \tag{3}$$

The controllers under consideration depend on the continuous state $x \in \mathbb{R}^n$ and also on a discrete variable $s_d \in \mathcal{N}$, where $\mathcal{N}$ is a countable subset of $\mathbb{N}$. According to the concept of a hybrid system of [17], we introduce the following definition.

Definition 2.3: A hybrid feedback is a 4-tuple (C, D, k, k_d), where

- C and D are subsets of $\mathbb{R}^n \times \mathcal{N}$;
- $k : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ is a function;
- $k_d : \mathbb{R}^n \times \mathcal{N} \to \mathcal{N}$ is a function.

The sets C and D are respectively called the *controlled continuous evolution set* and the *controlled discrete evolution set*.

We next recall the notion of robustness to small noises (see [36]). Consider two functions e and d satisfying the following *regularity assumptions*:

$$\begin{aligned} &e(\cdot,\cdot), d(\cdot,\cdot) \in L^\infty_{loc}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n), \\ &e(\cdot, t), d(\cdot, t) \in C^0(\mathbb{R}^n, \mathbb{R}^n),\ \forall t \in [0, +\infty). \end{aligned} \tag{4}$$

We introduce these functions as a measurement noise e and an external disturbance d. Below, we define the perturbed hybrid system $\mathcal{H}_{(e,d)}$. The notion of solution of such hybrid perturbed systems has been well studied in the literature (see *e.g.* [7], [8], [23], [39], [29], [30]). Here, we consider the notion of solution given in [17], [18].

Definition 2.4: Let $S = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$, where $J \in \mathbb{N} \cup \{+\infty\}$ and $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$. The domain S is said to be a a hybrid time domain. A map $(x, s_d) : S \to \mathbb{R}^n \times \mathcal{N}$ is said to be a solution of $\mathcal{H}_{(e,d)}$ with the initial condition (x_0, s_0) if

- the map x is continuous on S;
- for every j, $0 \le j \le J-1$, the map $x : t \in (t_j, t_{j+1}) \mapsto x(t, j)$ is absolutely continuous;
- for every j, $0 \le j \le J - 1$ and almost every $t \ge 0$, $(t, j) \in S$, we have

$$(x(t, j) + e(x(t, j), t), s_d(t, j)) \in C, \tag{5}$$

and

$$\begin{aligned} \dot{x}(t, j) &= f(x(t), k(x(t, j) + e(x(t, j), t), s_d(t, j))) \\ &\quad + d(x(t, j), t), \\ \dot{s}_d(t, j) &= 0; \end{aligned} \tag{6}$$

(where the dot stands for the derivative with respect to the time variable t)

- for every $(t,j) \in S$, $(t,j+1) \in S$, we have

$$(x(t,j) + e(x(t,j),t), s_d(t,j)) \in D, \tag{7}$$

and

$$\begin{aligned} x(t,j+1) &= x(t,j), \\ s_d(t,j+1) &= k_d(x(t,j) + e(x(t,j),t), s_d(t,j)); \end{aligned} \tag{8}$$

- $(x(0,0), s_d(0,0)) = (x_0, s_0)$.

In this context, we next define the concept of stabilization of (3) by a minimal time hybrid feedback law sharing a robustness property with respect to measurement noises and external disturbances. The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. Recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0,+\infty) \to [0,+\infty)$ which is continuous, increasing, satisfying $\delta(0) = 0$ and $\lim_{R\to+\infty} \delta(R) = +\infty$.

Definition 2.5: Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

$$\rho(x) > 0,\ \forall x \neq 0. \tag{9}$$

We say that the *completeness assumption for* ρ holds if, for all (e,d) satisfying the regularity assumptions (4), and so that,

$$\begin{aligned} &\sup_{[0,+\infty)} |e(x,\cdot)| \le \rho(x),\ \forall x \in \mathbb{R}^n, \\ &\mathrm{esssup}_{[0,+\infty)} |d(x,\cdot)| \le \rho(x),\ \forall x \in \mathbb{R}^n, \end{aligned} \tag{10}$$

for every $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$, there exists a maximal solution on $[0,+\infty)$ of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0).

Definition 2.6: We say that the *uniform finite time convergence property* holds if there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying (9), such that the completeness assumption for ρ holds, and if there exists a function δ : $[0,+\infty) \to [0,+\infty)$ of class $\mathcal{K}_\infty$ such that, for every $R > 0$, there exists $\tau = \tau(\mathrm{diam}(R)) > 0$, for all functions e, d satisfying the regularity assumptions (4) and inequalities (10) for this function ρ, for every $x_0 \in B(0,R)$, and every $s_0 \in \mathcal{N}$, the maximal solution (x, s_d) of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0) satisfies

$$|x(t,j)| \le \delta(R),\ \forall t \ge 0,\ (t,j) \in S, \tag{11}$$

and

$$x(t,j) = 0,\ \forall t \ge \tau,\ (t,j) \in S. \tag{12}$$

Definition 2.7: The origin is said to be a *semi-global minimal time hybrid robust stabilizable equilibrium* for the system (3) if, for every $\varepsilon > 0$ and every compact subset $K \subset \mathbb{R}^n$, there exists a hybrid feedback law (C, D, k, k_d) satisfying the constraint

$$\|k(x, s_d)\| \le 1, \tag{13}$$

where $\|\cdot\|$ stands for the Euclidian norm in $\mathbb{R}^m$, such that:

- the uniform finite time convergence property holds;
- there exists a continuous function $\rho_{\varepsilon,K} : \mathbb{R}^n \to \mathbb{R}$ satisfying (9) for $\rho = \rho_{\varepsilon,K}$, such that, for every $(x_0, s_0) \in K \times \mathcal{N}$, all functions e, d satisfying the regularity assumptions (4) and inequalities (10) for $\rho = \rho_{\varepsilon,K}$, the maximal solution of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0) reaches O within time $T(x_0) + \varepsilon$, where $T(x_0)$ denotes the minimal time to steer the system (3) from x_0 to 0, under the constraint $\|u\| \le 1$.

C. Main result

Theorem 2.1: Assume that Hörmander's condition holds. If there exists no nontrivial singular minimal time solution of (1), (2), starting from 0, then the origin is a semi-global minimal time hybrid robust stabilizable equilibrium for the system (1), under the constraint (2).

Remark 2.3: The problem of global robust minimal time stabilization (*i.e.* $K = \mathbb{R}^n$ in Definition 2.7) cannot be achieved a priori because measurement noises may then accumulate and slow down the solution reaching 0 (compare with [9]).

Remark 2.4: The assumption of the absence of nontrivial singular minimizing solutions is classical in optimal control theory. Notice the following facts.

- If $m \ge n$, then there exists no singular solution.
- Denote by $\mathcal{F}_m$ the set of m-uples of linearly independent vector fields $(f_1, \ldots, f_m)$, endowed with the C^∞ Whitney topology. If $m \ge 3$, there exists an open dense subset of $\mathcal{F}_m$, such that any control system of the form (1), associated to a m-tuple of this subset, admits no nontrivial singular minimizing solution (see [11], [12], see also [2] for the existence of a dense set only).
- If there exist singular minimizing solutions, then the conclusion on subanalyticity of the function T may fail, and we cannot a priori prove that the set $\mathcal{S}$ of singularities of T is a stratifiable manifold, which is the crucial fact in order to define a hybrid strategy.

III. Sketch of proof

Due to the space limitation, we only sketch the proof (for a complete proof, see [32]).

In Subsection III-A, we study the minimal time control problem with the Hörmander's condition and we define an optimal control law k^{opt}, which is smooth on $\mathbb{R}^n$ except on the so-called cut locus set. Thus, outside of any given Ω neighborhood of this cut locus, we have a natural robustness property.

In Subsection III-B, we define the components of the hysteresis. First, using the minimal time function, the optimal controller is defined, outside a singular set which is a stratified submanifold of codimension greater than or equal to one. Then, the second component of the hysteresis is defined; it consists of a set of controllers, defined in a neighborhood of the singular set.

The main part of the proof, not described here, actually consists in uniting these controllers using an adapted hysteresis strategy, and describing the properties of the closed-loop system with this hybrid feedback law.

A. The optimal controller

We first interpret the minimal time control problem for the system (1) with the constraint (2) as a sub-Riemannian problem.

1) Sub-Riemannian distance: Recall that the sub-Riemannian distance (also called Carnot-Carathéodory distance) is defined as follows in $\mathbb{R}^n$ (see [6]). Let m an integer such that $1 \leq m \leq n$, and $f_1, \ldots, f_m$ be smooth vector fields on $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, set

$$g(x, v) := \inf \left\{ \sum_{i=1}^m u_i^2 \mid u_1, \ldots, u_m \in \mathbb{R}, \sum_{i=1}^m u_i f_i(x) = v \right\}.$$

Then $g(x, \cdot)$ is a positive definite quadratic form on the subspace spanned by $f_1(x), \ldots, f_m(x)$. Outside this subspace we set $g(x, v) = +\infty$. The form g is called *sub-Riemannian metric* associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$. Let $\mathcal{AC}([0,1], \mathbb{R}^n)$ denote the set of absolutely continuous paths in $\mathbb{R}^n$ defined on $[0, 1]$. Define the *length* of $\gamma \in \mathcal{AC}([0,1], \mathbb{R}^n)$ as $l(\gamma) = \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))} dt$. Note that the length of a path does not depend on its parametrization.

Recall that *Hörmander's condition* holds if the Lie algebra spanned by the vector fields $f_1, \ldots, f_m$, is equal to $\mathbb{R}^n$ at every point x. It is well-known that under this condition any two points of $\mathbb{R}^n$ can be joined by an absolutely continuous path with finite length.

Definition 3.1: The *sub-Riemannian distance* associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$, between two points x_0, x_1, is defined by

$$\begin{aligned} d_{SR}(x_0, x_1) \quad = \quad \inf \Big\{ & l(\gamma) \mid \gamma \in \mathcal{AC}([0,1], \mathbb{R}^n), \\ & \gamma(0) = x_0, \gamma(1) = x_1 \Big\}. \end{aligned}$$

A path $\gamma \in \mathcal{AC}([0,1], \mathbb{R}^n)$ is said to be *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 3.1: If Hörmander's condition holds, then any two points can be joined by a minimizing path, and the topology defined by the sub-Riemannian distance d_{SR} coincides with the usual topology of $\mathbb{R}^n$.

2) Optimal control formulation: Since the length of a path does not depend on its parametrization, if a path $x(\cdot) : [0, T] \to \mathbb{R}^n$ is parametrized by its arc-length, then there holds almost everywhere on $[0, T]$

$$\dot{x}(t) = \sum_{i=1}^m u_i(t) f_i(x(t)), \ \sum_{i=1}^m u_i(t)^2 = 1,$$

and the length of $x(\cdot)$ is equal to T. Therefore, the sub-Riemannian problem is equivalent to the minimal time problem for the system (1) with the (non convex) constraint $\sum_{i=1}^m u_i^2 = 1$. Obviously, it is also equivalent to the minimal time problem for the system (1) with the (convex) constraint (2). We sum up the situation in the following result.

Lemma 3.1: The minimal time problem for the system (1) with the constraint (2) is equivalent to the sub-Riemannian problem associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$. Moreover, the minimal time $T(x)$ needed to steer the system (1) with the constraint (2) from 0 to a point $x \in \mathbb{R}^n$ is equal to the sub-Riemannian distance $d_{SR}(0, x)$ of x to 0.

In particular, minimal time solutions of (1), (2), are exactly minimizing paths of the previous sub-Riemannian problem. This equivalence permits to work in the framework of sub-Riemannian geometry.

Singular solutions (resp. singular controls) are defined in the same way as previously (see Definition 2.2).

3) Computation of minimizing solutions: Let $x \in \mathbb{R}^n$. The sub-Riemannian problem of determining a minimizing solution steering 0 to x can be easily seen (up to reparametrization, and using the Cauchy-Schwarz inequality) to be equivalent to the *optimal control problem* of finding a control $u(\cdot) \in \mathcal{U}$ such that the solution of the control system (1) steers 0 to x in time 1, and minimizes the *cost function*

$$C(u(\cdot)) = \int_0^1 \sum_{i=1}^m u_i(t)^2 \, dt. \tag{14}$$

If a control $u(\cdot)$ associated to a solution $x(\cdot)$ so that $x(0) = 0$ is optimal, then there exists a nontrivial *Lagrange multiplier* $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\psi . dE_1(u(\cdot)) = -\psi^0 dC(u(\cdot)), \tag{15}$$

where $dE_1(u(\cdot))$ (resp. $dC(u(\cdot))$) denotes the Fréchet derivative of E_1 (resp. C) at the point $u(\cdot)$. Moreover, according to Pontryagin's maximum principle (see [27]), the solution $x(\cdot)$ is the projection of an *extremal*, that is a quadruple $(x(\cdot), p(\cdot), p^0, u(\cdot))$, solution of the constrained Hamiltonian system

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)), \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)), \\ \frac{\partial H}{\partial u}&(x(t), p(t), p^0, u(t)) = 0, \end{aligned}$$

almost everywhere on $[0, 1]$, where

$$H(x, p, p^0, u) = \langle p, \sum_{i=1}^m u_i f_i(x) \rangle + p^0 \sum_{i=1}^m u_i^2$$

is the *Hamiltonian* of the optimal control problem, $p(\cdot)$ (called *adjoint vector*) is an absolutely continuous mapping on $[0, 1]$ such that $p(t) \in \mathbb{R}^n$, and p^0 is a real nonpositive constant. Moreover, there holds $(p(1), p^0) = (\psi, \psi^0)$, up to a multiplying scalar. If $p^0 < 0$ then the extremal is said to be *normal*, and in this case it is normalized to $p^0 = -1/2$. If $p^0 = 0$ then the extremal is said to be *abnormal*.

Remark 3.2: Any singular solution is the projection of an abnormal extremal, and conversely.

Using the previous normalization, controls associated to normal extremals can be computed as

$$u_i(t) = \langle p(t), f_i(x(t)) \rangle, \ i = 1, \ldots, m.$$

Hence, normal extremals are solutions of the Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_1}{\partial p}(x(t), p(t)), \ \dot{p}(t) = -\frac{\partial H_1}{\partial x}(x(t), p(t)), \tag{16}$$

where

$$H_1(x, p) = \frac{1}{2} \sum_{i=1}^m \langle p, f_i(x) \rangle^2.$$

4) The cut locus:

Definition 3.2: A point $x \in \mathbb{R}^n$ is not a *cut point* if there exists a minimizing solution joining 0 to x, which is the strict restriction of a minimizing solution starting from 0. The *cut locus*, denoted by $\mathcal{L}$, is the set of all cut points.

In other words, a cut point is a point at which a minimizing solution ceases to be optimal.

The following result on the cut locus is crucial (see [32] for a proof).

Proposition 3.2: Assume that the m vector fields $f_1, \ldots, f_m$ are analytic, and that there exists no singular minimizing solution starting from 0. Then, the set of points where the sub-Riemannian distance to 0 is not analytic is equal to the cut locus, that is, $\text{Sing } d_{SR}(0, \cdot) = \mathcal{L}$.

Proposition 3.2 provides the key argument of the proof. Indeed, on the one part, in the absence of nontrivial minimizing solutions, the mapping $d_{SR}(0, \cdot)$ is subanalytic outside 0, and in particular its singular set $\mathcal{S} := \text{Sing}(d_{SR}(0, \cdot))$ is a subanalytic manifold of codimension greater than or equal to one. Outside this singular set, this feedback law is analytic and defines our feedback law k^{opt}.

B. Components of the hysteresis, and hybrid strategy

The function $T(\cdot)$, which coincides with the function $d_{SR}(0, \cdot)$, is subanalytic outside 0, and hence, its singular set $\mathcal{S}$ (*i.e.*, the analytic singular support of $T(\cdot)$) is a stratified submanifold of $\mathbb{R}^n$, of codimension greater than or equal to 1. The objective is then to construct neighborhoods of $\mathcal{S} \setminus \{0\}$ in $\mathbb{R}^n$ whose complements share invariance properties for the optimal flow.

1) The optimal controller: Outside the singular set $\mathcal{S}$, the function $T(\cdot)$ is analytic, and the minimal time controllers steering a point $x \in \mathbb{R}^n \setminus \mathcal{S}$ to 0 are given by the closed-loop formula

$$u_i(x) = -\frac{1}{2}\langle \nabla(T(x)^2), f_i(x)\rangle, \ i = 1, \ldots, m. \qquad (17)$$

The smoothness of this optimal controller outside the submanifold $\mathcal{S}$ ensures a robustness property of the stability outside $\mathcal{S}$. The following lemma, yieldings invariance properties of the optimal flow, is proved in [32].

Lemma 3.3: For every neighborhood Ω of $\mathcal{S} \setminus \{\bar{x}\}$ in $\mathbb{R}^n$, there exists a neighborhood Ω' of $\mathcal{S} \setminus \{\bar{x}\}$ in $\mathbb{R}^n$, satisfying

$$\Omega' \subsetneq \text{clos}(\Omega') \subsetneq \Omega, \qquad (18)$$

such that every trajectory of the closed-loop system (1) with the optimal controller, joins a point $x \in \mathbb{R}^n \setminus \Omega$ to $\bar{x}$, and is contained in $\mathbb{R}^n \setminus \Omega'$.

A switching strategy must be defined in order to connect the optimal controller to other controllers, defined next, which have to be continuous in the neighborhood Ω' of the submanifold $\mathcal{S}$. The switching strategy is achieved by adding a dynamical discrete variable s_d and using a hybrid feedback law.

Robustness properties of the system in closed-loop with the optimal controller are investigated in [32].

2) The second component of the hysteresis: The second component of the hysteresis consists of a set of controllers, defined in a neighborhood of $\mathcal{S}$. Since $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^n$ of codimension greater than or equal to one, there exists a partition $(M_i)_{i\in\mathbb{N}}$ of $\mathcal{S}$, where M_i is a stratum, *i.e.*, a locally closed submanifold of $\mathbb{R}^n$.

C. End of the proof

To conclude the proof, we unit have the optimal controller together with the second component of the hysteresis defined previously, using an hysteresis. Note that, it is proved in [33] that this class of feedbacks has natural generic robustness properties with respect to measurement, actuator noise and external disturbances. It is also the case of our hybrid feedback. However, since, in our context, we are interested in a quasi-optimal property, we cannot apply [33] directly. This procedure is similar to the one of [29], where an infinite number of state-feedbacks are united.

In [32], a family of three nested patchy vector fields is required to prove the main result. Some of the patches define the dynamics of the discrete component of the hybrid controller, and the others are used for technical reasons to handle the measurement noises. The precise definition of this hysteresis procedure, so as the description of the properties of the closed-loop system with this hybrid feedback law, actually represent the main part of the proof of the main result (see [32] for details).

Note that this program was achieved in an explicit way in [31]; in particular, in this reference, the hybrid feedback law is completely explicit, and defined in a very simple way. The result of [32], whose proof is sketched here, is rather an existence result, though we explain how to derive an hysteresis feedback law, in function of the Lie bracket configuration of the system. The main result, Theorem 2.1, however points out the main assumption under which it is possible to achieve the stabilization process, namely, the absence of singular minimal time solutions of the control system (1), under the constraint (2).

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