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On a Nonanticipative Filippov Theorem for Control Systems with state constraints

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Abstract— **An important consequence of the famous Filippov's Theorem for a control system, states that given a fixed initial point y_0, and a control u_0, for any initial point y_1, it is possible to find a control u_1 such that the trajectory associated with (y_1, u_1) approximates the trajectory associated with (y_0, u_0) (with a suitable estimation of exponential type). The goal of the present paper is to obtain such a result when trajectories of the control system have to fullfill furthermore a given in advance state constraint. The main specificity of our results lies in the fact that u_1 is built from u_0 in a nonanticipative way i.e. at every time t, $u_1(t)$ depends only values of u_0 on the interval $[0, t]$. This nonanticipative property is crucial in particular for further applications to differential games.**

I. Introduction

We consider the following control system

$$\begin{cases} y'(t) & = g(y(t), u(t)), \quad u(t) \in U \\ y(t_0) & = y_0 \in K \end{cases} \tag{1}$$

where $t_0 \in [0, T]$ and the controller choosing has to ensure the state constraint $y(t) \in K$ to be fulfilled for any $t \in [t_0, T]$.

As is usual in state-constraint problems, the main difficulty comes from the fact that the players have to use *admissible* controls, and in particular that the set of controls allowed to a player strongly depends on its position. To overcome this difficulty, one has to be able, for two fixed initial positions , to approximate a given admissible control at one position by some admissible control at the other position. This problem has been successfully handled for control problems by several authors, under various assumptions in [1], [11], [12]. However, in the case of differential games, it is very important to build the approximating control in a nonanticipative way. Unfortunately, the constructions of the above quoted papers are all anticipative. We refer to [4], [7], [13] for application of this procedure to differential games. The main result of the present paper is the construction of such nonanticipative approximating control which furnishes a kind of generalization of the famous Filippov's Theorem [9].

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II. Preliminaries

A. Notations and assumptions

We first introduce some notations. Throughout this paper, $|\cdot|$ denotes the euclidean norm of $\boldsymbol{R}^l$. If K is a subset of $\boldsymbol{R}^l$, $d_K(x)$ denotes the distance of x from K, i.e., $d_K(x) = \inf_{y \in K} |y - x|$. We also denote by B the closed unit ball. If K is a subset of $\boldsymbol{R}^l$ and $r > 0$, we denote by $K + rB$ the set of points $x \in \boldsymbol{R}^l$ such that $d_K(x) \leq r$.

We denote by **(H)** the following assumptions concerning with the vector fields of the dynamics:

- (i) U is compact subset of some finite dimensional space
- (ii) $g : \boldsymbol{R}^l \times U \to \boldsymbol{R}^l$ is continuous and Lipschitz continuous (with Lipschitz constant M) with respect to $y \in \boldsymbol{R}^l$;
- (iii) $\bigcup_u g(y, u)$ is convex for any y.
- (iv) $K = \{y \in \boldsymbol{R}^l,\ \phi(y) \leq 0\}$ with $\phi \in \mathcal{C}^2(\boldsymbol{R}^l; \boldsymbol{R})$, $\nabla\phi(y) \neq 0$ if $\phi(y) = 0$
- (v) $\forall y \in \partial K,\ \exists u \in U\ < \nabla\phi(y), g(y, u) > <\ 0$

For any $y \in \boldsymbol{R}^l$, we set

$$U(y) := \{u \in U |\ g(y, u) \in T_K(y)\}$$

where $T_K(y)$ is the tangent half-space to the set K. Notice that under assumptions (H) the set-valued map $y \mapsto g(y, U(y))$ is lower semicontinuous with convex compact values ([3]).

For any starting point $y_0 \in K$, for any initial time $t_0 \in [0, T]$ and for any measurable control $u(\cdot) : [t_0, T] \to U$, we denote by $y[t_0, y_0; u(\cdot)](t)$ the solution of system (1).

The controller $u(\cdot)$, has to ensure that $y(t) \in K$ for any $t \geq 0$. We n introduce the notions of admissible controls: $\forall y_0 \in K$, and $\forall t_0 \in [0, T]$ we define

$$\mathcal{U}(t_0, y_0) := \{u(\cdot) : [t_0, +\infty) \to U \text{ measurable } | \\ y[t_0, y_0; u(\cdot)](t) \in K \quad \forall t \geq t_0\}.$$

Under the assumptions (H), it is well known that for all $y_0 \in K$ the set $\mathcal{U}(t_0, y_0)$ is not empty.

B. Non-anticipative maps

An important problem in order to get suitable estimations on constrained trajectories, is to obtain a kind of Filippov Theorem with constraints. Namely a result which allows to approach - in a suitable sense - a given trajectory of the dynamics by a constrained trajectory. Namely a result which allows to approach - in a suitable sense - a given trajectory of the dynamics by a constrained trajectory. Note that similar results exists in the literature (cf [1], [11], [12]) but in the present paper we wish a construction of the constrained trajectory in a nonanticipative way.

Let us recall the fundamental notion of *non-anticipative strategies* due to Varayia-Lin-Roxin-Elliot-Kalton (cf for instance [5]. A map $\sigma : \mathcal{U}(t_0, y_0) \to \mathcal{U}(t_0, y_1)$ is non-anticipative (for the point $(t_0, y_0, y_1) \in \mathbb{R}^+ \times K \times K$) if, for any $\tau > 0$, for all controls $u_1(\cdot)$ and $u_2(\cdot)$ belonging to $\mathcal{U}(t_0, y_0)$, which coincide a.e. on $[t_0, t_0 + \tau]$, $\sigma(u_1(\cdot))$ and $\sigma(u_2(\cdot))$ coincide almost everywhere on $[t_0, t_0 + \tau]$.

Non-anticipative strategies are very important for differential games theory, because their generalizes the feedback strategies and they enables to prove the existence of the value of the differential games.

III. Main Result

In this section we state the main result of this paper, namely the generalized non-anticipative Filippov's Theorem with stste-constraint.

Theorem 3.1: Assume that conditions (H) are satisfied. For any $R > 0$ there exist $C_0 = C_0(R) > 0$ such that for any initial time $t_0 \in [0, T]$, for any y_0, $y_1 \in K$ with $|y_0|$, $|y_1| \leq R$, there is a nonanticipative strategy $\sigma : \mathcal{U}(t_0, y_0) \longrightarrow \mathcal{U}(t_0, y_1)$ with the following property: for any $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$ and for $t \in [t_0, T]$ we have

$$\begin{aligned}&\int_{t_0}^{t} |g(y_0(s), u_0(s)) - g(y_1(s), \sigma(u_0(\cdot))(s))| \, ds \\ &+ |y_0(t) - y_1(t)| \leq C_0 |y_0 - y_1| e^{C_0(t - t_0)}\end{aligned} \tag{2}$$

where we have set for simplicity $y_0 = y[t_0, y_0; u_0(\cdot)]$ and $y_1 = y[t_0, y_1; \sigma(u_0(\cdot))](t)$.

Corollary 3.2: In particular if g is affine with respect to the control u, namely

$$g(y, u) = g_1(y)u + g_2(y)$$

where $g_1(y)$ is an invertible matrix with a Lipschitz continuous inverse, then we have

$$\begin{aligned}&\int_{t_0}^{t} |u_0(s) - \sigma(u_0(\cdot))(s)| \, ds \\ &+ |y_0(t) - y_1(t)| \leq C_1 |y_0 - y_1| e^{C_1(t - t_0)}.\end{aligned} \tag{3}$$

for some constant $C_1 = C_1(R) > 0$.

IV. Proof of the Main Result

Fix an admissible control $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$, and let us set $y_0(\cdot) = y[t_0, y_0; u_0(\cdot)](\cdot)$, take a new starting point $\bar{y} \in K$. We wish to build in a nonanticipative way a control $\bar{u}(\cdot)$ satisfying:

$$\begin{aligned}&\int_{t_0}^{t} |g(y[t_0, y_0; u_0(\cdot)](s), u_0(s)) - g(y[t_0, \bar{y}; \bar{u}(\cdot)](s), \bar{u}(s))| \, ds \\ &+ |y[t_0, y_0; u_0(\cdot)](t) - y[t_0, \bar{y}; \bar{u}(\cdot)](t)| \\ &\leq C_0 |y_0 - \bar{y}| e^{C_0(t - t_0)}.\end{aligned}$$

To this end, we consider the system:

$$\begin{cases} y'(t) = \pi_{G(y(t)) \cap T_K(y(t))}\big(g(y(t), u_0(t))\big) \\ y(t_0) = \bar{y} \in K, \end{cases} \tag{4}$$

where we denote $G(y) := g(y, U)$ and $\pi_{G(y(t)) \cap T_K(y(t))}\big(g(y(t), u_0(t))\big)$ denotes the projection of $g(y(t), u_0(t))$ onto $G(y(t)) \cap T_K(y(t))$. Notice that $\pi_{G(y) \cap T_K(y)}\big(g(y, u_0(t))\big) = g(y, u_0(t))$ whenever y belongs to the interior of K or if y is on the boundary of K and

$$\langle g(y(t), u_0(t)), \nabla \phi(y(t)) \rangle \leq 0 \, .$$

Let us also underline that, since the set $G(y) \cap T_K(y)$ is convex, the projection onto $G(y) \cap T_K(y)$ is unique. We denote it by $g(y, \bar{u}(y, u))$ and we note that the control $\bar{u}(y, u) \in U$ is not necessarily unique. Our goal is to show that $\bar{u}(y, u)$ is a suitable feedback, which enables us to build the control $\bar{u}$ in a nonanticipative way. First we show that there is a solution to (4).

Lemma 4.1: System (4) admits at least one solution.

Proof of Lemma 4.1. We claim that the set of solutions of system (4) is the same as the set of solutions of the following system

$$\begin{cases} y'(t) \in \tilde{G}(t, y(t)), \; y(t) \in K \\ y(t_0) = \bar{y} \in K, \end{cases} \tag{5}$$

where

$$\tilde{G}(t, y) := \begin{cases} g(y, u_0(t)) \text{ if } y \in \mathrm{Int}(K) \\ \overline{\mathrm{co}}\,\{g(y, u_0(t)); g(y, \bar{u}(y, u_0(t)))\} \text{ if } y \in \partial K \end{cases}$$

Before proving the claim, let us note that, since the set-valued function $\tilde{G}$ is clearly Lebesgue-Borel measurable in (t, y) and upper semicontinuous with respect to y, by the measurable viability Theorem of [10], we obtain that system (5) and, so, according to the claim, also system (4), has a solution for any starting point $\bar{y} \in K$ at any initial time t_0.

We now prove the claim. Since

$$\pi_{G(y) \cap T_K(y)}\big(g(y, u_0(t))\big) \subset \tilde{G}(t, y)$$

then any solution of (4) is also a solution of (5). Conversely, suppose that $y(\cdot)$ is a solution of (5) and consider the set

$$\mathcal{D} := \{t \mid \exists y'(t) \text{ with } y'(t) \notin \pi_{G(y(t)) \cap T_K(y(t))}\big(g(y(t), u_0(t))\big)\}$$

We have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ where $\mathcal{D}_1 = \{t \in \mathcal{D} \mid y(t) \in \partial K\}$ and $\mathcal{D}_2 = \{t \in \mathcal{D} \mid y(t) \in \mathrm{Int}(K)\}$. The measure of $\mathcal{D}_2$ is zero because

$$\tilde{G}(t, y)\Big|_{\mathrm{Int}(K)} \equiv \pi_{G(y) \cap T_K(y)}\big(g(y, u_0(t))\big).$$

On the other hand, the measure of $\mathcal{D}_1$ is also zero; indeed, for almost every t such that $y(t) \in \partial K$ the derivative $y'(t)$ exists, and, since t is a local maximum for the function $s \mapsto \phi(y(s))$, the derivative of $\phi(y(s))$ with respect to s at time t vanishes:

$$0 = \frac{d}{ds} \phi(y(s)) \Big|_{s=t} = \langle y'(t), \nabla \phi(y(t)) \rangle.$$

If $\langle g(y(t), u_0(t)), \nabla \phi(y(t)) \rangle > 0$ then we obtain $y'(t) = g(y(t), \bar{u}(y(t), u_0(t)))$; otherwise $g(y(t), \bar{u}(y(t), u_0(t))) = g(y(t), u_0(t))$. In any case we get

$$y'(t) \in \pi_{G(y) \cap T_K(y)}\big(g(y(t), u_0(t))\big).$$

□

Next Lemma allows to compare $g(y,u)$ and $g(y,\bar{u}(y,u))$:

Lemma 4.2: Under assumption (H), for any $R>0$, there is a constant $C>0$ such that for any $y\in\partial K$ with $|y|\le R$ and any $u\in U$, we have

$$|g(y,\bar{u}(y,u))-g(y,u)|\le C\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+ \ , \quad (6)$$

where $(x)_+=\max\{x,0\}$.

Proof of Lemma 4.2. From (H), we can choose $\eta>0$ such that:

$$\sup_{y\in\partial K,\ |y|\le R}\ \inf_{u\in U}\langle g(y,u),\nabla\phi(y)\rangle<-\eta<0.$$

Fix $y\in\partial K$ and consider $u_1\in U$ such that

$$< g(y,u_1),\nabla\phi_U(y)> \ < -\eta<0\ .$$

Let us set

$$\lambda=\frac{\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+}{\eta+\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+}\ .$$

Note that $\lambda\in[0,1]$. From the convexity of $g(y,U)$, we can find some $u_\lambda\in U$ such that

$$g(y,u_\lambda)=(1-\lambda)g(y,u)+\lambda g(y,u_1)\ .$$

Then

$$\langle g(y,u_\lambda),\nabla\phi(y)\rangle\le-\eta\lambda+(1-\lambda)\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+=0.$$

Hence, observing that $\lambda\le\frac{1}{\eta}\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+$, we obtain

$$\begin{aligned}&|g(y,\bar{u}(y,u))-g(y,u)|\le|g(y,u_\lambda)-g(y,u)|\le\\&\le\lambda|g(y,u_1)-g(y,u)|\le\frac{M}{\eta}\left(\langle g(y,u),\nabla\phi(y)\rangle\right)_+\ ,\end{aligned}$$

for some constant $M=M(R)$. □

If $\bar{y}(\cdot)$ is a solution of (4), by the measurable selection theorem there exists an admissible control $\bar{u}(\cdot)$ such that

$$\begin{cases}\bar{y}'(t)=g(\bar{y}(t),\bar{u}(t))=\pi_{G(\bar{y}(t))\cap T_K(\bar{y}(t))}\big(g(\bar{y}(t),u_0(t))\big)\\ y(t_0)=\bar{y}\in K,\end{cases} \quad (7)$$

Lemma 4.3: Assume that conditions (H) hold. For any positive constant R there exists a positive $\tilde{C}=\tilde{C}(R)$ such that for any y_0, $\bar{y}\in K$ with $|y_0|$, $|\bar{y}|\le R$ and for any admissible control $u_0(\cdot)\in\mathcal{U}(t_0,y_0)$, the admissible control $\bar{u}(\cdot)\in\mathcal{U}(t_0,\bar{y})$ is such that

$$\begin{aligned}&\int_{t_0}^{t}|g(\bar{y}(s),\bar{u}(s))-g(y_0(s),u_0(s))|\ ds\\&\le\tilde{C}\left(|\bar{y}-y_0|+\int_{t_0}^{t}|\bar{y}(s)-y_0(s)|\ ds\right),\end{aligned} \quad (8)$$

where $\bar{y}(s):=y[t_0,\bar{y};\bar{u}(\cdot)](s)$ and $y_0(s):=y[t_0,y_0;u_0(\cdot)](s)$.

Proof. Recall that $(\bar{u}(\cdot),\bar{y}(\cdot))$ denotes the couple control-trajectory which satisfies system (7).

In order to fix the ideas, let us assume that $\bar{y}\in\mathrm{Int}(K)$. The case in which $\bar{y}$ belongs to the boundary of K can be treated similarly. Let us define the following set:

$$O:=\{s\in(t_0,t)\mid\bar{y}(s)\in\mathrm{Int}(K)\}=\{s\in(t_0,t)\mid\phi(\bar{y}(s))<0\}.$$

The set O is open in $[t_0,t]$ and it is an enumerable union of open disjoint intervals, I_n,

$$O=\bigcup_{n\in\boldsymbol{N}}I_n.$$

For any $\varepsilon>0$ we can choose a finite number of these intervals, say I_i for $i=1,\dots,k$, such that

$$\left|O\setminus\bigcup_{i=1}^{k}I_i\right|\le\varepsilon.$$

Let us call $O_k:=\mathrm{Int}\left(\overline{\bigcup_{i=1}^{k}I_i}\right)$; notice that $O_k=\bigcup_{j=0}^{h}J_j$, where J_j are open intervals: $J_j=]t_{2j},t_{2j+1}[$ with $t_{2j+1}\le t_{2j+2}$. Observe that

$$\left|O_k\ \triangle\left(\bigcup_{i=1}^{k}I_i\right)\right|=0$$

and that $\phi(\bar{y}(t_{2j}))=\phi(\bar{y}(t_{2j+1}))=0$ for any j. Moreover we have

$$O_k^c=[t_0,t]\setminus O_k=[t_0,t]\setminus\bigcup_{j=0}^{h}J_j=\bigcup_{j=0}^{h}[t_{2j+1},t_{2j+2}],$$

where $t_{2h+2}=t$.

We claim that there is a constant C, independent of the control u_0 and of the initial positions y_0 and $\bar{y}$, such that for almost every $s\in O^c$ we have

$$|g(\bar{y}(s),\bar{u}(s))-g(\bar{y}(s),u_0(s))|\le C\langle g(\bar{y}(s),u_0(s)),\nabla\phi(\bar{y}(s))\rangle\ . \quad (9)$$

For this we apply Lemma 4.2, for the constant $\tilde{R}$ such that any solution staring from a point $y\in RB\cap K$ remains in $\tilde{R}B$ on the time interval $[0,T]$. We have now to explain how to remove the “plus” in the inequality (6) of Lemma 4.2. Let E be the set where the derivative of $\bar{y}(\cdot)$ exists. For any $s\in E\cap O^c$, we obtain

$$0=\frac{d}{d\tau}\phi(\bar{y}(\tau))\bigg|_{\tau=s}=\langle\bar{y}'(s),\nabla\phi(\bar{y}(s))\rangle \quad (10)$$

because s is a local maximum for $\tau\longmapsto\phi(\bar{y}(\tau))$. Since $|E\cap O^c|=|O^c|$, by (10) we obtain that for almost every $s\in O^c$ either $\bar{u}(s)=u_0(s)$ and, hence,

$$\langle g(\bar{y}(s),u_0(s)),\nabla\phi(\bar{y}(s))\rangle=\langle g(\bar{y}(s),\bar{u}(s)),\nabla\phi(\bar{y}(s))\rangle=0,$$

or $\bar{u}(s)\ne u_0(s)$ and, so, $\bar{u}(s)=\bar{u}(\bar{y}(s),u_0(s))$ and

$$\langle g(\bar{y}(s),u_0(s)),\nabla\phi(\bar{y}(s))\rangle>\langle g(\bar{y}(s),\bar{u}(s)),\nabla\phi(\bar{y}(s))\rangle=0.$$

Thanks to Lemma 4.2 we have (9).

Now, by using (9), we obtain

$$\int_{t_0}^{t}|g(\bar y(s),\bar u(s))-g(y_0(s),u_0(s))|\,ds\leq$$
$$\leq\int_{O^c}|g(\bar y(s),\bar u(s))-g(\bar y(s),u_0(s))|\,ds$$
$$+\int_{t_0}^{t}|g(\bar y(s),u_0(s))-g(y_0(s),u_0(s))|\,ds\leq$$
$$\leq C\int_{O^c}\langle g(\bar y(s),u_0(s)),\nabla\phi_U(\bar y(s))\rangle\,ds$$
$$+M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds$$
$$=C[\int_{O_k^c}\langle g(\bar y(s),u_0(s)),\nabla\phi_U(\bar y(s))\rangle\,ds$$
$$-\int_{O_k^c\setminus O^c}\langle g(\bar y(s),u_0(s)),\nabla\phi_U(\bar y(s))\rangle\,ds]+$$
$$+M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds\leq$$
$$\leq C[\int_{O_k^c}\langle g(\bar y(s),u_0(s)),\nabla\phi(\bar y(s))\rangle\,ds+$$
$$\varepsilon\|\langle g(\bar y,u_0),\nabla\phi(\bar y)\rangle\|_\infty]++M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds=$$
(because $|O_k^c\setminus O^c|\leq\varepsilon$)
$$=C[\sum_{j=0}^{h}\int_{t_{2j+1}}^{t_{2j+2}}\langle g(\bar y(s),u_0(s)),\nabla\phi_U(\bar y(s))\rangle\,ds$$
$$+\varepsilon\bar M]+M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds\leq$$
$$\leq C\left[\sum_{j=0}^{h}\int_{t_{2j+1}}^{t_{2j+2}}\langle g(y_0(s),u_0(s)),\nabla\phi_U(y_0(s))\rangle\,ds\right.$$
$$\left.+K_0\sum_{j=0}^{h}\int_{t_{2j+1}}^{t_{2j+2}}|\bar y(s)-y_0(s)|\,ds+\varepsilon\bar M\right]$$
$$+M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds,$$

where K_0 and $\bar M$ are suitable constants depending on R. Observe that

$$\sum_{j=0}^{h}\int_{t_{2j+1}}^{t_{2j+2}}\langle g(y_0(s),u_0(s)),\nabla\phi_U(y_0(s))\rangle\,ds$$
$$=\sum_{j=0}^{h}(\phi(y_0(t_{2j+2}))-\phi(y_0(t_{2j+1})))=\phi(y_0(t_{2h+2}))-\phi(y_0(t_1))$$
$$-\sum_{j=1}^{h}(\phi(y_0(t_{2j+1}))-\phi(y_0(t_{2j})))=\phi(y_0(t_{2h+2}))-\phi(y_0(t_1))$$
$$-\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}\langle g(y_0(s),u_0(s)),\nabla\phi(y_0(s))\rangle\,ds\leq-\phi(y_0(t_1))$$
$$-\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}\langle g(\bar y(s),u_0(s)),\nabla\phi(\bar y(s))\rangle\,ds$$
$$+K_0\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}|\bar y(s)-y_0(s)|\,ds=-\phi(y_0(t_1))-$$
$$\sum_{j=1}^{h}(\phi(\bar y(t_{2j+1}))-\phi(\bar y(t_{2j})))+$$
$$K_0\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}|\bar y(s)-y_0(s)|\,ds=-\phi(y_0(t_1))$$
$$+K_0\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}|\bar y(s)-y_0(s)|\,ds,$$

because the points $\bar y(t_{2j})$ and $\bar y(t_{2j+1})$ for each $j=1,\dots,h$ belong to the boundary. Moreover, we have:

$$-\phi(y_0(t_1))\leq-\phi(\bar y(t_1))+K_0|\bar y(t_1)-y_0(t_1)|\leq$$
$$K_0\left[|\bar y-y_0|+\int_{t_0}^{t_1}|\bar y(s)-y_0(s)|\,ds\right]$$

because $-\phi(\bar y(t_1))=0$ and applying Gronwall Lemma. Then, finally, we obtain:

$$\int_{t_0}^{t}|g(\bar y(s),\bar u(s))-g(y_0(s),u_0(s))|\,ds$$
$$\leq C\Big\{K_0\left[|\bar y-y_0|+\int_{t_0}^{t_1}|\bar y(s)-y_0(s)|\,ds\right]$$
$$+K_0\sum_{j=1}^{h}\int_{t_{2j}}^{t_{2j+1}}|\bar y(s)-y_0(s)|\,ds$$
$$+K_0\sum_{j=0}^{h}\int_{t_{2j+1}}^{t_{2j+2}}|\bar y(s)-y_0(s)|\,ds+\varepsilon\bar M\Big\}+$$
$$+M\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds\leq$$
$$C\left\{K_1\left[|\bar y-y_0|+\int_{t_0}^{t}|\bar y(s)-y_0(s)|\,ds\right]+\varepsilon\bar M\right\},$$

for some constant $K_1>0$. This gives (8) because ε is arbitrary. □

Let us end the proof of Theorem 3.1. For any admissible control $u_0(\cdot)\in\mathcal{U}(t_0,y_0)$, we claim that it is possible to construct (in a nonanticipative way) an admissible control $u_1(\cdot)\in\mathcal{U}(t_0,y_1)$ such that $\forall t\in[t_0,T]$

$$\int_{t_0}^{t}|g(y_1(s),u_1(s))-g(y_0(s),u_0(s))|\,ds+|y_1(t)-y_0(t)|\leq C_0|y_1-y_0|e^{C_0(t-t_0)}\tag{11}$$

where $y_0(t) = y[t_0, y_0; u_0(\cdot)](t)$ and $y_1(t) = y[t_0, y_1; u_1(\cdot)](t)$. Indeed, let $(u_1(\cdot), y_1(\cdot))$ the couple control-trajectory which satisfies system (7) with the starting point $y_1 = \bar{y}$. We get

$$\begin{aligned} |y_1(t) - y_0(t)| \leq |y_1 - y_0| + \\ \int_{t_0}^{t} |g(y_1(s), u_1(s)) - g(y_0(s), u_0(s))| \, ds \\ \leq (1 + \tilde{C})|y_1 - y_0| + \tilde{C} \int_{t_0}^{t} |y_1(s) - y_0(s)| \, ds, \end{aligned} \tag{12}$$

invoking Lemma 4.3. Thus

$$|y_1(t) - y_0(t)| + \int_{t_0}^{t} |g(y_1(s), u_1(s)) - g(y_0(s), u_0(s)| \, ds \leq C_0|y_1 - y_0| + C_0 \int_{t_0}^{t} |y_1(s) - y_0(s)| \, ds$$

for some positive constant C_0 and, thanks to the Gronwall's Lemma, we obtain (11).

Finally, it is easy to check that the set-valued map $\Sigma : \mathcal{U}(t_0, y_0) \mapsto \mathcal{U}(t_0, y_1)$ defined by:

$$\Sigma(u_0(\cdot)) := \{u(\cdot) \in \mathcal{U}(t_0, y_1) \mid (u(\cdot), y(\cdot)) \text{ solves } (7) \}$$

is nonexpansive with nonempty (*)-closed values (in the sense of [8]). Hence, by Plaskacz Lemma (see Lemma 2.7 of [8]) there exists a nonanticipative selection σ with the following property: $\sigma(u_0(\cdot)) \in \Sigma(u_0(\cdot))$ for any $u_0(\cdot) \in \mathcal{U}(t_0, y_0)$.

The proof of (3) is a direct consequence of the assumptions and of (2).

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