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A Global Non-linear Control Design for a PVTOL Vehicle with Aerodynamics

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***Abstract*— Non-linear stabilisation and control of a PVTOL vehicle has received much attention in the literature for the low speed case, where aerodynamic effects can be neglected. In this paper, a non-linear control design is presented for a three degree of freedom "Hovering Rocket" with aerodynamic effects included, such that the controller remains functional at high speeds. Control design is based on a non-linear, decoupling change of coordinates presented by many authors in the past. The decoupled system is then controlled using an optimised cascade approach.**

I. INTRODUCTION

The task of stabilisation and control of a planar, vertical take-off and landing (PVTOL) aerospace vehicle has received much attention throughout the previous decade. The quantity of research may be attributed to the difficulties involved. Namely, the system is multi-variable, under-actuated and highly non-linear. Furthermore it is also non-minimum phase, the severity of which depends on the degree of input coupling. Such input coupling is observed in rotatory wing aircraft and the vehicle investigated in this paper.

An early approach applied to the PVTOL control problem was published by [1], who in 1992 designed a stabilising controller using conventional feedback linearization techniques and a non-minimum phase approximation of a PVTOL system. [2] used the same approximation, and a backstepping approach to control the translational dynamics through the cascaded roll subsystem. In general, methods using a minimum-phase approximation exhibit good performance for small values of coupling, however larger values such as those exhibited by the subject vehicle of this paper, lead to diminished performance and instability.

An alternative approach taken by a number of authors is to first apply a non-linear change of coordinates such that the new system representation is minimum phase. This change of coordinates was discovered by [3] through searching for differentially flat outputs of the system [4]. It was shown that this corresponded to changing the outputs of the system from translation at the vehicle's center of gravity to translation at its *Huygen centre of oscillation.* A state tracker was then designed using feedback linearization to control the new system outputs, and flatness* was exploited to indirectly control motion at the centre of gravity through generation of appropriate trajectory demands on the *centre of oscillation*. A similar approach was also used by [5], who used an approximating sequence for trajectory generation. [6] used a similar change of coordinates corresponding to outputs at a point with constant offset from the *centre of oscillation*, coincident with the center of gravity when at equilibrium. The backstepping approach presented in [2] was then used to globally stabilise the system. More recently, Ref. [7] used the same change of variables proposed by [6] and a Lyapunov analysis method to globally stabilise a four rotor mini-aircraft with bounded thrust.

To the authors' knowledge, PVTOL non-linear control has only been considered for the low speed case where aerodynamic forces on the vehicle body can be neglected. This paper presents a non-linear control design for a three degree of freedom "Hovering Rocket". Aerodynamic forces on the vehicle body are taken into account during controller design, such that performance is maintained at high velocities. The approach used here is derived from the change of variables given in [6], and an optimised version of the cascade control method presented in [2].

II. SYSTEM DYNAMICS

The subject of this paper is a three degree of freedom thrust-vectoring rocket, as depicted in Fig. 1 where flight is constrained to the zox plane. The rocket is intended to operate at low horizontal speed as compared to conventional missiles, and hence is primarily reliant on powered lift. The system can be considered as having two inputs; primary thrust T and control thrust T_c. The state equations of this system have the same structure as those investigated in previous literature [1]. However, as the vehicle is required to track nonzero velocity trajectories, a more accurate system description will model the effects of aerodynamic drag. The dominant aerodynamic mode, is drag normal to the vehicle, N. Axial drag will be significantly lower, can be simply compensated through input augmentation and it is therefore not considered. Aerodynamic pitching moment can be neglected due to vehicle symmetry. The state equations representing vehicle motion at the center of gravity are displayed in Eqn. (1), where m is the vehicle mass, J the moment of inertia about the y axis and l the control thrust

*As the new outputs are flat, there is a one-to-one correspondence between trajectories at the center of gravity and at its *centre of oscillation.*

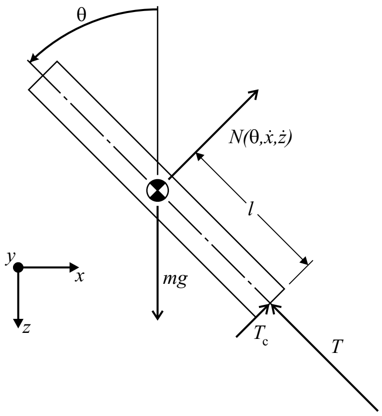


Fig. 1. The VTOL vehicle. T is primary thrust, T_c control thrust, m vehicle mass, g gravitational acceleration, θ vehicle roll and N normal aerodynamic drag.

moment arm about the centre of gravity.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{T}{m}\sin(\theta) + \frac{T_c}{m}\cos(\theta) + \frac{N(\theta, z_2, x_2)}{m}\cos(\theta) \\
\dot{z}_1 &= z_2 \\
z_2 &= -\frac{T}{m}\cos(\theta) - \frac{T_c}{m}\sin(\theta) - \frac{N(\theta, z_2, x_2)}{m}\sin(\theta) + g \\
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{l}{J}T_c
\end{aligned} \tag{1}
$$

Here $x_1,\ z_1,\ \theta$ are horizontal, vertical and roll displacements, and $x_2,\ z_2\ ,\omega$ are horizontal, vertical and roll rates respectively.

III. Control System Design

The control system design used here relies on the same non-linear change of coordinates presented in [3], [8] and first generalised in [6]. This acts to decouple the three second-order subsystems with respect to the control thrust input. The change of coordinates is shown below:

$$
\begin{aligned}
r_1 &= x_1 - \frac{J}{lm}\sin(\theta) \\
r_2 &= x_2 - \frac{J}{lm}\cos(\theta)\,\omega \\
h_1 &= z_1 - \frac{J}{lm}(\cos(\theta) - 1) \\
h_2 &= z_2 + \frac{J}{lm}\sin(\theta)\,\omega \\
\varepsilon_1 &= \theta \\
\varepsilon_2 &= \omega,
\end{aligned} \tag{2}
$$

where r_1 and h_1 are the horizontal and vertical displacements at a distance of $\frac{J}{lm}$ vertically below the vehicle's *centre of oscillation* (the "control point") respectively, and r_2 and h_2 are their corresponding rates†. Applying this change of variables converts the state equations to:

$$
\begin{aligned}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= \frac{1}{m}\bar{T}\sin(\varepsilon_1) + \frac{N(\theta, z_2, x_2)}{m}\cos(\varepsilon_1) = v_1 \\
\dot{h}_1 &= h_2 \\
\dot{h}_2 &= \frac{1}{m}\bar{T}\cos(\varepsilon_1) - \frac{N(\theta, z_2, x_2)}{m}\sin(\varepsilon_1) + g = v_2 \\
\dot{\varepsilon}_1 &= \varepsilon_2 \\
\dot{\varepsilon}_2 &= \frac{J}{l}T_c,
\end{aligned} \tag{3}
$$

where $\bar{T} = \frac{J}{l}\varepsilon_2^2 - T$ is the augmented control input to remove h and r subsystem dependence on ε_2^2. The system now takes the form of two second-order translational subsystems h and r controlled directly by input $\bar{T}$ and indirectly by T_c through the cascaded roll subsystem ε. The remainder of the control system design is an adaptation and modification of the work done by [6], [8].

It should be noted that if it were possible to directly force the values of v_1 and v_2 to p_1 and p_2, given by:

$$
\begin{aligned}
p_1 &\triangleq c_{11}e_{r_1} + c_{12}e_{r_2} \\
p_2 &\triangleq c_0\tanh(c_{21}e_{h_1} + c_{22}e_{h_2}) \qquad |c_0| < g,
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
e_{r_1} &= r_1 - r_1^d \\
e_{r_2} &= r_2 - r_2^d \\
e_{h_1} &= h_1 - h_1^d \\
e_{h_2} &= h_2 - h_2^d,
\end{aligned} \tag{5}
$$

$c_{1j},\ c_{2j}$ for $j = 1, 2$ are coefficients of stable polynomials and c_0 bounds $|p_2| < g$, the vehicle translation would be forced to track the reference trajectory defined by $\left(r_1^d, r_2^d, h_1^d, h_2^d\right)$. Trajectory demands will therefore be on the "control point" rather than the centre of gravity. However, trajectories at the two points will be similar (specifically in r_1 and h_1) due to the bound distance between them [3].

It is not possible to directly set the values of v_1 and v_2 through $\bar{T}$ and T_c due to the finite convergence speed of the ε subsystem. The control strategy used here is to analytically determine values of $\bar{T}$ and ε_1 (labelled k_1 and k_2 respectively) that would set the values of the translational acceleration v_1 and v_2 to the desired p_1 and p_2 . A standard backstepping procedure is then used to force ε_1 to converge to the desired value of k_2, while feeding the system with an appropriate value of $\bar{T}$.

The equation for normal aerodynamic force on a cylinder in cross flow is defined as [9]:

$$
N = \frac{1}{2}C_N S \rho U^2, \tag{6}
$$

where C_N is the coefficient of drag, S a reference area, ρ the fluid density and U the fluid velocity. Taking U as the component of velocity perpendicular to the vehicle

†It is important to realise that the velocity and acceleration at the *centre of oscillation* and the "control point" will be equal, as they are at a constant displacement from each other.

longitudinal axis:

$$N\left(\varepsilon_1, r_2, h_2\right) = \frac{1}{2} C_N S \rho \left(r_2^2 + h_2^2\right) \times \cos\left(\operatorname{atan2}\left(-h_2, -r_2\right) + \theta\right). \quad (7)$$

Two simplifications have been made in this representation such that it can be incorporated into the controller design. Firstly, instead of using the more accurate square of the cosine of the angle of incidence of the fluid with the vehicle, we have simply used the cosine function to allow a single analytic solution to k_2 ‡. If the square of the cosine were used, the sign (direction) of N would have to be made conditional on the value of $\operatorname{atan2}\left(-h_2, -r_2\right) + \theta$, otherwise N would always be positive. Secondly, although it would be more accurate to consider drag as a function of velocity at the center of pressure, for the purpose of controller design presented here, it must be taken as a function of velocity at the "control point" (r_2, h_2). If it were taken at any other point fixed to the vehicle body, the change of co-ordinates given in Eqn. (2) would couple the r and h subsystems to the ε subsystem through ε_2, and the required cascade structure would not be achieved. Furthermore, taking N as a function of (r_2, h_2) will preserve differential flatness of the system§.

It should also be noted that C_N could be expressed more accurately as a polynomial function of r_2 and h_2. However, the control design presented is only restricted by the way in which N depends on θ. Hence, for the purpose of simplicity this paper assumes C_N is constant.

A single analytic solution is sought for k_1 and k_2 to the system of equations:

$$\begin{aligned} p_1 &= \frac{1}{m} k_1 \sin\left(k_2\right) + \frac{1}{2m} S C_n \rho \left(r_2^2 + h_2^2\right) \times \\ &\quad \cos\left(\operatorname{atan}(-h_2, -r_2) + k_2\right) \cos\left(k_2\right) \\ p_2 &= \frac{1}{m} k_1 \cos\left(k_2\right) - \frac{1}{2m} S C_n \rho \left(r_2^2 + h_2^2\right) \times \\ &\quad \cos\left(\operatorname{atan}(-h_2, -r_2) + k_2\right) \sin\left(k_2\right) + g. \end{aligned} \quad (8)$$

This is a difficult problem to solve analytically, compounded by the non-surjective nature of the sine and cosine functions and the multiple solutions introduced if an approach polynomial in $\sin\left(k_2\right)$ is used. The key to arriving at a single analytic solution is to realise what the equations physically represent; an acceleration balance at the vehicle's *centre of oscillation*. Hence, a graphical approach can be taken. It is also helpful to think of k_1 and k_2 as the desired vehicle orientation ε_1, and thrust $\bar{T}$ to achieve the desired vehicle acceleration (p_1, p_2) at the current velocity. The key to producing the vector diagram in Fig. 2 is the plotting of N^* (a vector of magnitude equal to the normal force on the vehicle if its longitudinal axis were perpendicular to the velocity: $\frac{1}{2} C_N S \rho \left(r_2^2 + h_2^2\right)$, in the direction of current velocity (r_2, h_2)). The desired vehicle orientation k_2 is then obtained by construction of a line joining the gravity mg and

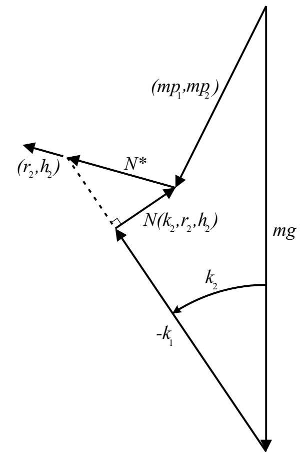


Fig. 2. Force balance at *centre of oscillation*. Here (mp_1, mp_2) is the desired force imbalance. The vector N^* is a vector of magnitude equal to the normal force on the vehicle if its longitudinal axis were perpendicular to the velocity: $\frac{1}{2} C_N S \rho \left(r_2^2 + h_2^2\right)$, in the direction of current velocity (r_2, h_2). Quantities k_1 and k_2 are the desired values of $\bar{T}$ and ε_1 respectively to produce this imbalance, and $N\left(k_2, r_2, h_2\right)$ is the aerodynamic force on the vehicle at the current velocity, if $\varepsilon_1 = k_2$.

N^* vectors. The magnitude of the line perpendicular to this, through the end of the desired force imbalance (mp_1, mp_2) will then be equal to the normal force N at the desired orientation, and current velocity.

The resulting solutions to k_1 and k_2 are¶;

$$k_2 = \operatorname{atan2}\left(A, B\right) \quad (9)$$

$$k_1 = -m\sqrt{\left(g - p_2\right)^2 + p_1^2} \times \cos\left(\operatorname{atan2}\left(A, B\right) - \operatorname{atan}\left(\frac{-p_1}{g - p_2}\right)\right) \quad (10)$$

where

$$\begin{aligned} A &= \left(\left|N^*\right| \sin\left(\operatorname{atan2}\left(-h_2, -r_2\right) + k_2\right) + k_1\right) \sin\left(k_2\right) \\ &= -p_1 - \frac{r_2}{\sqrt{r_2^2 + h_2^2}} \left(r_2^2 + h_2^2\right) \frac{1}{2m} L D C_n \rho \end{aligned} \quad (11)$$

$$\begin{aligned} B &= \left(\left|N^*\right| \sin\left(\operatorname{atan2}\left(-h_2, -r_2\right) + k_2\right) + k_1\right) \cos\left(k_2\right) \\ &= g - p_2 - \frac{h_2}{\sqrt{r_2^2 + h_2^2}} \left(r_2^2 + h_2^2\right) \frac{1}{2m} L D C_n \rho. \end{aligned} \quad (12)$$

Fig. 2 shows the primary reason for bounding $|p_2| < g$. This acts to significantly reduce the instances where $|k_2| > \frac{\pi}{2}$ (i.e. when $m r_2 + p_2 > g$), such that the vehicle remains upright as desired.

The next step in the design process is to determine the required T_c feedback such that the vehicle orientation ε_1 is forced to track that desired k_2. This is obtained via the standard backstepping procedure [10], using simple quadratic

‡It should be noted that this assumption is less erroneous at high speeds where N is more significant, as compared to low speeds.

§This is easily shown by direct calculation, indicating the system with $N\left(\varepsilon_1, r_2, h_2\right)$ has no zero dynamics.

¶NOTE: Only require arctan function of $\frac{-p_1}{g - p_2}$ since $-\frac{\pi}{2} < \operatorname{atan}\left(\frac{-p_1}{g - p_2}\right) < \frac{\pi}{2}$. (i.e $g - p_2 > 0$)

control Lyapunov functions. The following feedback control law results:

$$T_c = \ddot{k}_2 - (c_1 + c_2)\left(\varepsilon_2 - \dot{k}_2\right) - (c_1 c_2 + 1)(\varepsilon_1 - k_2) \qquad c_1, c_2 \in \mathbb{R}^+ \tag{13}$$

$$= \ddot{k}_2 - d_2\left(\varepsilon_2 - \dot{k}_2\right) - d_1(\varepsilon_1 - k_2) \qquad d_1, d_2 \in \mathbb{R}^+. \tag{14}$$

It is possible to determine $\dot{k}_2$ and $\ddot{k}_2$ as a static function of inputs and state variables, as k_2 depends only on differentially flat outputs of the system.

It would be adequate to simply set $\bar{T} = k_1$, as done by [6], [2]. However, as ε_1 will generally not equal the desired k_2, it follows that k_1 would not be the optimal value for $\bar{T}$. The method proposed here is to minimise the error between desired and actual acceleration introduced by $\varepsilon_1 - k_2 \neq 0$. The h and r subsystem dynamics can be written as follows:

$$\begin{aligned}\dot{\gamma} &= \Psi(\gamma, \varepsilon_1 - k_2) \\ &= \Psi(\gamma, 0) + \Psi_e(\gamma, \varepsilon_1 - k_2), \qquad \gamma = [r_1, r_2, h_1, h_2]^T,\end{aligned} \tag{15}$$

where

$$\Psi(\gamma, 0) = \begin{bmatrix} r_2 \\ p_1 \\ h_2 \\ p_2 \end{bmatrix}, \qquad \Psi_e(\gamma, \varepsilon_1 - k_2) = \begin{bmatrix} 0 \\ v_1 - p_1 \\ 0 \\ v_2 - p_2 \end{bmatrix}.$$

Here $\Psi(\gamma, 0)$ represents the desired closed loop dynamics (i.e if $v_1 = p_1$ and $v_2 = p_2$). Ψ_e represents the error due to finite convergence speed of $\varepsilon_1 - k_2$. It is therefore desired to minimise the norm of Ψ_e over $\bar{T}$. As $\parallel \Psi_e \parallel$ is quadratic in $\bar{T}$, it is convex and can be easily minimised by setting $\frac{\partial \|\Psi_e\|}{\partial \bar{T}} = 0$ and solving for $\bar{T}$, resulting in:

$$\bar{T}_{opt} = m p_1 \sin(\varepsilon_1) + m(p_2 - g)\cos(\varepsilon_1). \tag{16}$$

$\bar{T}_{opt}$ can be found graphically by the construction of a line from the end of the desired acceleration vector, perpendicularly through the line along the vehicle longitudinal axis. Fig. 3 shows how setting $\bar{T} = \bar{T}_{opt}$, will always result in a lower value of $\parallel \psi_e \parallel$ than if $\bar{T} = k_1$. It can also be seen that at most configurations, this minimisation will also result in $\bar{T}_{opt} < k_1$, and hence lower input energy. Furthermore, if aerodynamic drag is small enough to be neglected, it can be shown that $\bar{T}_{opt} < k_1$ will always hold. Using $\bar{T} = \bar{T}_{opt}$ introduces the possibility of instances where $\bar{T} < 0$, which is impossible for this vehicle. However, as mentioned earlier, bounding $|p_2| < g$ will result in $|k_2| > \frac{\pi}{2}$ for the majority of state configurations, and $\bar{T} < 0$ can only occur when $|\theta| > \frac{\pi}{2}$. Hence $\bar{T} \geq 0$ for the majority of state configurations.

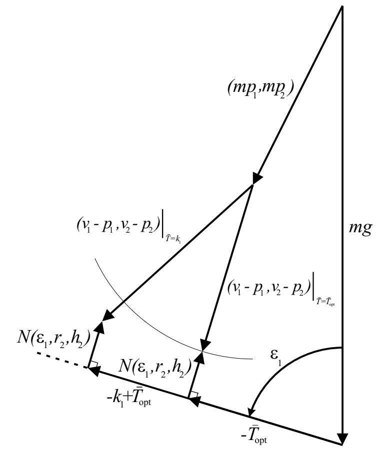


Fig. 3. $\bar{T}$ is found by the construction of a line from the desired force imbalance (mp_1, mp_2), perpendicularly through the vehicle longitudinal axis. The vector $N(\varepsilon_1, h_1, r_1)$ is the current aerodynamic drag on the vehicle. It can be seen graphically that the magnitude of the error between the desired and achieved acceleration $\parallel (v_1 - p_1, v_2 - p_2) \parallel$ when $\bar{T} = \bar{T}_{opt}$, will always be less than if $\bar{T} = k_1$.

IV. Results

Simulations were carried out for a cylindrical vehicle of 10kg, 1m in length and 0.1m in width at standard sea-level atmospheric conditions. A value of $C_N = 0.8$ was assumed. Gain selection was as follows; c_{ij} gains were chosen such that the response of $\Psi(\gamma, 0)$ to desired trajectories was as desired (critically damped in this case) and could be tracked open-loop with reasonable actuator demands. The gains d_1 were chosen such that the response predicted by the second-order subsystem in $\varepsilon_1 - k_2$ could be tracked with allowable T_c demands, assuming reasonable values of $\dot{k}_2$ and $\ddot{k}_2$. Too large $\bar{T}$ demands were alleviated by reducing c_{ij} values, and T_c by reducing d_i.

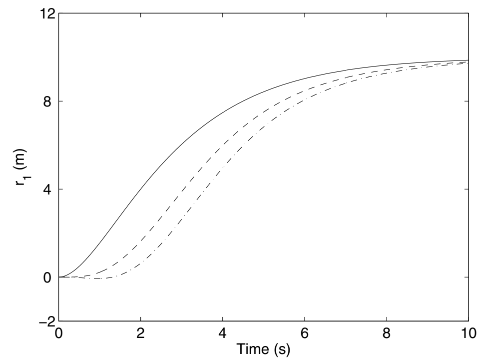
The first simulation demonstrates the benefits of using $\bar{T} = \bar{T}_{opt}$ as described above, in comparison to $\bar{T} = k_1$ as in [6]. Fig. 4 shows results of a 10m (horizontal) step demand in r_2 for two respective controllers. The controller utilising $\bar{T}_{opt}$ results in a trajectory far closer to the desired $\Psi(\gamma, 0)$ response. Actuator demands are also lower than those when $\bar{T} = k_2$, as expected[‖]. These observations are more pronounced with decreasing d_i, corresponding to slower $\varepsilon_1 - k_2$ convergence.

Fig. 5 demonstrates the need for controller design to consider the aerodynamic force N when reference trajectories include high velocities, in this case a 50m.s^{-1} step change in r_2. Results are shown for two controllers, one that considers N as above, and another that assumes $C_N = 0$ using the same design procedure. The controller accounting for N tracks the trajectory with a smooth, desirable response. However, the controller neglecting N performs poorly at high speeds and will in general fall into a nonlinear limit cycle.

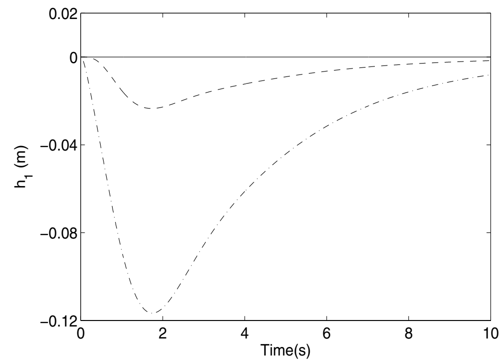
V. Conclusion

A global, non-linear control design has been presented for a PVTOL vehicle with limited aerodynamic effects. The primary contributions of this work was to extend and modify the controller design presented by [6], [2] to more optimal feedback, and situations where aerodynamic forces cannot be neglected. A control design that considers the

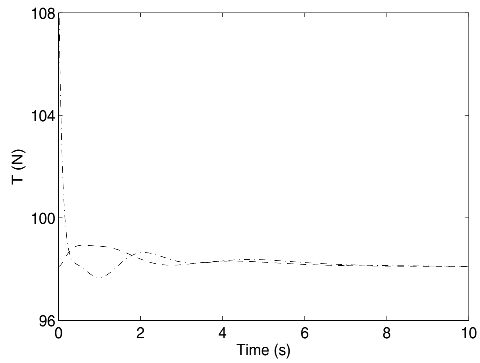
[‖]Particularly at times when $\varepsilon_1 - k_2$ is large such as $t = 0$.



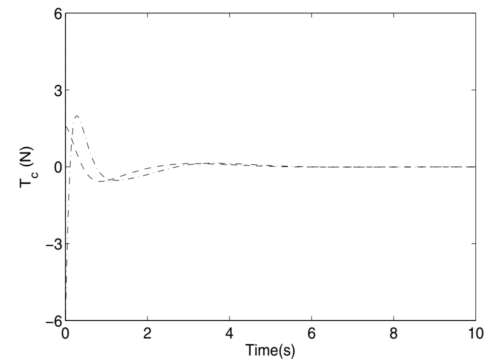
(a) Horizontal displacement response r_1.



(b) Vertical displacement response h_1.



(c) Primary thrust demand T .



(d) Control thrust demand T_c.

Fig. 4. Horizontal step response comparison for different $\bar{T}$ feedback. —— : desired system response given by $\psi(\gamma, 0)$, $- - -$: system response if $\bar{T} = \bar{T}_{opt}$, $\cdot - \cdot -$: system response if $\bar{T} = k_1$.

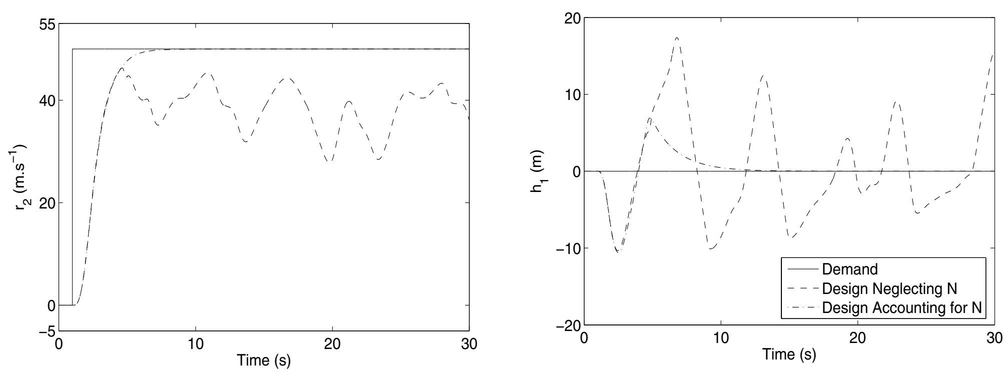


Fig. 5. Response to a 50m.s^{-1} step demand in r_2. Comparison between controller designed to account for normal aerodynamic drag N, and controller design neglecting it.

primary aerodynamic mode of a cylindrical vehicle has been presented. An optimal approach to primary thrust feedback has been presented. Simulations have shown the benefits of the proposed thrust feedback, and the superiority at high speeds of the controller that considers system aerodynamics.

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