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Costs of Competition in General Networks

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Abstract— **In this paper, we present an analysis of competition in congested networks. We consider the problem of routing flows across multiple paths controlled by serial and parallel service providers that charge prices for transmission. We study the efficiency properties of oligopoly equilibria. Our measure of efficiency is the difference between users' willingness to pay and delay costs. Under the assumption that delay costs without transmission (latencies at zero) are equal to zero, we show that, irrespective of the number of serial and parallel providers, the efficiency of oligopoly equilibria in pure strategies is no worse than 1/2 times the efficiency of the social optimum. When latencies at zero can be positive, the efficiency of oligopoly equilibria can be arbitrarily low.**

I. INTRODUCTION

There has been growing interest in pricing as a method of allocating scarce network resources (see, for example, [13], [15]). Although prices may be set to satisfy some network objectives, in practice many prices are controlled by for-profit service providers that charge prices, at least in part, to increase their revenues and profits.

Research to date suggests that profit-maximizing pricing may improve the allocation of resources in communication networks. Let the metric of efficiency be the difference between users' willingness to pay and delay costs in the equilibrium relative to that in the social optimum (which would be chosen by a a network planner with full information and full control over users). Acemoglu and Ozdaglar [2] show that with inelastic and homogeneous users, pricing by a monopolist controlling all links in a parallel-link network always achieves efficiency (i.e., the efficiency metric is equal to one). Huang, Ozdaglar and Acemoglu [11] extend this result to a general network topology. Acemoglu and Ozdaglar [1] show that in a parallel-link network with inelastic and homogeneous users, the efficiency metric with an arbitrary number of competing network providers is lower-bounded by 5/6 when there is zero latency at zero flow and by $2\sqrt{2}-2$ with positive latency at zero flow.

This paper shows that the efficiency of equilibrium with competing network providers is considerably lower when we depart from the parallel-link topology. To illustrate this, we consider a parallel-serial topology where an origin-destination pair is linked by multiple parallel paths, each potentially consisting of an arbitrary number of serial links. Congestion costs are captured by a link-specific non-decreasing convex latency functions, denoted by $l_i(\cdot)$. Each link is owned by a different service provider. All users are inelastic and homogeneous.

This environment induces the following two-stage game: each service provider simultaneously sets the price for transmission of bandwidth on its link, denoted by p_i. Observing all the prices, in the second stage users route their information through the path with the lowest effective cost, where effective cost consists of the sum of prices and latencies of the links along a path [i.e., sum of $p_i + l_i(\cdot)$'s over the links comprising a path].

We characterize the pure strategy subgame perfect equilibria of this game and show that when latency without any traffic is equal to zero [i.e., $l_i(0) = 0$], there is a tight bound of 1/2 on the efficiency metric irrespective of the number of paths and service providers in the network. This bound is reached by simple examples. This establishes that the performance of general network under competition can be significantly worse than the 5/6 bound established for the parallel topology in Acemoglu and Ozdaglar [1].

The reason for this degradation of performance is interesting. As Example 1 below illustrates, the bound 1/2 is reached when there are multiple serial providers along a path with zero (total) latency. Serial providers do not take into account that when they charge higher prices, they reduce the profits of other serial providers on their path. This increases prices along paths with multiple serial providers, and when it is socially optimal to transmit most of the data through such paths, it raises the scope for inefficiency.

More strikingly, when the assumption that $l_i(0) = 0$ is relaxed, we find that the efficiency is arbitrarily small relative to the social optimum. This result sheds doubt on the conjecture that unregulated competition among service providers might achieve good network performance in general.

Related work include studies quantifying efficiency losses of selfish routing without prices (e.g., Koutsoupias and Papadimitriou [14], Roughgarden and Tardos [18], Correa, Schulz, and Stier-Moses [7], Perakis [17], and Friedman [9]); of resource allocation by different market mechanisms (e.g., Johari and Tsitsiklis [12], Sanghavi and Hajek [19]); and of network design (e.g., Anshelevich et. al. [4]). More closely related are the works of Basar and Srikant [5], who analyze monopoly pricing in a network context under specific assumptions on the utility and latency functions; He and Walrand, [10], who study competition cooperation among Internet service providers under specific demand models; as well as Acemoglu, Ozdaglar, and Srikant [3], who study resource allocation in a wireless network under fixed pricing. None of these papers, except our previous work, Acemoglu

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and Ozdaglar [1], consider the performance of a network with competing providers.

II. Model

We consider a network with I parallel paths that connect two nodes. Each path i consists of n_i links. Let $\mathcal{I} = \{1, \ldots, I\}$ denote the set of paths and $\mathcal{N}_i$ denote the set of links on path i. Let x_i denote the flow on path i, and $x = [x_1, \ldots, x_I]$ denote the vector of path flows. Each link in the network has a flow-dependent latency function $l_i(x_i)$, which measures the delay as a function of the total flow on link i. We denote the price per unit flow (bandwidth) of link j by p_j. Let $p = [p_j]_{j \in \mathcal{N}_i,\ i \in \mathcal{I}}$ denote the vector of prices.

We are interested in the problem of routing d units of flow across the I paths. We assume that this is the aggregate flow of many "small" users and thus adopt the Wardrop's principle (see [20]) in characterizing the flow distribution in the network; i.e., the flows are routed along paths with minimum effective cost, defined as the sum of the latencies and prices of the links along that path (see the definition below). Wardrop's principle is used extensively in modelling traffic behavior in transportation networks ([6], [8], [16]) and communication networks ([18], [7]). We also assume that users have a *reservation utility* R and decide not to send their flow if the effective cost exceeds the reservation utility.

Definition 1: For a given price vector $p \geq 0$, a vector $x^{WE} \in \mathbb{R}^I_+$ is a *Wardrop equilibrium* (WE) if

$$\begin{aligned} \sum_{j \in \mathcal{N}_i} l_j(x_i^{WE}) + p_j &= \min_{k \in \mathcal{I}} \Big\{ \sum_{j \in \mathcal{N}_k} l_j(x_k^{WE}) + p_j \Big\} \\ & \qquad\qquad \forall\ i \text{ with } x_i^{WE} > 0, \qquad (1) \\ \sum_{j \in \mathcal{N}_i} l_j(x_i^{WE}) + p_j &\leq R, \qquad \forall\ i \text{ with } x_i^{WE} > 0, \end{aligned}$$

and $\sum_{i \in \mathcal{I}} x_i^{WE} \leq d$, with $\sum_{i \in \mathcal{I}} x_i^{WE} = d$ if $\min_{k \in \mathcal{I}} \big\{ \sum_{j \in \mathcal{N}_k} l_j(x_k^{WE}) + p_j \big\} < R$. We denote the set of WE at a given p by $W(p)$.

We adopt the following assumption on the latency functions throughout the paper except in Section IV-C.

Assumption 1: For each $i \in \mathcal{I}$, the latency function $l_i : [0, \infty) \mapsto [0, \infty)$ is convex, continuously differentiable, nondecreasing, and satisfies $l_i(0) = 0$.

The following proposition establishes the existence and continuity properties of the WE as a function of the prices. The proof idea is standard, but is included briefly here for use in subsequent analysis.

Proposition 1: **(Existence and Continuity)** Let Assumption 1 hold. For any price vector $p \geq 0$, the set of WE, $W(p)$, is nonempty. Moreover, the correspondence $W : \mathbb{R}^I_+ \rightrightarrows \mathbb{R}^I_+$ is upper semicontinuous.

Proof sketch: Given any $p \geq 0$, the proof is based on using Assumption 1 (in particular the convexity assumption) to show that the set of WE is given by the set of optimal solutions of the following optimization problem

$$\begin{aligned} \text{maximize}_{x \geq 0} \quad & \textstyle\sum_{i \in \mathcal{I}} \Big(\big(R - \sum_{j \in \mathcal{N}_i} p_i \big) x_i - \\ & \qquad \int_0^{x_i} \sum_{j \in \mathcal{N}_i} l_i(z) dz \Big) \qquad (2) \\ \text{subject to} \quad & \sum_{i=1}^{I} x_i \leq d. \end{aligned}$$

The results follow by the fact that the optimal solution set of the preceding problem is nonempty and by using the Theorem of the Maximum. **Q.E.D.**

For a given price vector p, the WE need not be unique in general. Under further restrictions on the l_i, we obtain the following result.

Proposition 2: **(Uniqueness)** Let Assumption 1 hold. Assume further that the l_i are strictly increasing. For any price vector $p \geq 0$, the set of WE, $W(p)$, is a singleton. Moreover, the function $W : \mathbb{R}^I_+ \mapsto \mathbb{R}^I_+$ is continuous.

We next define the social problem and the social optimum, which is the routing (flow allocation) that would be chosen by a central network planner that has full control and information about the network.

Definition 2: A flow vector x^S is a *social optimum* if it is an optimal solution of the *social problem*

$$\begin{aligned} \text{maximize}_{x \geq 0} \quad & \sum_{i \in \mathcal{I}} \Big(R - \sum_{j \in \mathcal{N}_i} l_j(x_i) \Big) x_i \qquad (3) \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} x_i \leq d. \end{aligned}$$

In view of Assumption 1, the social problem has a continuous objective function and a compact constraint set, guaranteeing the existence of a social optimum, x^S. Moreover, using the optimality conditions for a convex program, we see that a vector $x^S \in \mathbb{R}^I_+$ is a social optimum if and only if $\sum_{i \in \mathcal{I}} x_i^S \leq d$ and there exists a $\lambda^S \geq 0$ such that $\lambda^S \big(\sum_{i=1}^{I} x_i^S - d \big) = 0$ and for each $i \in \mathcal{I}$,

$$\begin{aligned} R - \sum_{j \in \mathcal{N}_i} l_j(x_i^S) - x_i^S \sum_{j \in \mathcal{N}_i} l'_j(x_i^S) \quad & \leq \lambda^S \quad \text{if } x_i^S = 0, \\ & = \lambda^S \quad \text{if } x_i^S > 0. \end{aligned}$$

For future reference, for a given vector $x \in \mathbb{R}^I_+$, we define the value of the objective function in the social problem,

$$\mathbb{S}(x) = \sum_{i \in \mathcal{I}} \Big(R - \sum_{j \in \mathcal{N}_i} l_j(x_i) \Big) x_i, \qquad (4)$$

as the *social surplus*, i.e., the difference between the users' willingness to pay and the total latency.

III. Oligopoly Pricing and Equilibrium

We assume that there are multiple service providers, each of which owns one of the links on the paths in the network. Service provider j charges a price p_j per unit bandwidth on link $j \in \mathcal{N}_i$. Given the vector of prices of links owned by other service providers, $p_{-j} = [p_k]_{k \neq j}$, the profit of service provider j with $j \in \mathcal{N}_i$ is

$$\Pi_j(p_j, p_{-j}, x) = p_j x_i,$$

where $x \in W(p_j, p_{-j})$.

The objective of each service provider is to maximize profits. Because their profits depend on the prices set by other service providers, each service provider forms conjectures about the actions of other service providers, as well as the behavior of users, which they do according to the notion of subgame perfect Nash equilibrium. We refer to the game among service providers as the *price competition game*.

Definition 3: A vector $(p^{OE}, x^{OE}) \geq 0$ is a (pure strategy) *Oligopoly Equilibrium* (OE) if $x^{OE} \in W\left(p_j^{OE}, p_{-j}^{OE}\right)$ and for all $i \in \mathcal{I}$, $j \in \mathcal{N}_i$,

$$\Pi_j(p_j^{OE}, p_{-j}^{OE}, x^{OE}) \geq \Pi_j(p_j, p_{-j}^{OE}, x), \quad \forall\ p_j \geq 0,\ \forall\ x \in W(p_j, p_{-j}^{OE}). \tag{5}$$

We refer to p^{OE} as the *OE price*.

The next proposition shows that for linear latency functions, there exists a pure strategy OE.

Proposition 3: Let Assumption 1 hold, and assume further that the latency functions are linear. Then the price competition game has a pure strategy OE.

Proof: Let $l_j(x) = a_j x$ for some $a_j \geq 0$. Define the set

$$\mathcal{I}_0 = \{i \in \mathcal{I} \mid \sum_{j \in \mathcal{N}_i} a_j = 0\},$$

(or equivalently, $\mathcal{I}_0$ is the set of $i \in \mathcal{I}$ such that $a_j = 0$ for all $j \in \mathcal{N}_i$). Let I_0 denote the cardinality of set $\mathcal{I}_0$. There are two cases to consider:

- $I_0 \geq 2$: Then it can be seen that a vector (p^{OE}, x^{OE}) with $p_j^{OE} = 0$ for all $i \in \mathcal{I}_0,\ j \in \mathcal{N}_i$ and $x^{OE} \in W(p^{OE})$ is an OE.
- $I_0 \leq 1$: For some $j \in \mathcal{N}_i$, let $B_j(p_{-j}^{OE})$ be the set of p_j^{OE} such that

$$(p_j^{OE}, x^{OE}) \in \arg \max_{\substack{p_j \geq 0 \\ x \in W(p_j, p_{-j}^{OE})}} p_j x_i. \tag{6}$$

Let $B(p^{OE}) = [B_j(p_{-j}^{OE})]$. In view of the linearity of the latency functions, it follows that $B(p^{OE})$ is an upper semicontinuous and convex-valued correspondence. Hence, we can use Kakutani's fixed point theorem to assert the existence of a p^{OE} such that $B(p^{OE}) = p^{OE}$. To complete the proof, it remains to show that there exists $x^{OE} \in W(p^{OE})$ such that (5) holds.

If $\mathcal{I}_0 = \emptyset$, we have by Proposition 2 that $W(p^{OE})$ is a singleton, and therefore (5) holds and $(p^{OE}, W(p^{OE}))$ is an OE.

Assume finally that $I_0 = 1$, and that without loss of generality $1 \in \mathcal{I}_0$. We show that for all $\bar{x},\ \tilde{x} \in W(p^{OE})$, we have $\bar{x}_i = \tilde{x}_i$, for all $i \neq 1$. Let

$$EC(x, p^{OE}) = \min_{k \in \mathcal{I}} \big\{ \sum_{j \in \mathcal{N}_k} l_j(x_k) + p_j^{OE} \big\}.$$

If at least one of

$$EC(\tilde{x}, p^{OE}) < R, \qquad \text{or} \qquad EC(\bar{x}, p^{OE}) < R$$

holds, then one can show that $\sum_{i=1}^{I} \tilde{x}_i = \sum_{i=1}^{I} \bar{x}_i = d$. Substituting $x_1 = d - \sum_{i \in \mathcal{I},\ i \neq 1} x_i$ in problem (2), we see that the objective function of problem (2) is strictly convex in $x_{-1} = [x_i]_{i \neq 1}$, thus showing that $\tilde{x} = \bar{x}$. If both $EC(\tilde{x}, p^{OE}) = R$ and $EC(\bar{x}, p^{OE}) = R$, then $\bar{x}_i = \tilde{x}_i = l_i^{-1}(R - p_i^{OE})$ for all $i \neq 1$, establishing our claim.

For some $x \in W(p^{OE})$, consider the vector $x^{OE} = \left(d - \sum_{i \neq 1} x_i, x_{-1}\right)$. Since x_{-1} is uniquely defined and x_1 is chosen such that the providers on link 1 have no incentive to deviate, it follows that (p^{OE}, x^{OE}) is an OE.

Q.E.D.

Although the proof of the existence of a pure strategy OE cannot be extended to arbitrary convex latency functions, existence of a mixed strategy OE can be established along the lines of the analysis in Acemoglu and Ozdaglar [1].

We next provide an explicit characterization of the OE prices, which will be essential for the efficiency analysis of Section IV. The proof relies on showing that all path flows are positive at the OE (which allows us to write the optimization problems for each provider in terms of equality and inequality constraints) and using the first order optimality conditions to characterize the OE prices. Details of the proof for the parallel-link topology are provided in [1].

Proposition 4: Let Assumption 1 hold. Let (p^{OE}, x^{OE}) be an OE such that $p_j^{OE} x_i^{OE} > 0$ for some $i \in \mathcal{I}$, $j \in \mathcal{N}_i$. Then, for all $i \in \mathcal{I}$, $j \in \mathcal{N}_i$, we have

$$p_j^{OE} = \begin{cases} x_i^{OE} \sum_{k \in \mathcal{N}_i} l'_k(x_i^{OE}), & \text{if } l'_k(x_s^{OE}) = 0 \\ & \text{for } k \neq j,\ s \neq i, \\ \min\left\{ \frac{1}{n_i}\Big[R - \sum_{k \in \mathcal{N}_i} l_k(x_i^{OE})\Big], \right. & \\ \quad x_i^{OE}\Big[\sum_{k \in \mathcal{N}_i} l'_k(x_i^{OE}) & \\ \quad\quad \left. + \frac{1}{\sum_{s \neq i} \frac{1}{\sum_{k \in \mathcal{N}_s} l'_k(x_s^{OE})}}\Big] \right\}, & \text{otherwise.} \end{cases} \tag{7}$$

In particular, for two links, when the minimum effective cost is less than R, for $i = 1, 2,\ j \in \mathcal{N}_i$, the OE prices are given by

$$p_j^{OE} = x_i^{OE} \left[\sum_{k \in \mathcal{N}_1} l'_k(x_1^{OE}) + \sum_{k \in \mathcal{N}_2} l'_k(x_2^{OE}) \right].$$

IV. Efficiency Analysis

In this section, we study the efficiency properties of oligopoly pricing. We take as our measure of efficiency the ratio of the social surplus at the equilibrium flow allocation to the social surplus at the social optimum, $\mathbb{S}(x^{OE})/\mathbb{S}(x^S)$ [cf. (4)]. We consider price competition games that have pure strategy equilibria (this set includes, but is larger than, games with linear latency functions, see Section III). Given a parallel-link network with I paths, n_i links on path i, and latency functions $\{l_j\}_{(j \in \mathcal{N}_i,\ i \in \mathcal{I})}$, let $\overrightarrow{OE}(\{l_j\})$ denote the set of flow allocations $x^{OE} = [x_i^{OE}]_{i \in \mathcal{I}}$ at an OE. We define the efficiency metric at some $x^{OE} \in \overrightarrow{OE}(\{l_j\})$ as

$$r_I(\{l_j\}, x^{OE}) = \frac{R \sum_{i \in \mathcal{I}} x_i^{OE} - \sum_{i \in \mathcal{I}} \Big(\sum_{j \in \mathcal{N}_i} l_j(x_i^{OE}) \Big) x_i^{OE}}{R \sum_{i \in \mathcal{I}} x_i^{S} - \sum_{i \in \mathcal{I}} \Big(\sum_{j \in \mathcal{N}_i} l_j(x_i^{S}) \Big) x_i^{S}}, \qquad (8)$$

where x^S is a social optimum given the latency functions $\{l_j\}$ and R is the reservation utility. Following the literature on the "price of anarchy," (see [14]), we are interested in the worst performance of an oligopoly equilibrium, so we look for a lower bound on

$$\inf_{\{l_j\}} \inf_{x^{OE} \in \overrightarrow{OE}(\{l_j\})} r_I(\{l_j\}, x^{OE}).$$

A. Two Paths

We consider a two path network, with n_i links on path $i = 1, 2$, where each link is owned by a different provider. We first study the following example, which illustrates that the efficiency loss can be worse than that in parallel link networks without serial links (which was shown to be bounded below by 5/6 in [1]).

Example 1: Consider a two path network, which has n links on path 1 with identically 0 latency functions and one link on path 2 with latency function $l(x_2) = x_2/2$. Let the total flow be $d = 1$ and the reservation utility be $R = 1$.

The unique social optimum for this example is $x^S = (1, 0)$. Using Proposition 4 and the definition of a WE, it follows that the flow allocation at the OE, x^{OE}, satisfies

$$\sum_{j \in \mathcal{N}_1} l_j(x_1^{OE}) + x_1^{OE} \Big[\sum_{j \in \mathcal{N}_1} l'_j(x_1^{OE}) + \sum_{j \in \mathcal{N}_2} l'_j(x_2^{OE}) \Big]$$
$$= \sum_{j \in \mathcal{N}_2} l_j(x_2^{OE}) + x_2^{OE} \Big[\sum_{j \in \mathcal{N}_1} l'_j(x_1^{OE}) + \sum_{j \in \mathcal{N}_2} l'_j(x_2^{OE}) \Big].$$

Substituting for the latency functions and solving the above together with $x_1^{OE} + x_2^{OE} = 1$ shows that the flow allocation at the unique oligopoly equilibrium is

$$x^{OE} = \Big(\frac{2}{n+2}, \frac{n}{n+2} \Big),$$

which goes to $(0, 1)$ as $n \to \infty$. The social surplus at the social optimum is 1, while the social surplus at the oligopoly equilibrium tends to $1/2$ as $n \to \infty$.

The next theorem provides a tight lower bound on $r_2(\{l_j\}, x^{OE})$ [cf. (8)]. In the following, we assume without loss of generality that $d = 1$.

Theorem 1: Consider a two path network, with n_i links on path i, where link j has a latency function l_j. Assume that each link is owned by a different provider and the price competition game has a pure strategy OE. Then

$$r_2(\{l_j\}, x^{OE}) \geq \frac{1}{2}, \qquad \forall\ x^{OE} \in \overrightarrow{OE}(\{l_j\}). \qquad (9)$$

Moreover, the bound above is tight, i.e., there exists $\{l_j\}$ and $x^{OE} \in \overrightarrow{OE}(\{l_j\})$ that attains the lower bound in (9).

Proof: The proof is an extension of the proof of efficiency for parallel-link topology in [1].

Step 1: We are interested in finding a lower bound for the problem

$$\inf_{\{l_j\}} \inf_{x^{OE} \in \overrightarrow{OE}(\{l_j\})} r_2(\{l_j\}, x^{OE}). \qquad (10)$$

Given $\{l_j\}$, let $x^{OE} \in \overrightarrow{OE}(\{l_j\})$ and let x^S be a social optimum. We can assume without loss of generality that $\sum_{i=1}^2 x_i^{OE} = \sum_{i=1}^2 x_i^S = 1$ (see [1]). This implies that there exists some i such that $x_i^{OE} \leq x_i^S$. Since the problem is symmetric, we can restrict ourselves to $\{l_j\}$ for which $x_1^{OE} \leq x_1^S$. We claim

$$\inf_{\{l_i\} \in \mathcal{L}_2} \inf_{x^{OE} \in \overrightarrow{OE}(\{l_i\})} r_2(\{l_i\}, x^{OE}) \geq r_2^{OE}, \qquad (11)$$

where

$$r_2^{OE} = \operatorname*{minimize}_{\substack{l_{i,j}^S, (l_{i,j}^S)' \geq 0 \\ l_{i,j}, l'_{i,j} \geq 0 \\ y_i^S, y_i^{OE} \geq 0}} \frac{R - y_1^{OE} \Big(\sum_{j \in \mathcal{N}_1} l_{1,j} \Big) - y_2^{OE} \Big(\sum_{j \in \mathcal{N}_2} l_{2,j} \Big)}{R - y_1^{S} \Big(\sum_{j \in \mathcal{N}_1} l_{1,j}^S \Big) - y_2^{S} \Big(\sum_{j \in \mathcal{N}_2} l_{2,j}^S \Big)} \qquad (E)$$

subject to

$$l_{i,j}^S \leq y_i^S (l_{i,j}^S)', \qquad i = 1, 2,\ j \in \mathcal{N}_i, \qquad (12)$$

$$\Big(\sum_{j \in \mathcal{N}_2} l_{2,j}^S \Big) + y_2^S \Big(\sum_{j \in \mathcal{N}_2} (l_{2,j}^S)' \Big) = \Big(\sum_{j \in \mathcal{N}_1} l_{1,j}^S \Big) + y_1^S \Big(\sum_{j \in \mathcal{N}_1} (l_{1,j}^S)' \Big), \qquad (13)$$

$$\Big(\sum_{j \in \mathcal{N}_1} l_{1,j}^S \Big) + y_1^S \Big(\sum_{j \in \mathcal{N}_1} (l_{1,j}^S)' \Big) \leq R, \qquad (14)$$

$$\sum_{i=1}^2 y_i^S = 1, \qquad (15)$$

$$l_{1,j} \leq l_{1,j}^S, \qquad \forall\ j \in \mathcal{N}_1, \qquad (16)$$

$$l_{i,j} \leq y_i^{OE} l'_{i,j}, \qquad i = 1, 2,\ j \in \mathcal{N}_i, \tag{17}$$

$$\sum_{i=1}^{2} y_i^{OE} = 1, \tag{18}$$

$$+ \text{ Oligopoly Equilibrium Constraints.} \tag{19}$$

Problem (E) can be viewed as a finite dimensional problem that captures the equilibrium and social optimum characteristics of the infinite dimensional problem given in (10). This implies that instead of optimizing over the entire function l_j for some $j \in \mathcal{N}_i$, $i \in \mathcal{I}$, we optimize over the possible values of $l_j(\cdot)$ and $l'_j(\cdot)$ at the equilibrium and the social optimum, which we denote by $l_{i,j}, l'_{i,j}, l^S_{i,j}, (l^S_{i,j})'$. The constraints of the problem guarantee that these values satisfy the necessary optimality conditions for a social optimum and an OE. In particular, conditions (12) and (17) capture the convexity assumption on $l_j(\cdot)$ by relating the values $l_{i,j}, l'_{i,j}$ and $l^S_{i,j}, (l^S_{i,j})'$ (note that the assumption $l_j(0) = 0$ is essential here). Condition (13) is the optimality condition for the social optimum. Condition (16) captures the nondecreasing assumption on the latency functions; since we are considering $l_j(\cdot)$ such that $x_1^{OE} \leq x_1^S$, we must have $l_{1,j} \leq l^S_{1,j}$ for all j. Finally, the last set of constraints are the necessary conditions for a pure strategy OE. In particular, for a two path network, using Proposition 4, the Oligopoly Equilibrium Constraints are given by

$$n_1 y_1^{OE}\Big[\sum_{j\in\mathcal{N}_1} l'_{1,j} + \sum_{j\in\mathcal{N}_2} l'_{2,j}\Big] + \sum_{j\in\mathcal{N}_1} l_{1,j}$$
$$= n_2 y_2^{OE}\Big[\sum_{j\in\mathcal{N}_1} l'_{1,j} + \sum_{j\in\mathcal{N}_2} l'_{2,j}\Big] + \sum_{j\in\mathcal{N}_2} l_{2,j},$$

[and therefore n_1 and n_2 are also decision variables in problem (E)]. Note that given any feasible solution of problem (10), we have a feasible solution for problem (E) with the same objective function value. Therefore, the optimum value of problem (E) is indeed a lower bound on the optimum value of problem (10).

Step 2: Consider the following change of variables for problem (E)

$$l_1^S = \sum_{j\in\mathcal{N}_1} l^S_{1,j}, \qquad l_2^S = \sum_{j\in\mathcal{N}_2} l^S_{2,j}$$
$$l_1 = \sum_{j\in\mathcal{N}_1} l_{1,j}, \qquad l_2 = \sum_{j\in\mathcal{N}_2} l_{2,j},$$

and rewrite problem (E) as

$$r_2^{OE} = \underset{\substack{l_i^S, (l_i^S)' \geq 0 \\ l_i, l_i' \geq 0 \\ y_i^S, y_i^{OE} \geq 0}}{\text{minimize}} \ \frac{R - l_1 y_1^{OE} - l_2 y_2^{OE}}{R - l_1^S y_1^S - l_2^S y_2^S} \tag{E'}$$

subject to

$$l_i^S \leq y_i^S (l_i^S)', \qquad i = 1, 2,$$
$$l_2^S + y_2^S (l_2^S)' = l_1^S + y_1^S (l_1^S)',$$
$$l_1^S + y_1^S (l_1^S)' \leq R,$$
$$\sum_{i=1}^{2} y_i^S = 1,$$
$$l_1 \leq l_1^S,$$
$$l_i \leq y_i^{OE} l'_i, \qquad i = 1, 2,$$
$$\sum_{i=1}^{2} y_i^{OE} = 1,$$
$$+ \text{ Oligopoly Equilibrium Constraints.}$$

Let $(\bar{l}_i^S, (\bar{l}_i^S)', \bar{l}_i, \bar{l}'_i, \bar{y}_i^S, \bar{y}_i^{OE})$ denote the optimal solution of problem (E'). We have shown in [1] that $\bar{l}_i^S = 0$ for $i = 1, 2$.

Step 3: Using $\bar{l}_i^S = 0$ for $i = 1, 2$ and $\bar{l}_1 = 0$, we see that

$$r_2^{OE} =$$

$$\begin{aligned} \underset{\substack{l_2, l_2' \\ y_1^{OE},\ y_2^{OE} \geq 0 \\ n_1,\ n_2 \geq 1}}{\text{minimize}} \quad & 1 - \frac{l_2 y_2^{OE}}{R} \\ \text{subject to} \quad & l_2 \leq y_2^{OE} l'_2, \\ & l_2 + n_2 y_2^{OE} l'_2 = n_1 y_1^{OE} l'_2, \\ & n_1 y_1^{OE} l'_2 \leq R. \\ & \sum_{i=1}^{2} y_i^{OE} = 1. \end{aligned}$$

It is straightforward to show that the optimal solution of this problem is $(\bar{l}_2, \bar{l}'_2, \bar{y}_1^{OE}, \bar{y}_2^{OE}) = (\frac{R}{2}, \frac{R}{2}, 0, 1)$, and therefore the optimum value is $r_2^{OE} = 1/2$. By (11), this implies that

$$\inf_{\{l_j\}} \inf_{x^{OE} \in \overrightarrow{OE}(\{l_j\})} r_2(\{l_j\}, x^{OE}) \geq \frac{1}{2}.$$

Finally, Example 1 shows that this bound is tight, i.e.,

$$\min_{\{l_j\}} \min_{x^{OE} \in \overrightarrow{OE}(\{l_j\})} r_2(\{l_j\}, x^{OE}) = \frac{1}{2}.$$

Q.E.D.

To see why efficiency is worse in this case than in the parallel-link topology considered in [2], suppose that in Example 1 all n links along path 1 are owned by the same service provider. This makes the example equivalent to a parallel-link topology. It is straightforward to see that in this case the unique OE flows are given by $x_1^{OE} = 2/3$ and $x_2^{OE} = 1/3$, and this example reaches the 5/6 bound of [2]. The reason for the substantially worse performance with multiple serial links is that each provider along path 1 has a greater incentive to increase its price (relative to the benchmark where all these links are owned by the same provider), because it does not internalize the reduction in the profits of the other link owners along the same path. Consequently, for a network with serial and parallel links, there are higher prices along path 1, and this induces greater fraction of users to choose path 2 and increases inefficiency.

B. Multiple Paths

We next consider the general case where we have an I path network, with n_i links on path i, where each link is owned by a different provider. The following example illustrates the efficiency properties of an I path network which has positive flows on all links at the OE.

Example 2: Consider an I path network, which has n links on path 1 with identically 0 latency functions and one link on each of the paths $2, \dots, I$ with the same latency function $l(x) = x(I-1)/2$. Let the total flow be $d = 1$ and the reservation utility be $R = 1$.

Clearly, the unique social optimum for this example is $x^S = [1, 0, \dots, 0]$. Using Proposition 4 and the definition of a WE, it can be seen that the flow allocation at the unique OE is

$$x^{OE} = \Big[\frac{2/n}{1+2/n}, \frac{1}{(I-1)(1+2/n)}, \dots, \frac{1}{(I-1)(1+2/n)}\Big].$$

Hence the efficiency metric for this example is

$$r_I(\{l_j\}, x^{OE}) = 1 - \frac{1}{2}\left(\frac{1}{1+2/n}\right)^2,$$

which goes to 1/2 as $n \to \infty$.

The next theorem generalizes Theorem 1. The proof uses similar ideas to the proof of Theorem 1 and is omitted for brevity.

Theorem 2: Consider a general I path parallel link network, with n_i links on path i, where link j, $j \in \mathcal{N}_i$, has a latency function l_j. Assume that each link is owned by a different provider and the price competition game has a pure strategy OE. Then

$$r_I(\{l_j\}, x^{OE}) \geq \frac{1}{2}, \qquad \forall\, x^{OE} \in \overrightarrow{OE}(\{l_j\}). \tag{20}$$

Moreover, the bound above is tight, i.e., there exists $\{l_j\}$ and $x^{OE} \in \overrightarrow{OE}(\{l_j\})$ that attains the lower bound in Eq. (20).

C. Positive Latency at 0 Congestion

We finally study the implications of the assumption $l_i(0) = 0$ on efficiency. Consider the following example:

Example 3: Consider a two path network, which has n links on path 1 with identically 0 latency functions and one link on path 2 with latency function $l(x_2) = \epsilon x_2 + b$ for some scalars $\epsilon > 0$ and $b > 0$. Again the unique social optimum is $\bar{x}^S = (1, 0)$. The flows at the unique OE are given by

$$\bar{x}^{OE} = \left(\frac{2\epsilon + b}{\epsilon(n+2)}, \frac{\epsilon n - b}{\epsilon(n+2)}\right).$$

Let $\epsilon = b/\sqrt{n}$. Then, as $b \to 1$ and $n \to \infty$, we have that $\bar{x}^{OE} \to (0, 1)$, and the efficiency metric $r_2(\{l_j\}, x^{OE}) \to 0$.

This example shows that the efficiency loss could be arbitrarily high for a network that involves parallel and serial links if the assumption $l_i(0) = 0$ is relaxed.

Interestingly, it is straightforward to see that in the same example with the parallel-link topology (i.e., all n links along path 1 owned by the same provider), we would have $x^{OE} = \left(\frac{b+2\epsilon}{3\epsilon}, \frac{\epsilon-b}{3\epsilon}\right)$ if $\epsilon \geq b$ and $x^{OE} = (1, 0)$ otherwise. Consequently, $b \to 1$ and $\epsilon \to 0$, we have that $x^{OE} \to (1, 0)$, and $r_2(\{l_j\}, x^{OE}) \to 1$. Therefore, the highly inefficient equilibrium is a result of the parallel-serial topology, not of the assumption that there is positive latency at 0 congestion. This fact was established in [1], i.e., with parallel topology but positive latency at zero congestion, there is a tight bound of $2\sqrt{2} - 2$ on efficiency, which is quite close to, but slightly lower than 5/6.

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