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Efficient Market Mechanisms for Network Resource Allocation

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Abstract— **We study the interaction between buyers and sellers of several indivisible goods (or items). A buyer wants a combination of items while each seller offers only one type of item. The setting is motivated by communication networks in which buyers want to construct routes using several links and sellers offer transmission capacity on individual links. Agents are strategic and may not be truthful, so a competitive equilibrium may not be realized. To ensure a good outcome among strategic agents, we propose a combinatorial double auction. We show that a Nash equilibrium exists for the associated game with complete information, and more surprisingly, the resulting allocation is efficient. We then consider competitive analysis in the continuum model of the auction setting and show that the auction outcome is a competitive equilibrium.**

I. Introduction

We study the interaction among buyers and sellers of several *indivisible goods* (or *items*). The motivation is to investigate the strategic interaction between internet service providers who lease transmission capacity (or bandwidth) from owners of individual links to form desired routes. Bandwidth is traded in indivisible amounts, say multiples of 100 Mbps. Thus, the buyers want bandwidth on combinations of several links available in multiples of some indivisible unit. This makes the problem *combinatorial*. We consider the interaction in several settings.

The setting of a conventional market economy, in which there is perfect competition, was considered in [8]. It was shown that the interaction among agents results in a competitive equilibrium if their utilities are linear in bandwidth (and money) and they truthfully reveal them, and the desired routes form a tree. The latter requirement is needed for the existence of an equilibrium in the presence of indivisibility.

Strategic agents, however, have an incentive not to be truthful. We propose a 'combinatorial sellers' bid double auction' (c-SeBiDA) mechanism that achieves a socially desirable interaction among strategic agents. The mechanism requires both buyers and sellers to make bids. It is *combinatorial* because buyers make bids on combinations of items, such as several links that form a route. Each seller, however, offers to sell only a single type of item, e.g., bandwidth on a single link. The mechanism takes all buy and sell bids, solves a mixed-integer program that matches bids to maximize the social surplus, and announces prices at which the matched (i.e., accepted) bids are settled. The settlement price for a link is the highest price asked by a matched seller (hence 'sellers' bid' auction). As a result there is a *uniform* price for each item.

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The outcome of strategic behavior in the auction is modelled as a Nash equilibrium. It is shown that under complete information a Nash equilibrium exists; it is not generally a competitive equilibrium. Nevertheless, the Nash equilibrium is efficient. Moreover, it is a dominant strategy for all buyers and for all sellers except the matched seller with the highest-ask price to be truthful.

Following Aumann [2], we then consider the continuum model. It was shown in [8] that a competitive equilibrium exists in a continuum exchange economy with indivisible goods and money (a divisible good). Here we show that the c-SeBiDA auction outcome is a competitive equilibrium [18] in the continuum model even without money. This is accomplished by casting the mechanism in an optimal control framework and appealing to Pontryagin's maximum principle to conclude existence of competitive prices. This suggests that the auction outcome in a finite setting approximates a competitive equilibrium in the continuum model (see [1] for approximate competitive equilibrium). The proposed mechanism has been implemented in a web-based software testbed and available for use (see `http://auctions.eecs.berkeley.edu`).

Previous Work and Our Contribution

When items are indivisible, a competitive equilibrium may not exist. However, when the utility functions are linear and the demand-supply constraint matrix has a special structure (such as the totally unimodular property [8]), a competitive equilibrium does exist [29]. However, the realization of the competitive equilibrium still requires agents to truthfully report their utilities. But strategic agents (aware of their 'market power') may not be truthful. Thus, many auction mechanisms are designed to elicit truthful reporting following Vickrey's fundamental result [28].

Attention in the auction theory literature has focused on one-sided, single-item auctions [14] but combinatorial bids arise in many contexts, and a growing body of research is devoted to combinatorial auctions [29]. The interplay between economic, game-theoretic and computational issues has sparked interest in algorithmic mechanism design [22]. Some iterative, ascending price combinatorial auctions achieve efficiencies close to the Vickrey auction [3], [19], [24]. It is however well-known that generalized Vickrey auction mechanisms for multiple heterogeneous items may not be computationally tractable [22], [20]. Thus, mechanisms which rely on approximation of the integer program (though with restricted strategy spaces such as "bounded"

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or "myopic rationality") [20] or linear programming (when there is a particular structure such as "gross" or "agent substitutability") [4] have been proposed.

In [5] one of the first multi-item auction mechanisms is introduced. However, it is not combinatorial and consideration is only given to computation of equilibria among truth-telling agents. An auction for single items is presented in [25]. It is similar in spirit to what we present but cannot be generalized to multiple items. In [31] a modified Vickrey double auction with participation fees is presented, while [6] considers truthful double auction mechanisms and obtains upper bounds on the profit of any such auction. But the setting in both [6], [25] is non-combinatorial since each bid is for an individual item only.

Ours is one of few proposals for a *combinatorial double auction* mechanism. It appears to be the only combinatorial market mechanism for strategic agents with unrestricted strategy spaces. We are able to achieve efficient allocations. Furthermore, the mechanism's linear integer program structure makes the computation manageable for many practical applications [11].

The results here also relate to recent efforts in the network pricing [12], [16], [26] and congestion games literature [15], [23]. There is an ongoing effort to propose mechanisms for network resource allocation through auctions [13] and to understand the worst case Nash equilibrium efficiency loss of such mechanisms when users act strategically [10], [17]. Optimal mechanisms that minimizes this efficiency loss has also been proposed [30] though not extended to the case of multiple items. Most of this literature regards the good (in this case, bandwidth) as divisible, with complete information for all players. The case of indivisible goods or incomplete information case is harder. This paper considers indivisible goods and combinatorial buy-bids. The case of incomplete information is presented in [9].

The results in this paper are significant from several perspectives. It is well known that the only known positive result in the mechanism design theory is the VCG class of mechanisms [18]. The generalized Vickrey auction (GVA) (with complete information) is ex post individual rational, dominant strategy incentive compatible and efficient. It is however not budget-balanced. The incomplete information version of GVA (dAGVA) is Bayesian incentive compatible, efficient and budget-balanced. It is, however, not ex post individual rational. Indeed, there exists no mechanism which is efficient, budget-balanced, ex post individual rational and dominant strategy incentive compatible (Hurwicz impossibility theorem). Moreover, there exists no mechanism which is efficient, budget-balanced, ex post individual rational and Bayesian incentive compatible (Myerson-Satterthwaite impossibility theorem).

In this paper, we provide a non-VCG combinatorial (market) mechanism which in the complete information case is always efficient, budget-balanced, ex post individual rational and "almost" dominant strategy incentive compatible. In the incomplete information case, it is budget-balanced, ex post individual rational and asymptotically efficient and Bayesian incentive compatible.

Moreover, we have shown that *any* Nash equilibrium allocation (say of a network resource allocation game) is always efficient (zero efficiency loss) and *any* Bayesian-Nash equilibrium allocation is asymptotically efficient. This seems to be the only known combinatorial double-auction mechanism with these properties.

It is worth noting that a one-sided auction is a special case of a double auction when there is only one seller with zero costs. The network and congestion games [12], [15] are all one-sided auctions.

The second result we present in this paper concerns the competitive analysis of the c-SeBIDA auction mechanism. We considered the continuum model and showed that within that model c-SeBiDA outcome is a competitive equilibrium. This suggests that in the finite setting, the auction outcome is close to efficient. showed that a competitive equilibrium exists in a continuum model We have tested the proposed mechanism c-SeBiDA through human-subject experiments. Those results can be found elsewhere [11].

II. The Combinatorial Sellers' Bid Double Auction

In this section, we present the combinatorial seller's bid double auction (c-SeBiDA) mechanism.

A buyer places buy bids for a bundle of items such as a set of links that form a route. A buyer's bid is *combinatorial*: he must receive all items in his bundle or nothing. A buy-bid consists of a buy-price per unit of the bundle and maximum demand, the maximum units of the bundle that the buyer needs. On the other hand, each seller makes *non-combinatorial* bids. A sell-bid consists of an ask-price and maximum supply, the maximum units the seller offers for sale.

The mechanism collects all announced bids, matches a subset of these to maximize the 'surplus' (equation (1), below) and declares a settlement price for each item at which the matched buy and ask bids—which we call the winning bids—are transacted. This constitutes the payment rule. As will be seen, each matched buyer's buy bid is larger, and each matched seller's ask bid is smaller than the settlement price, so the outcome respects individual rationality.

There is an asymmetry: buyers make multi-item combinatorial bids, but sellers only offer one type of item. This yields uniform settlement prices for each item.

Players' bids may not be truthful. They know how the mechanism works and formulate their bids to maximize their individual returns.

A player can make multiple bids. The mechanism treats these as XOR bids, so at most one bid per player is a winning bid. Therefore the outcome is the same as if a matched player only makes (one) winning bid. Thus, in the formal description of the *combinatorial sellers' bid double auction* (c-SeBiDA), each player places only one bid. c-SeBiDA is a 'double' auction because both buyers and sellers bid; it is a 'sellers' bid' auction because the settlement price depends only on the matched sellers' bids, as we will see.

Formal mechanism.

There are L items $l_1, \cdots, l_L$, m buyers and n sellers. Buyer i has (true) reservation value v_i per unit for a bundle of items $R_i \subseteq \{l_1, \cdots, l_L\}$, and submits a buy bid of b_i per unit and demands up to δ_i units of the bundle R_i. Thus, the buyers have quasi-linear utility functions of the form $u_i^b(x; \omega, R_i) = \bar{v}_i(x) + \omega$ where ω is money and

$$\bar{v}_i(x) = \begin{cases} x \cdot v_i, & \text{for } x \leq \delta_i, \\ \delta_i \cdot v_i, & \text{for } x > \delta_i. \end{cases}$$

Seller j has (true) per unit cost c_j and offers to sell up to σ_j units of l_j at a unit price of a_j. Denote $L_j = \{l_j\}$. Again, the sellers have quasi-linear utility functions of the form $u_j^s(x; \omega, L_j) = -\bar{c}_j(x) + \omega$ where ω is money and

$$\bar{c}_j(x) = \begin{cases} x \cdot c_j, & \text{for } x \leq \sigma_j, \\ \infty, & \text{for } x > \sigma_j. \end{cases}$$

The mechanism receives all these bids, and matches some buy and sell bids. The possible matches are described by integers x_i, y_j: $0 \leq x_i \leq \delta_i$ is the number of units of bundle R_i allocated to buyer i and $0 \leq y_j \leq \sigma_j$ is the number of units of item l_j sold by seller j.

The mechanism determines the allocation (x^*, y^*) as the solution of the surplus maximization problem **MIP**:

$$\begin{aligned} \max_{x,y} \quad & \textstyle\sum_i b_i x_i - \sum_j a_j y_j & (1) \\ s.t. \quad & \textstyle\sum_j y_j \mathbb{1}(l \in L_j) - \sum_i x_i \mathbb{1}(l \in R_i) \geq 0, \\ & \forall l \in [1:L], x_i \in [0:\delta_i], \forall i, \quad y_j \in [0, \sigma_j], \forall j, \end{aligned}$$

where $\mathbb{1}(\cdot)$ is the indicator function. MIP is a mixed integer program: Buyer i's bid is matched up to his maximum demand δ_i; Seller j's bid will also be matched up to his maximum supply σ_j. x_i^* is constrained to be integral; y_j^* will be integral due to the demand less than equal to supply constraint.

The settlement price is the highest ask-price among matched sellers,

$$\hat{p}_l = \max\{a_j : y_j^* > 0, l \in L_j\}. \tag{2}$$

The payments are determined by these prices. Matched buyers pay the sum of the prices of items in their bundle; matched sellers receive a payment equal to the number of units sold times the price for the item. Unmatched buyers and sellers do not participate. This completes the mechanism description.

If i is a matched buyer ($x_i^* > 0$), it must be that his bid $b_i \geq \sum_{l \in R_i} \hat{p}_l$; for otherwise, the surplus (1) can be increased by eliminating the corresponding matched bid. Similarly, if j is a matched seller ($y_j^* > 0$), and $l \in L_j$, his bid $a_j \leq \hat{p}_l$, for otherwise the surplus can be increased by eliminating his bid. Thus the outcome of the auction respects *individual rationality*.

It is easy to understand how the mechanism picks matched sellers. For each item j, a seller with lower ask bid will be matched before one with a higher bid. So sellers with bid $a_j < \hat{p}_l$ sell all their supply ($y_j^* = \sigma_j$). At most one seller with ask bid $a_j = \hat{p}_l$ sells only a part of his total supply ($y_j^* < \sigma_j$). On the other hand, because their bids are combinatorial, the matched buyers are selected only after solving the MIP.

The proposed mechanism resembles the k-double auction mechanism [25]. We designed c-SeBiDA so that its outcome mimics a competitive equilibrium with a particular interest in the combinatorial case. It was later discovered that the single item version SeBiDA resembles the k-double auction (a special case being called the buyer's bid double auction [25], [27]). But the two mechanisms differ in how the prices are determined. It is not clear what a generalization of the k-double auction would be to the combinatorial case. Moreover, as we will see SeBiDA has certain incentive-compatibility properties lacking in the k-double auction.

III. Nash Equilibrium Analysis: c-SeBiDA is Efficient

We first focus on how strategic behavior of players affects price when they have complete information. We will assume that players don't strategize over the quantities (namely, δ_i, σ_j), which will be considered fixed in the players' bids. A strategy for buyer i is a buy bid b_i, a strategy for seller j is an ask bid a_j. Let θ denote a collective strategy. Given θ, the mechanism determines the allocation (x^*, y^*) and the prices $\{\hat{p}_l\}$. So the payoff to buyer i and seller j is, respectively,

$$u_i^b(\theta) = \bar{v}_i(x_i^*) - x_i^* \cdot \sum_{l \in R_i} \hat{p}_l, \tag{3}$$

$$u_j^s(\theta) = y_j^* \cdot \sum_{l \in L_j} \hat{p}_l - \bar{c}_j(y_j^*). \tag{4}$$

The bids b_i, a_j may be different from the true valuations v_i, c_j, which however figure in the payoffs.

A collective strategy θ^* is a *Nash equilibrium* if no player can increase his payoff by unilaterally changing his strategy.

We now construct a Nash equilibrium for the game described by (1)-(4) for multiple items with single unit bids.

Theorem 1: (i) A Nash equilibrium (b^*, a^*) exists in the c-SeBiDA game. (ii) Except for the matched seller with the highest bid on each item, it is a dominant strategy for each player to bid truthfully. (iii) Any Nash equilibrium allocation is efficient.

Proof: For the sake of clarity, we change some of the notation. As before, buyer i demands the bundle R_i with reservation value v_i. Let seller (l, j) be the j-th seller offering item l ($l \in L_j$ in the previous notation) with reservation cost $c_{l,j}$, and assume $c_{l,1} \leq \cdots \leq c_{l,n_l}$, in which n_l is the number of sellers offering item l.

We will iteratively construct a set of strategies to consider as Nash equilibrium.

Set $a_{l,0} = c_{l,0} = 0, b_0 = v_0 = 1$. Consider the surplus maximization problem (1) with true valuations and costs. Let I be the set of matched buyers and k_l the number of matched sellers offering item l determined by the **MIP**. Set $b_i^* = v_i$ for all i; $a_{l,j}^0 = c_{l,j}$; $\gamma_i^t = b_i^* - \sum_{l \in R_i} a_{l,k_l}^t$, the surplus of a matched buyer i at stage $t \geq 0$, and

$$\hat{l} \in \arg\min_{l}\{\min_{i \in I: l \in R_i} \gamma_i^t : \gamma_i^t > 0\}, \tag{5}$$

the item with the smallest surplus among the matched buyers at stage t. Denote the corresponding surplus by $\gamma_{\hat{l}}^t$. Now, define

$$a_{\hat{l},k_{\hat{l}}}^{t+1} := \min\{a_{\hat{l},k_{\hat{l}}+1}^t, a_{\hat{l},k_{\hat{l}}}^t + \gamma_{\hat{l}}^t\}, \tag{6}$$

which is the strategy of seller $(\hat{l}, k_{\hat{l}})$ at the t-th stage: His ask bid is increased to decrease the surplus of the matched buyer with the smallest surplus up to the ask bid of the unmatched seller with the lowest bid. For all other $(l, j) \neq (\hat{l}, k_{\hat{l}})$, the ask bid remains the same, $a_{l,j}^{t+1} = a_{l,j}^t$. This procedure is repeated until the strategies converge with each $\hat{l}$ being picked only once. In fact, it is repeated at most L times. Observe that at each stage, the matches and the allocations from the MIP using the current bids (b^*, a^t) do not change. Let a^* denote the seller ask bids when the procedure converges.

We prove that (b^*, a^*) is a Nash equilibrium, by showing that no player has an incentive to deviate.

First, an unmatched seller offering item l has no incentive to bid lower than a_{l,k_l}^*: Because his reservation cost is higher than that, by bidding lower than his reservation cost, it may get matched but his payoff will be negative. Next, consider a matched seller $(l, j) \neq (l, k_l)$ offering item l. By bidding higher or lower he cannot change the price of the item but may end up getting unmatched. Thus, it is the dominant strategy of all sellers except the 'marginal' seller (l, k_l) to bid truthfully.

Now, consider this marginal matched seller (l, k_l). If he bids lower then a_{l,k_l}^*, his payoff will decrease. He could bid higher but because of (6), either there is an unmatched seller of the item with the same ask bid, or there is a marginal buyer whose surplus has been made zero by (6). So if he bids higher than a_{l,k_l}^*, either he will become unmatched and the first unmatched seller of the item will become matched, or the 'marginal' buyer with zero surplus will become unmatched causing this marginal seller to be unmatched as well. Thus, a_{l,k_l}^* is a Nash strategy of the marginal seller given that all other players (except the marginal sellers of the other items) bid truthfully.

Now, consider the buyers. First, an unmatched buyer i has no incentive to bid lower than b_i^* since he wouldn't match anyway. And if he bids higher, he may become matched but his payoff will become negative. Next, a matched buyer with a positive payoff has no incentive to bid lower since by bidding lower he can lower the prices but only when he becomes unmatched. Also, he certainly has no incentive to bid higher since by so doing he will not be able to lower the price. Lastly, consider the 'marginal' matched buyers with zero payoff: Clearly, if they bid higher, their payoff will become negative; and if they bid lower, they will become unmatched. Thus, it is the dominant strategy of all buyers to bid truthfully.

The Nash equilibrium allocation (x^*, y^*) as determined above is efficient since it maximizes (1).

We now show that any Nash equilibrium allocation is efficient.

Suppose $(\tilde{x}, \tilde{y})$ is another Nash equilibrium allocation which is not efficient. Either there is a buyer or a seller which goes from being matched in (x^*, y^*) to being unmatched in $(\tilde{x}, \tilde{y})$, or vice-versa. If there is a seller that goes from being matched to unmatched then either there is a matched seller in (x^*, y^*) replaced by another seller in $(\tilde{x}, \tilde{y})$ selling the same item (case (i)), or some unmatched sellers in (x^*, y^*) are matched in $(\tilde{x}, \tilde{y})$ with the set of matched sellers in (x^*, y^*) remaining matched. In this case, some unmatched buyer must also become matched (case (ii)). The rest of the cases can be argued similarly. Thus, the two Nash equilibrium allocations would differ in one of the five cases as we go from (x^*, y^*) to $(\tilde{x}, \tilde{y})$.

(i) A matched seller (l, j_1) is made unmatched and a unmatched seller (l, j_2) is made matched;

(ii) An unmatched buyer i demanding R_i is made matched and a set of unmatched sellers J such that $\{l : (l, j_l) \in J\} = R_i$ are made matched;

(iii) A matched buyer i demanding R_i is made unmatched and a set of matched sellers J such that $\{l_j : j \in J\} = R_i$ are made unmatched;

(iv) An unmatched buyer i demanding R_i is made matched and a set of matched buyers J with $j \in J$ demanding R_j such that $\cup_{j \in J} R_j = R_i$ are made unmatched;

(v) A matched buyer i demanding R_i is made unmatched and a set of unmatched buyers J with $j \in J$ demanding R_j such that $\cup_{j \in J} R_j = R_i$ are made matched;

Case (i) We must have $c_{l,j_1} < c_{l,j_2}$ and the new bids must satisfy $\tilde{a}_{l,j_2} < \tilde{a}_{l,j_1}$. But then either (l, j_2)'s payoff is negative or (l, j_1) can also bid just above (l, j_2)'s bid. In either case $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium allocation.

Case (ii) We must have $v_i < \sum_{(l,j_l) \in R_i} c_{l,j_l}$ and the new bids must satisfy $\tilde{b}_i > \sum_{(l,j_l) \in R_i} \tilde{a}_{l,k_l}$ with $\tilde{a}_{l,j_l} < \tilde{a}_{l,k_l}$. This means that either the buyer or at least one seller has a negative payoff. Thus, $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium allocation.

Case (iii) The argument for this case is similar to case (ii).

Case (iv) We must have $v_i < \sum_{j \in J} v_j$ and the new bids must satisfy $\tilde{b}_i > \sum_{j \in J} \tilde{b}_j$. But then either i's payoff is negative or any $j \in J$ can bid high enough to outbid i. In either case $(\tilde{x}, \tilde{y})$ cannot be a Nash equilibrium allocation.

Case (v) The argument for this case is similar to case (iv).

Thus, every Nash equilibrium allocation is efficient. This proves (iii).

■

It is obvious that if the minimum in step (5) is not unique, the Nash equilibrium will not be unique. However, any Nash equilibrium allocation will still be efficient.

IV. Competitive Analysis of c-SeBiDA in the Continuum Model

We now present competitive analysis of the c-SeBiDA mechanism. Since competitive equilibria may not exist for the setting considered, we investigate the behavior of the outcome of the c-SeBiDA auction when the number of players is large enough such that no single player by itself can affect the outcome. An idealization is a continuum of agents. Such a setting was first considered by Aumann [2] in a general equilibrium setting and others have used this approach to analysis of games [7].

Assume the continuum of buyers is indexed by $t \in [0,1]$, and the continuum of sellers is indexed by $\tau \in [0,1]$. There are m types of buyers and n types of sellers. Let $B_1, \cdots, B_m$ and $S_1, \cdots, S_n$ partition $[0,1]$ so that all buyers in B_i demand the same set of items R_i (corresponding say to a route), and all sellers in S_j offer the same item l_j, $L_j = \{l_j\}$. We assume that the partitions B_i's and S_j's are subintervals.

A buyer $t \in B_i$ has true value $v(t)$, bids $p(t)$ per unit for the set R_i, and demands $\delta(t) \in [0, D]$ units. Suppose $v(t), p(t) \in [0, V]$. A seller $\tau \in S_j$ has true cost $c(\tau)$ and asks $q(\tau)$ for the item(s) L_j with supply $\sigma(\tau) \in [0, S]$ units, with $c(\tau), q(\tau) \in [0, C]$. Let $x(t)$ and $y(\tau)$ be the decision variables, i.e. buyer t's $x(t)$ is 1, if his bid is accepted, 0 otherwise. And similarly seller τ's $y(\tau)$ is 1 if his offer is accepted, 0 otherwise. We assume that within each partition B_i, the buyers' bid function $b(t)$ is non-increasing, and within each partition S_j, the sellers' bid function $q(\tau)$ is nondecreasing.

Note that while in section II, we assumed that buyers specify a maximum demand and they may be allocated any integral units up to the maximum demand, here we will assume that their bundles are *all-or-none* kind: All demand must be met or none. Similarly, for sellers, all supply must be accepted or none.

Denote the indicator function by $\mathbb{1}(\cdot)$ and as before, consider the surplus maximization problem **cLP**:

$$\begin{aligned} \sup_{x,y} \quad & \int_0^1 \sum_{i=1}^m x(t)\delta(t)p(t)\mathbb{1}(t \in B_i)dt \\ & - \int_0^1 \sum_{j=1}^n y(\tau)\sigma(\tau)q(\tau)\mathbb{1}(\tau \in S_j)d\tau \\ \text{s.t.} \quad & \int_0^1 \sum_{j=1}^n y(\tau)\sigma(\tau)\mathbb{1}(l \in L_j, \tau \in S_j)d\tau \\ & - \int_0^1 \sum_{i=1}^m x(t)\delta(t)\mathbb{1}(l \in R_i, t \in B_i)dt \geq 0, \\ & \forall l \in [1:L],\ x(t), y(\tau) \in \{0,1\}, \forall t, \tau \in [0,1]. \end{aligned} \tag{7}$$

The mechanism determines $((x^*, y^*), \hat{p})$ where (x^*, y^*) is the solution of the above continuous linear integer program and for each $l \in [1:L]$,

$$\hat{p}_l = \sup\{q(\tau) : y(\tau) > 0, \tau \in S_l\}, \tag{8}$$

and

$$\check{p}_l = \inf\{q(\tau) : y(\tau) = 0, \tau \in S_l\}. \tag{9}$$

The mechanism announces prices $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_L)$; the matched buyers (those for which $x^*(t) = 1$) pay the sum of the prices of the items in their bundle while the matched sellers (those for which $y^*(\tau) = 1$) get a payment equal to the number of their items sold times the price of the item. When buyers and sellers bid truthfully, the following result holds.

Theorem 2: If the bid function of the sellers $q : [0,1] \to [0, C]$ is continuous and nondecreasing in each partition S_j of $[0,1]$, then (x^*, y^*) is a competitive allocation and $\hat{p}$ is a competitive price.

Proof: We first show the existence of (x^*, y^*) and $(\lambda_1^*, \cdots, \lambda_L^*)$, the dual variables corresponding to the demand less than equal to supply constraints. We do this by casting the cLP above as an optimal control problem and then appeal to Pontryagin's maximum principle [21]. Define

$$\begin{aligned} \dot{\zeta}(t) = \quad & \Sigma_{i=1}^m x(t)\delta(t)p(t)\mathbb{1}(t \in B_i) \\ & -\Sigma_{j=1}^n y(t)\sigma(t)q(t)\mathbb{1}(t \in S_j), \\ \dot{\xi}_l(t) = \quad & \sum_{j=1}^n y(t)\sigma(t)\mathbb{1}(l \in L_j, t \in S_j) \\ & - \sum_{i=1}^m x(t)\delta(t)\mathbb{1}(l \in R_i, t \in B_i), \\ \theta(t) = \quad & (\xi_1(t), \cdots, \xi_L(t), \zeta(t))', \end{aligned}$$

respectively, where θ is the state of the system, x and y are controls, and $\zeta(t)$ and $\xi(t)$ describe the state evolution as a function of the controls. The objective is to find the optimal control (x^*, y^*) which maximizes $\zeta(1)$. Let $\Sigma(t)$ denote

$$\{\dot{\theta}(t) : x_l(0) = 0, \forall l \text{ and } x(t), y(t) \in \{0,1\}, \forall t \in [0,1]\}.$$

Observe that $\Sigma(t)$ has cardinality at most 2^{L+1} in $\mathbb{R}^{L+1}$. $\int_0^1 \Sigma(\tau)d\tau$ is the set of reachable states under the set of all allowed control functions, namely, all measurable functions x and y such that $x(\tau), y(\tau) \in \{0,1\}$. Note that $\zeta(1)$ defines our total surplus; i.e., buyer surplus minus seller surplus, and $\xi_l(1)$ defines the excess supply for item l; i.e., total supply minus total demand for item l. Define

$$\Gamma := \{\theta(1) \in \mathbb{R}^{L+1} : \theta(1) \in \int_0^1 \Sigma(\tau)d\tau, \xi_l(1) \geq 0, \forall l\},$$

the set of final reachable states under all control functions such that state evolution happens according to the equations above, and excess supply is non-negative.

Lemma 1: Γ is a compact, convex set.

Proof: By assumption, $\delta(t), p(t), \sigma(t)$, and $q(t)$ are bounded. By Lyapunov's theorem [2], $\int_0^1 \Sigma(\tau)d\tau$ is a closed and convex set. Since x and y are bounded functions, the integral is bounded as well. Thus, it is also compact. Moreover, $\xi_l(1)$ is a hyperplane, and $\xi(1) \geq 0$ defines a closed subset of $\mathbb{R}^L$. Therefore, $\{\theta(1) : \theta(1) \in \int_0^1 \Sigma(\tau)d\tau\} \bigcap \{\theta(1) : \xi_l(1) \geq 0, l = 1, \cdots, L\}$ is a compact, convex set. ∎

Now, our optimal control problem is: $\sup_{\theta(1) \in \Gamma} \zeta(1)$. But observe that one component of $\theta(1)$ is $\zeta(1)$. Since Γ is compact and convex, the supremum is achieved and an optimal control (x^*, y^*) exists in Γ. By the maximum principle [21], there exist adjoint functions $p_0^*(t)$ and $p_l^*(t)$,

$l = 1, \cdots, L$ such that $\dot{p}_0^*(t) = 0$, and $\dot{p}_l^*(t) = 0$, (i.e., $p_l^*(t) = \lambda_l^*$, a constant) for $l = 0, \cdots, L$.

Defining the Lagrangian over the objective function and the demand less than equal to supply constraint

$$L(x, y; \lambda) = \zeta(1) + \sum_{l=1}^{L} \lambda_l \xi_l(1), \tag{10}$$

we get from the saddle-point theorem,

$$L(x, y; \lambda^*) \leq L(x^*, y^*; \lambda^*) \leq L(x^*, y^*; \lambda). \tag{11}$$

We use this saddle point inequality to conclude the existence of a competitive equilibrium.

Lemma 2: If $((x^*, y^*), \lambda^*)$ is a saddle point satisfying the inequality (11) above, then the λ^* are competitive equilibrium prices. Moreover, $\hat{p}_l \leq \lambda_l^* \leq \check{p}_l, \forall l = 1, \cdots, L$.

Proof: Let $((x^*, y^*), \lambda^*)$ be the saddle point satisfying the above inequality. Rewrite the Lagrangian $L(x, y; \lambda)$ as

$$\sum_{i=1}^{m} \int_{B_i} \delta(t) x(t) (p(t) - \sum_{l \in R_t} \lambda_l) dt$$

$$+ \sum_{j=1}^{n} \int_{S_j} \sigma(\tau) y(\tau) (\lambda_{l(\tau)} - q(\tau)) d\tau$$

where $l(\tau)$ is the item offered by seller τ. Now, using the first saddle-point inequality, we get that $x^*(t) = \mathbb{1}(p(t) > \Sigma_{l \in R_t} \lambda_l^*)$ and $y^*(\tau) = \mathbb{1}(q(\tau) < \lambda_{l(\tau)}^*)$, which implies that the Lagrange multipliers are competitive equilibrium prices. To prove the second part, note that by definition, for a given τ, $y(\tau) > 0$ implies that $q(\tau) \leq \lambda_l^*$ for $\tau \in S_l$, which implies the first inequality. Again from definition, we get that $y(\tau) = 0$ implies that $q(\tau) \geq \lambda_l^*$ for $\tau \in S_l$, which implies the second inequality. ■

To conclude the proof of the theorem, we observe that if q is continuous and non-decreasing in each interval S_j of $[0, 1]$, then $\hat{p}_l = \check{p}_l$ for each l, which then equals λ_l^* by Lemma 2. ■

The implication of this result is that as the number of players becomes large, the outcome of the above auction approximates the competitive equilibria of the associated continuum exchange economy. We will defer discussion of the relationship between the Nash equilibria and the competitive equilibria to the conclusions section.

We now show that the assumption that the sellers' bid function is piecewise continuous and nondecreasing is necessary for the c-SeBiDA's price to be a competitive price.

Example 1: Suppose that there is only one item. Buyers $t \in [0, 0.5]$ have reservation value 3 while buyers $t \in (0.5, 1]$ have reservation value 4. Sellers $t \in [0, 0.5]$ have reservation cost 5 while sellers $t \in (0.5, 1]$ have reservation cost 2. Then, it is clear that the buyers in $(0.5, 1]$ and sellers $(0.5, 1]$ will be matched with surplus $0.5 \times 2 = 1$. Thus, $\hat{p} = 2$ which is not equal to $\check{p} = 3$. As can be easily checked, the competitive price is $\lambda^* = 3$ different from $\hat{p}$.

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