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Integral control in the presence of hysteresis: an input-output approach

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***Abstract*— Using an input-output approach, it is shown that under certain mild and natural assumptions, application of integral control to the series interconnection of a hysteretic input nonlinearity, an L^2-stable, time-invariant linear system and a non-decreasing globally Lipschitz static output nonlinearity guarantees tracking of constant reference signals, provided the positive time-dependent integrator gain is ultimately smaller than a certain constant determined by a positivity condition in the frequency domain. The input-output result is applied in a general state-space setting wherein the linear component of the interconnection is given by a strongly stable well-posed infinite-dimensional system.**

I. INTRODUCTION

Consider the system shown in Fig. 1, where u is the input, Φ is a hysteresis nonlinearity, G is an L^2-stable time-invariant linear system, the signal $g \in L^2(\mathbb{R}_+)$ models the effect of non-zero initial conditions of the system with input-output operator G, ψ is a non-decreasing globally Lipschitz static nonlinearity and y is the output. The operator Φ belongs to a class of hysteresis operators with certain natural monotonicity and Lipschitz continuity properties and which contains, in particular, backlash, elastic-plastic and, more generally, operators of Prandtl and Preisach type.

It is shown that applying integral control to the system in Fig. 1 guarantees tracking of constant reference signals, in the presence of output disturbances, provided that a number of natural assumptions are satisfied. In particular, it is assumed that (a) the steady-state gain of the linear part of the plant is positive, (b) the positive time-dependent integrator gain is ultimately smaller than some constant determined by a positivity condition in the frequency domain, (c) the output disturbance is of a particular class which encompasses sums of constant signals and weighted L^2-signals and (d) the reference value is feasible in a natural sense to be made precise in due course. This input-output result is applied in a general state-space setting wherein the linear component of the interconnection is given by a strongly stable well-posed infinite-dimensional system. Our results complement and substantially extend earlier work in [4], where a state-space approach to low-gain integral control of exponentially stable regular infinite-dimensional systems with input hysteresis was developed.

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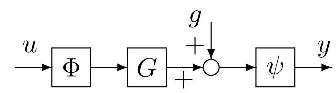

Fig. 1.

II. A CLASS OF HYSTERESIS OPERATORS

A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *time transformation* if it is continuous and non-decreasing with $f(0) = 0$ and $\lim_{t\to\infty} f(t) = \infty$; in other words, f is a time transformation if it is continuous, non-decreasing and surjective. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is called *rate independent* if, for every time transformation f,

$$(\Phi(u \circ f))(t) = (\Phi(u))(f(t)), \ \forall\, u \in C(\mathbb{R}_+), \ \forall\, t \in \mathbb{R}_+ .$$

We say that $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if Φ is causal and rate independent. The *numerical value set* $\mathrm{NVS}\,\Phi$ of a hysteresis operator Φ is defined by

$$\mathrm{NVS}\,\Phi := \{(\Phi(u))(t) : u \in C(\mathbb{R}_+),\, t \in \mathbb{R}_+\} .$$

A function $u \in C(\mathbb{R}_+)$ is called *ultimately non-decreasing* (*non-increasing*) if there exists $\tau \in \mathbb{R}_+$ such that u is non-decreasing (non-increasing) on $[\tau, \infty)$; u is said to be *approximately ultimately non-decreasing* (*non-increasing*), if for all $\varepsilon > 0$, there exists an ultimately non-decreasing (non-increasing) function $v \in C(\mathbb{R}_+)$ such that

$$|u(t) - v(t)| \le \varepsilon , \quad \forall\, t \in \mathbb{R}_+ .$$

For $w \in C([0, \alpha])$ (with $\alpha \ge 0$) and $\gamma, \delta > 0$, we define

$$\mathcal{C}(w; \delta, \gamma) := \{v \in C([0, \alpha + \gamma]) : v|_{[0,\alpha]} = w, \sup_{t\in[\alpha,\alpha+\gamma]} |v(t) - w(\alpha)| \le \delta\} .$$

We impose the following six conditions on hysteresis operators $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$.

(N1) $\Phi(W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$;

(N2) Φ is monotone in the sense that, for all $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$,

$$(\Phi(u))'(t)u'(t) \ge 0 , \quad \text{a.e. } t \in \mathbb{R}_+ .$$

(N3) There exists $\lambda > 0$ such that for all $\alpha \ge 0$ and $w \in C([0, \alpha])$, there exist numbers $\gamma, \delta > 0$ such that, for all $u, v \in \mathcal{C}(w; \delta, \gamma)$,

$$\sup_{t\in[\alpha,\alpha+\gamma]} |(\Phi(u))(t) - (\Phi(v))(t)| \le \lambda \sup_{t\in[\alpha,\alpha+\gamma]} |u(t) - v(t)| .$$

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(N4) For all $a > 0$ and all $u \in C([0,a))$, there exist $\gamma_1, \gamma_2 > 0$ such that, for all $\tau \in [0,a)$,

$$\sup_{t\in[0,\tau]} |(\Phi(u))(t)| \leq \gamma_1 + \gamma_2 \sup_{t\in[0,\tau]} |u(t)| .$$

(N5) If $u \in C(\mathbb{R}_+)$ is approximately ultimately non-decreasing and $\lim_{t\to\infty} u(t) = \infty$, then $\Phi(u)(t)$ and $\Phi(-u)(t)$ converge to $\sup \mathrm{NVS}\,\Phi$ and $\inf \mathrm{NVS}\,\Phi$, respectively, as $t \to \infty$;

(N6) If, for $u \in C(\mathbb{R}_+)$, $\lim_{t\to\infty}(\Phi(u))(t) \in \mathrm{int}\,\mathrm{NVS}\,\Phi$, then u is bounded.

It is not difficult to see that (N5) implies that NVS Φ is an interval. The set of all hysteresis operators satisfying (N1)-(N6) is denoted by $\mathcal{N}(\lambda)$, where $\lambda > 0$ is the constant associated with (N3). It is well-known that many standard hysteresis nonlinearities which are important in control engineering are contained in $\mathcal{N}(\lambda)$ for some suitable $\lambda > 0$: this applies in particular to backlash (or play), plastic-elastic (or stop) and large classes of Prandtl and Preisach operators (see [4], [5]). We remark that our treatment of hysteresis operators has been strongly influenced by Chapter 2 in [1].

III. Low-gain integral control in the presence of hysteresis

Consider the feedback system shown in Fig. 2, where $\rho \in \mathbb{R}$ is a constant, $\kappa\colon \mathbb{R}_+ \to \mathbb{R}$ is a time-varying gain, the operator $G : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is linear, bounded and shift-invariant, Φ is a hysteresis operator, $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing globally Lipschitz continuous function, the function $g \in L^2(\mathbb{R}_+)$ models the effect of non-zero initial conditions of the system with input-output operator G and $\vartheta + h$ is a disturbance consisting of a constant ϑ and a locally integrable function h.

Denoting the transfer function of G by $\mathbf{G}$, we have that $\mathbf{G} \in H^\infty(\mathbb{C}_+)$, that is, $\mathbf{G}$ is holomorphic and bounded in the open right-half plane. We assume that

(L) The limit $\mathbf{G}(0) := \lim_{s\to 0,\, \mathrm{Re}\, s>0} \mathbf{G}(s)$ exists, $\mathbf{G}(0) > 0$ and

$$\limsup_{s\to 0,\, \mathrm{Re}\, s>0} |(\mathbf{G}(s) - \mathbf{G}(0))/s| < \infty .$$

Since shift-invariance implies causality, G can be extended to a shift-invariant operator mapping $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ into itself. We will not distinguish notationally between G and its extension.

The feedback system shown in Fig. 2 is described by the following Volterra integro-differential equation

$$u' = \kappa(\rho - \vartheta - h - \psi(g + G\circ\Phi(u))) , \quad u(0) = u^0 \in \mathbb{R}. \quad (1)$$

Our objective is to determine gain functions κ such that the *tracking error*

$$e(t) := \rho - y(t) = \rho - \vartheta - h - \psi\big(g + ((G\circ\Phi)(u))(t)\big) \quad (2)$$

becomes small in a certain sense as $t \to \infty$. For example, we might want to achieve "tracking in measure", i.e., for every $\varepsilon > 0$, the Lebesgue measure of the set $\{\tau \geq t : |e(\tau)| \geq \varepsilon\}$ tends to 0 as $t \to \infty$, or the aim might be "asymptotic tracking", that is $\lim_{t\to\infty} e(t) = 0$. Trivially,

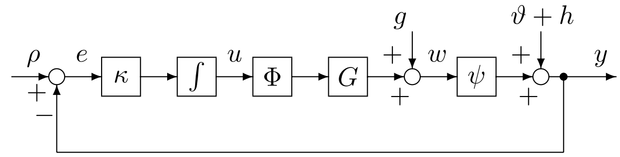

Fig. 2.

tracking in measure is guaranteed if e is of the form $e = e_1 + e_2$, where $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^p(\mathbb{R}_+)$ for some $p \in [1,\infty)$.

Set

$$f(G) := \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \mathrm{Re}(\mathbf{G}(i\omega)/i\omega) . \quad (3)$$

We claim that

$$-\infty < f(G) \leq 0.$$

Indeed, since $\mathbf{G}$ is bounded, we obtain $f(G) \leq 0$ by taking $|\omega| \to \infty$ in (3). The inequality $f(G) > -\infty$ follows from $\mathrm{Re}(\mathbf{G}(i\omega)/i\omega) = \mathrm{Re}([\mathbf{G}(i\omega) - \mathbf{G}(0)]/i\omega)$, assumption (L) and the boundedness of $\mathbf{G}$. Hence, the positivity condition

$$\frac{1}{a} + \operatorname*{ess\,inf}_{\omega\in\mathbb{R}} \mathrm{Re}(\mathbf{G}(i\omega)/i\omega) > 0, \quad (4)$$

holds, provided that

$$a < \frac{1}{|f(G)|} . \quad (5)$$

Equivalently, if (5) holds, then the operator

$$L^2_{\mathrm{loc}}(\mathbb{R}_+) \to L^2_{\mathrm{loc}}(\mathbb{R}_+) , \quad v \mapsto \frac{1}{a} v + \int_0^{\cdot} Gv$$

is strictly passive.

In the following, let θ denote the unit-step function, that is,

$$\theta(t) = 1 , \quad \forall\, t \in \mathbb{R}_+ .$$

The generality of the input and output nonlinearities Φ and ψ allows specific cases wherein tracking of all constant reference signals ρ and rejection of all constant disturbances ϑ may not be feasible. For this reason, we impose a restriction on the difference $\rho - \vartheta$; namely, it should belong to the following set:

$$\mathcal{R}(G,\Phi,\psi) := \{\psi(\mathbf{G}(0)v) \,:\, v \in \overline{\mathrm{NVS}}\,\Phi\} .$$

The intuition underlying $\mathcal{R}(G,\Phi,\psi)$ is as follows. If asymptotic tracking occurs, we would expect that $\Phi^\infty := \lim_{t\to\infty}(\Phi(u))(t)$ exists. Assuming that Φ^∞ is finite and that the final-value theorem holds for the linear system with input-output operator G, we may conclude that $\lim_{t\to\infty}(G\circ\Phi(u))(t) = \mathbf{G}(0)\Phi^\infty$. If, additionally, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = 0$, it follows from (2) that $\rho - \vartheta \in \mathcal{R}(G,\Phi,\psi)$. In fact, it has been shown in [3] that in the case of static input nonlinearities, if ψ is continuous and monotone, then $\rho - \vartheta \in \mathcal{R}(G,\Phi,\psi)$ is close to being a necessary condition for asymptotic tracking.

We are now in the position to state the main result of this paper.

Theorem 3.1: Assume that the following hold:

(a) $G : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a linear bounded shift-invariant operator with transfer function $\mathbf{G}$, satisfying assumption (L);
(b) $g \in L^2(\mathbb{R}_+)$;
(c) $\Phi \in \mathcal{N}(\lambda_1)$;
(d) $\psi : \mathbb{R} \to \mathbb{R}$ is non-decreasing and globally Lipschitz continuous with Lipschitz constant λ_2;
(e) $\rho - \vartheta \in \mathcal{R}(G, \Phi, \psi)$;
(f) h is such that $h \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and the function $t \mapsto \int_t^\infty |h(\tau)| d\tau$ is in $L^2(\mathbb{R}_+)$;
(g) $\kappa : \mathbb{R}_+ \to \mathbb{R}$ is measurable, non-negative and bounded with

$$\limsup_{t\to\infty} \kappa(t) < \frac{1}{\lambda_1 \lambda_2 |f(G)|},$$

where $1/0 := \infty$.

Then there exists a unique solution $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ of (1) and the following statements hold:

(i) $(\Phi(u))' \in L^2(\mathbb{R}_+)$ and the limit

$$\Phi^\infty := \lim_{t\to\infty} (\Phi(u))(t)$$

exists and is finite.

(ii) The signals $w = g + (G \circ \Phi)(u)$ and $y = \psi(w) + \vartheta + h$ (see Fig. 2) can be decomposed as

$$w = w_1 + w_2$$
$$y = y_1 + y_2,$$

where w_1, y_1 are continuous, $w_1(t)$, $y_1(t)$ have finite limits

$$w_1^\infty := \lim_{t\to\infty} w_1(t) = \mathbf{G}(0)\Phi^\infty,$$
$$y_1^\infty := \lim_{t\to\infty} y_1(t) = \psi(\mathbf{G}(0)\Phi^\infty) + \vartheta$$

and $w_2, y_2 \in L^2(\mathbb{R}_+)$. Under the additional assumptions that

$$\lim_{t\to\infty} \big(g(t) + (\Phi(u))(0)((G\theta)(t) - \mathbf{G}(0))\big) = 0 \quad (6)$$

and

$$\lim_{t\to\infty} h(t) = 0, \quad (7)$$

we have

$$\lim_{t\to\infty} w_2(t) = 0, \quad \lim_{t\to\infty} y_2(t) = 0.$$

(iii) If $\kappa \notin L^1(\mathbb{R}_+)$, then $y_1^\infty = \lim_{t\to\infty} y_1(t) = \rho$ and the error signal $e = \rho - y$ can be decomposed as

$$e = e_1 + e_2,$$

where e_1 is continuous with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2(\mathbb{R}_+)$. If, additionally, (6) and (7) hold, then

$$\lim_{t\to\infty} e(t) = 0.$$

(iv) If $\rho - \vartheta$ is an interior point of the set $\mathcal{R}(G, \Phi, \psi)$, then u is bounded.

Remarks. 1) Statement (iii) of Theorem 3.1 implies tracking in measure and, moreover, guarantees asymptotic tracking, provided that (6) and (7) hold. Trivially, if $\lim_{t\to\infty} g(t) = 0$ and

$$\lim_{t\to\infty} (G\theta)(t) = \mathbf{G}(0), \quad (8)$$

then (6) is satisfied. If the impulse response of G is a finite Borel measure μ, for example, if

$$\mu(ds) = f_a(s)ds + \sum_{i=0}^{\infty} f_i \delta_{t_i}(ds),$$

where $f_a \in L^1(\mathbb{R}_+)$, $\{f_i\} \in l^1(\mathbb{Z}_+)$, δ_{t_i} is the unit point mass at $t = t_i$ and $t_i \geq 0$, then (8) holds. Finally, since it follows from assumption (L) via the Paley-Wiener theorem that $G\theta - \mathbf{G}(0)\theta \in L^2(\mathbb{R}_+)$, we conclude that if the limit on the LHS of (8) exists, then it must be equal to $\mathbf{G}(0)$.

2) In general, ϑ is unknown, but it is reasonable to assume that $\vartheta \in [\vartheta_1, \vartheta_2]$, where ϑ_1 and ϑ_2 are known constants. The condition

$$\rho - \vartheta_1,\ \rho - \vartheta_2 \in \mathcal{R}(G, \Phi, \psi)$$

does not involve ϑ and is sufficient for assumption (e) to hold.

3) Note that it is not necessary to know $f(G)$ or the constants λ_1, λ_2 from (c) and (d), respectively, in order to apply Theorem 3.1. If κ is chosen such that $\kappa(t) \to 0$ and $\kappa \notin L^1(\mathbb{R}_+)$ (e.g., $\kappa(t) = (1+t)^{-p}$ with $p \in (0, 1]$), then the conclusions of statement (iii) hold. However, from a practical point of view, gain functions κ with $\lim_{t\to\infty} \kappa(t) = 0$ might not be appropriate, since the system essentially operates in open loop as $t \to \infty$. In [7] it has been shown how $|f(G)|$ (or upper bounds for $|f(G)|$) can be obtained from frequency-response experiments performed on the linear part of the plant.

4) Assumption (f) is satisfied if there exists $\alpha > 1$ such that the function $t \mapsto (1+t)^\alpha h(t)$ is in $L^2(\mathbb{R}_+)$.

IV. The Main Idea in the Proof of Theorem 3.1

We do not provide a full proof of Theorem 3.1 here, but give a brief description of the main idea in the proof; for more details see [6]. Consider the feedback system shown in Fig. 3, where N is a static, possibly time-varying, nonlinearity, G satisfies assumption (a) of Therem 3.1 and r is an input signal.

The equation describing the system in Fig. 3 is

$$v(t) = r(t) - \int_0^t G(N(\cdot, v(\cdot)))(\tau) d\tau \quad (9)$$

Lemma 4.1: Assume that the following hold:

(a) G satisfies hypothesis (a) of Theorem 3.1;
(b) $N : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a static nonlinearity, satisfying

$$0 \leq N(t, \xi)\xi \leq a\xi^2, \quad t \geq t_0,\ \xi \in \mathbb{R}$$

for some $0 < a < 1/|f(G)|$ and some $t_0 \geq 0$;
(c) $r \in L^2(\mathbb{R}_+) + \mathbb{R}$.

If v is a global solution of (9), then

(i) $v - r \in L^\infty(\mathbb{R}_+)$,
(ii) $N(\cdot, v) \in L^2(\mathbb{R}_+)$,
(iii) $\int_0^t N(\tau, v(\tau)) d\tau$ converges to a finite limit as $t \to \infty$.

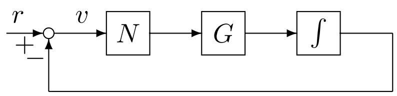

Fig. 3.

Lemma 4.1 is a special case of Theorem 3.3 in [2], the proof of which relies crucially on the positivity property (4) (satisfied for all sufficiently small a, see (5)).

To describe the main idea in the proof of Theorem 3.1, let u be the unique solution of (1) on $\mathbb{R}_+$. (It can be shown that a local solution exists and is unique by means of a contraction mapping argument. The extension of the solution to $\mathbb{R}_+$ can be proved using hypothesis (N4) and the global Lipschitz property of ψ.) The key idea is to apply Lemma 4.1 to the signal

$$w := g + (G \circ \Phi)(u),$$

modified by an offset which depends on ρ and ϑ. Since, by assumption (e), $\rho - \vartheta \in \mathcal{R}(G, \Phi, \psi)$, there exists $\Phi^\sharp \in \overline{\text{NVS}}\,\Phi$ satisfying

$$\psi(\mathbf{G}(0)\Phi^\sharp) = \rho - \vartheta\,.$$

We define

$$\tilde{w} := w - \mathbf{G}(0)\Phi^\sharp = g + (G \circ \Phi)(u) - \mathbf{G}(0)\Phi^\sharp,$$
$$\tilde{\psi}(\xi) := \psi(\xi + \mathbf{G}(0)\Phi^\sharp) - \rho + \vartheta\,, \quad \forall\, \xi \in \mathbb{R}\,.$$

It can be derived from (N1)–(N3) that

$$(\Phi(u))'(t) = d_u(t)u'(t)\,,$$

where $d_u : \mathbb{R}_+ \to \mathbb{R}$ is measurable and

$$0 \le d_u(t) \le \lambda_1, \quad \forall\, t \in \mathbb{R}_+\,,$$

see [4]. It is then easy to show that u satisfies

$$(\Phi(u))' = -N(\cdot, \tilde{w}) - \kappa d_u h\,, \tag{10}$$

where $N : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is given by

$$N(t, \xi) := \kappa(t)d_u(t)\tilde{\psi}(\xi).$$

From (10) we infer

$$(\Phi(u))(t) = (\Phi(u))(0) - \int_0^t N(\tau, \tilde{w}(\tau))d\tau + k(t)\,,$$

where

$$k(t) := -\int_0^t \kappa(\tau)d_u(\tau)h(\tau)d\tau\,, \quad \forall\, t \in \mathbb{R}_+\,.$$

By shift-invariance, G commutes with integration, and thus

$$((G \circ \Phi)(u))(t) = (\Phi(u))(0)(G\theta)(t) - \int_0^t \big(G(N(\cdot, \tilde{w}))\big)(\tau)d\tau + (Gk)(t).$$

By adding $g - \mathbf{G}(0)\Phi^\sharp$ to both sides of the above identity, we see that $\tilde{w}$ solves an equation of the form (9)

$$\tilde{w}(t) = r(t) - \int_0^t \big(G(N(\cdot, \tilde{w}))\big)(\tau)d\tau, \tag{11}$$

where

$$r := g - \mathbf{G}(0)\Phi^\sharp\theta + \Phi(u)(0)G\theta + Gk\,.$$

We observe that, by assumption (a) and the Paley-Wiener theorem, $G\theta \in L^2(\mathbb{R}_+) + \mathbb{R}$, and assumption (f) implies that $k \in L^2(\mathbb{R}_+) + \mathbb{R}$; therefore,

$$r \in L^2(\mathbb{R}_+) + \mathbb{R}\,.$$

Note that

$$0 \le N(t, \xi)\xi \le \lambda_1\lambda_2\kappa(t)\xi^2, \quad \forall\,(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}.$$

By assumption (g) on κ, there exist $0 < a < 1/|f(G)|$ and $t_0 \ge 0$ such that

$$0 \le N(t, \xi)\xi \le a\xi^2, \quad (t, \xi) \in [t_0, \infty) \times \mathbb{R}\,.$$

Applying Lemma 4.1 to (11) we obtain

$$(\Phi(u))' = -N(\cdot, \tilde{w}) - \kappa d_u h \in L^2(\mathbb{R}_+)\,,$$

and thus

$$\lim_{t\to\infty}(\Phi(u))(t) = (\Phi(u))(0) + \int_0^\infty \kappa(\tau)d_u(\tau)h(\tau)d\tau - \lim_{t\to\infty}\int_0^t N(\tau, \tilde{w}(\tau))d\tau$$

exists and is finite, which proves statement (i) of Theorem 3.1. For more details and the proof of statements (ii)–(iv), see [6].

V. Application to well-posed infinite-dimensional state-space systems

There are a number of equivalent definitions of well-posed systems, see [8]-[11]. We will be brief in the following and refer the reader to the above references for more details. We will consider a well-posed system Σ with state-space X (a real Hilbert space with norm denoted by $\|\cdot\|$), input space $U = \mathbb{R}$ and output space $Y = \mathbb{R}$, generating operators (A, B, C), input-output operator G and transfer function $\mathbf{G}$. Here A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\ge0}$ on X, $B \in \mathcal{B}(\mathbb{R}, X_{-1})$ and $C \in \mathcal{B}(X_1, \mathbb{R})$, where X_1 denotes the domain of A endowed with the norm $\|x\|_1 := \|x\| + \|Ax\|$ (the graph norm of A), whilst X_{-1} denotes the completion of X with respect to the norm $\|x\|_{-1} = \|(\zeta I - A)^{-1}x\|$, where $\zeta \in \text{res}(A)$, the resolvent set of A (different choices of ζ lead to equivalent norms). Clearly, $X_1 \subset X \subset X_{-1}$ and the canonical injections are bounded and dense. The semigroup $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1 and extends to a strongly continuous semigroup on X_{-1} with the exponential growth constant $\omega(\mathbf{T}) := \lim_{t\to\infty}\|\mathbf{T}_t\|/t$ being the same on all three spaces; the generator of the restriction (extension) of $\mathbf{T}$ is a restriction (extension) of A; we shall use the same symbol $\mathbf{T}$ (respectively, A) for the original semigroup (respectively, generator) and the associated restrictions and extensions: with this convention, we may write $A \in \mathcal{B}(X, X_{-1})$ (considered as a generator on X_{-1}, the domain of A is X). Moreover, the operators B and C are admissible control and observation operators for

$\mathbf{T}$, respectively. The transfer function $\mathbf{G}$ of Σ is related to the state-space operators A, B and C as follows:

$$\frac{1}{s-\zeta}(\mathbf{G}(s)-\mathbf{G}(\zeta)) = -C(sI-A)^{-1}(\zeta I-A)^{-1}B\,, \quad (12)$$

where $s,\zeta \in \mathbb{C}$ are such that $s \neq \zeta$ and $\operatorname{Re} s$, $\operatorname{Re}\zeta > \omega(\mathbf{T})$.

For $x^0 \in X$ and $v \in L^2_{\text{loc}}(\mathbb{R}_+)$, let x and w denote the state and output functions of Σ, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function v. Then $x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-\tau} Bv(s)ds$ for all $t \in \mathbb{R}_+$, $x(t) + A^{-1}Bv(t) \in \operatorname{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ and

$$x' = Ax + Bv\,, \quad x(0) = x^0\,, \quad (13a)$$

$$w = C_\Lambda\left(x - (\zeta I - A)^{-1}Bv\right) + \mathbf{G}(\zeta)v\,, \quad (13b)$$

where $\operatorname{Re}\zeta > \omega(\mathbf{T})$ and C_Λ denotes the so-called Λ-extension of C defined by

$$C_\Lambda z := \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI-A)^{-1}z\,.$$

Of course, (13) holds almost everywhere on $\mathbb{R}_+$ and the differential equation (13a) has to be interpreted in X_{-1}. In the following, we identify Σ and (13).

The well-posed system (13) is called *strongly stable* if the following four conditions are satisfied:

(i) G is L^2-stable, i.e., $G \in \mathcal{B}(L^2(\mathbb{R}_+))$, or, equivalently, $\mathbf{G} \in H^\infty(\mathbb{C}_+)$;

(ii) $\mathbf{T}$ is strongly stable, i.e., $\lim_{t\to\infty} \mathbf{T}_t z = 0$ for all $z \in X$;

(iii) B is an infinite-time admissible control operator, i.e., there exists $\alpha \geq 0$ such that

$$\left\|\int_0^\infty \mathbf{T}_\tau Bv(\tau)d\tau\right\| \leq \alpha\|v\|_{L^2(\mathbb{R}_+)}, \quad \forall\, v \in L^2(\mathbb{R}_+);$$

(iv) C is an infinite-time admissible observation operator, i.e., there exists $\beta \geq 0$ such that

$$\left(\int_0^\infty \|C\mathbf{T}_\tau z\|^2 d\tau\right)^{1/2} \leq \beta\|z\|, \quad \forall\, z \in X_1.$$

Obviously, exponential stability (i.e., $\omega(\mathbf{T}) < 0$) implies strong stability, but the converse is not true.

If the well-posed system (13) is *regular*, i.e., the following limit

$$\lim_{s\to\infty,\, s\in\mathbb{R}} \mathbf{G}(s) = D$$

exists and is finite, then $x(t) \in D(C_\Lambda)$ for almost every $t \in \mathbb{R}_+$, the output equation (13b) simplifies to

$$y(t) = C_\Lambda x(t) + Du(t)\,, \quad \text{a.e. } t \geq 0$$

and

$$(Gu)(t) = C_\Lambda \int_0^t \mathbf{T}_{t-\tau}Bu(\tau)d\tau + Du(t)\,, \qquad \forall\, u \in L^2_{\text{loc}}(\mathbb{R}_+)\,, \text{ a.e. } t \in \mathbb{R}_+\,.$$

Moreover, in the regular case, we have that $(sI-A)^{-1}B\mathbb{R} \subset D(C_\Lambda)$ for all $s \in \operatorname{res}(A)$ and

$$\mathbf{G}(s) = C_\Lambda(sI-A)^{-1}B + D\,, \quad \operatorname{Re} s > \omega(\mathbf{T})\,.$$

The number D is called the *feedthrough* of (13).

Assume that (13) is connected in series with a hysteretic input nonlinearity and a static output nonlinearity, the latter of which is subject to output disturbances. Application of integral control to the series interconnection (see Fig. 2) leads to the following feedback law

$$v = \Phi(u)\,, \quad (14a)$$

$$y = \psi(w) + \vartheta + h\,, \quad (14b)$$

$$\dot{u} = \kappa(\rho - y)\,, \quad u(0) = u^0\,, \quad (14c)$$

where Φ is a hysteresis operator, ψ is a static nonlinearity, κ is a time-varying gain, ρ, $\vartheta \in \mathbb{R}$ and h is a nonconstant part of the output disturbance. A *solution* of the feedback system, given by (13) and (14), on an interval $[0,\alpha)$ is a continuous function $(x,u) : [0,\alpha) \to X \times \mathbb{R}$ such that $(x(0),u(0)) = (x^0,u^0)$ and (x,u) is absolutely continuous as an $X_{-1}\times\mathbb{R}$-valued function and satisfies the feedback system equations (13) and (14) almost everywhere on $[0,\alpha)$.

The following theorem is a state-space version of Theorem 3.1. Before stating it, we remark that if (13) is strongly stable and $0 \in \operatorname{res(A)}$, then $\mathbf{G}$ can be analytically extended to a neighbourhood of 0 and so the evaluation $\mathbf{G}(0)$ of $\mathbf{G}(s)$ at $s = 0$ makes sense and (12) holds for $\zeta = 0$. Consequently, since $\omega(\mathbf{T}) \leq 0$ (by strong stability), we have that

$$\frac{\mathbf{G}(s) - \mathbf{G}(0)}{s} = C(sI-A)^{-1}A^{-1}B\,, \quad \operatorname{Re} s > 0\,. \quad (15)$$

Theorem 5.1: Assume that the following hold:

(a) System (13) is strongly stable, $0 \in \operatorname{res(A)}$ and $\mathbf{G}(0) > 0$;

(b) $\Phi \in \mathcal{N}(\lambda_1)$;

(c) $\psi : \mathbb{R} \to \mathbb{R}$ is non-decreasing and globally Lipschitz continuous with Lipschitz constant λ_2;

(d) $\rho - \vartheta \in \mathcal{R}(G,\Phi,\psi)$;

(e) h is such that $h \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and the function $t \mapsto \int_t^\infty |h(\tau)|d\tau$ is in $L^2(\mathbb{R}_+)$;

(f) $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded with

$$\limsup_{t\to\infty} \kappa(t) < \frac{1}{\lambda_1\lambda_2|f(G)|}\,,$$

where $1/0 := \infty$.

Then there exists a unique solution

$$(x,u) \in C(\mathbb{R}_+, X\times\mathbb{R}) \cap W^{1,1}_{\text{loc}}(\mathbb{R}_+, X_{-1}\times\mathbb{R})$$

of the feedback system given by (13) and (14) such that the following statements hold:

(i) $(\Phi(u))' \in L^2(\mathbb{R}_+)$ and the limit

$$\Phi^\infty := \lim_{t\to\infty}(\Phi(u))(t)$$

exists and is finite.

(ii) $\lim_{t\to\infty}\|x(t) + A^{-1}B\Phi^\infty\| = 0$.

(iii) The signals $w = C_\Lambda\mathbf{T}x^0 + (G\circ\Phi)(u)$ and $y = \psi(w) + \vartheta + h$ can be decomposed as

$$w = w_1 + w_2$$

$$y = y_1 + y_2,$$

where w_1, y_1 are continuous and have finite limits

$$w_1^\infty := \lim_{t\to\infty} w_1(t) = \mathbf{G}(0)\Phi^\infty,$$
$$y_1^\infty := \lim_{t\to\infty} y_1(t) = \psi(\mathbf{G}(0)\Phi^\infty) + \vartheta,$$

and $w_2, y_2 \in L^2(\mathbb{R}_+)$. If $\lim_{t\to\infty} h(t) = 0$ and, for some $t_0 \geq 0$,

$$\mathbf{T}_{t_0}\big(Ax^0 + B(\Phi(u))(0)\big) \in X \tag{16}$$

or

$$\mathbf{T}_{t_0}x^0 \in X_1 \quad \text{and} \quad \lim_{t\to\infty}(G\theta)(t) = \mathbf{G}(0)\,, \tag{17}$$

we have

$$\lim_{t\to\infty} w_2(t) = 0\,, \quad \lim_{t\to\infty} y_2(t) = 0\,.$$

(iv) If $\kappa \notin L^1(\mathbb{R}_+)$, then $\lim_{t\to\infty} y_1(t) = \rho$ and the error signal $e = \rho - y$ can be decomposed as

$$e = e_1 + e_2,$$

where e_1 is continuous with $\lim_{t\to\infty} e_1(t) = 0$ and $e_2 \in L^2(\mathbb{R}_+)$. If $\lim_{t\to\infty} h(t) = 0$ and, for some $t_0 \geq 0$, (16) or (17) holds, then

$$\lim_{t\to\infty} e(t) = 0\,.$$

(v) If $\rho - \vartheta$ is an interior point of the set $\mathcal{R}(G, \Phi, \psi)$, then u is bounded.

The proof of Theorem 5.1 involves an application of Theorem 3.1 (with $g = C_\Lambda \mathbf{T}x^0$) together with results from the theory of well-posed systems (see [6]).

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