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Stability of Nonlinear Switched Systems on the Plane

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***Abstract*— We consider the time-dependent nonlinear system $\dot{q}(t) = u(t)X(q(t)) + (1 - u(t))Y(q(t))$, where $q \in \mathbb{R}^2$, X and Y are two smooth vector fields, globally asymptotically stable at the origin, and $u : [0, \infty) \to \{0, 1\}$ is an arbitrary measurable function. Analysing the topology of the set where X and Y are parallel, we give some sufficient and some necessary conditions for global asymptotic stability, uniform with respect to $u(.)$. Such conditions can be verified without any integration or construction of a Lyapunov function, and they are robust under small perturbations of the vector fields.**

I. Introduction

A *switched system* is a family of continuous-time dynamical systems endowed with a rule that determines, at every time, which dynamical system is responsible for the time evolution. More precisely let $\{f_u \mid u \in U\}$ be a (possibly infinite) set of smooth vector fields on a manifold M, and consider, as u varies in U, the family of dynamical systems

$$\dot{q} = f_u(q)\,, \qquad q \in M\,. \tag{1}$$

A non-autonomous dynamical system is obtained by assigning a so-called *switching function* $u : [0, \infty) \to U$.

In this paper, the switching function models the behavior of a parameter which cannot be predicted a priori. It represents some phenomena (e.g., a disturbance) that it is not possible to control or include in the dynamical system model.

A typical problem related to switching systems is to obtain, out of a property which is shared by all the autonomous dynamical systems governed by the vector fields f_u, some, maybe weaker, property for the time-dependent system associated with an arbitrary switching function $u(.)$. For a discussion on various issues related to switched systems we refer the reader to [12], [13].

In this paper, we consider a two-dimensional nonlinear switched system of the type

$$\dot{q} = u\,X(q) + (1 - u)\,Y(q)\,, \quad q \in \mathbb{R}^2\,, \quad u \in \{0, 1\}\,, \tag{2}$$

where the two vector fields X and Y are smooth (say, $\mathcal{C}^\infty$) on $\mathbb{R}^2$. In order to define a proper non-autonomous system, we require the switching functions to be measurable.

Assume that $X(0) = Y(0) = 0$ and that the two dynamical systems $\dot{q} = X(q)$ and $\dot{q} = Y(q)$ are globally asymptotically stable at the origin. Our main aim is to study under which conditions on X and Y the origin is globally asymptotically stable for the system (2), uniformly with respect to the switching functions (GUAS for short). For the precise formulation of this and other stability properties, see Definition 1.

In order to study the stability of (2) it is natural to consider its convexification, i.e., the case in which u varies in the whole interval $[0, 1]$. It turns out that the stability properties of the two systems are equivalent (see Section II-B). The linear version of the system introduced above, namely,

$$\dot{q} = u\,A\,q + (1 - u)\,B\,q\,, \quad q \in \mathbb{R}^2\,, \quad u \in \{0, 1\}\,, \tag{3}$$

where the 2×2 real matrices A and B have eigenvalues with strictly negative real part, was studied in [6] (see also [14]). More precisely, the results in [6] establish a necessary and sufficient condition for GUAS in terms of three relevant parameters, two depending on the eigenvalues of A and B respectively, and the third one (namely, the cross ratio of the four eigenvectors of A and B in the projective line $\mathbb{C}P^1$) accounting for the interrelations among the two systems. The precise necessary and sufficient condition ensuring GUAS of (3) is quite technical and can be found in [6] (see also [14]). Notice that, in the linear case, asymptotic stability is equivalent to GUAS, which, in turns, is equivalent to the more often quoted GUES property, i.e., global exponential stability, uniform with respect to the switching rule (see, for example, [3] and references therein). For related results on linear switched systems, see [2], [5], [9], [11], [14].

For nonlinear systems, the problem of characterizing GUAS completely, without assuming the explicit knowledge of the integral curves of X and Y, is hopeless.

The problem, however, admits some partial solution. The purpose of this paper is to provide some sufficient and some necessary conditions for stability which are robust (with respect to small perturbations of the vector fields) and easily verifiable, directly on the vector fields X and Y, without requiring any integration or construction of a Lyapunov function.

Denote by $\mathcal{Z}$ the set on which X and Y are parallel. One of our main results is that, if $\mathcal{Z}$ reduces to the singleton $\{0\}$, then (2) is GUAS (Theorem 6). The proofs works by showing that an admissible trajectory starting from a point $p \in \mathbb{R}^2$ is forced to stay in a compact region bounded by the integral curves of X and Y from p. The fact that X and Y

are linearly independent outside the origin plays as a sort of drift which guarantees that the only possible accumulation point of an admissible trajectory is the origin.

When $\mathcal{Z}$ is just compact, we prove that (2) is at least bounded (see Theorem 9). Roughly speaking, this means that its trajectories do not escape to infinity. The idea of the proof is that, if we modify X and Y only in a compact region of the plane, then the boundedness properties of the system are left unchanged. Taking advantage of the result obtained in Theorem 6, we manage to prove the boundedness of (2) by reducing, using compact perturbations, $\mathcal{Z}$ to $\{0\}$, while preserving the global asymptotic stability of X and Y.

Other conditions can be formulated taking into account the relative position of X and Y along $\mathcal{Z}$. Assume that $\mathcal{Z}\setminus\{0\}$ contains at least one point q_0. Since both $X(q_0)$ and $Y(q_0)$ are different from zero, the property of pointing in the same or in the opposite versus can be stated unambiguously. If $X(q_0)$ and $Y(q_0)$ have opposite versus, then there exists a switching function, for the convexified system, whose output is the constant trajectory which stays in q_0. As a consequence, the system (2) is not GUAS.

Additional results can be obtained under the assumption that the pair of vector fields (X,Y) is generic. (For the notion of genericity appropriate to our aims, see Section II.) In particular, the genericity assumption can be used to guarantee that $\mathcal{Z}\setminus\{0\}$ is an embedded one-dimensional submanifold of the plane. Clearly, $\mathcal{Z}$ needs not to be connected. If the connected component of $\mathcal{Z}$ containing the origin reduces to $\{0\}$ and on all other components X and Y point in the same versus, transversally to $\mathcal{Z}$, then (2) is GUAS. This result is formulated in Theorem 8, which follows the pattern of proof of Theorem 6.

Conversely, Theorem 14 states that, if one connected component of $\mathcal{Z}\setminus\{0\}$ is unbounded and such that X and Y have opposite versus on it, then (2) admits a trajectory going to infinity. Intuitively, this happens because the orientation of $(X(p),Y(p))$ changes while p crosses $\mathcal{Z}\setminus\{0\}$. If $X(p)$ is not tangent to $\mathcal{Z}$ at p and $X(p)$ points in the opposite direction with respect to $Y(p)$, then one can embed $\mathcal{Z}$, locally near p, in a foliation made of admissible trajectories of (2), whose running direction is reversed while crossing $\mathcal{Z}$ (see Figure 1). Since, generically, the points where X is tangent to $\mathcal{Z}$ are isolated, it turns out that there exists an admissible trajectory which tracks globally the unbounded connected component of $\mathcal{Z}\setminus\{0\}$ on which X and Y have opposite versus.

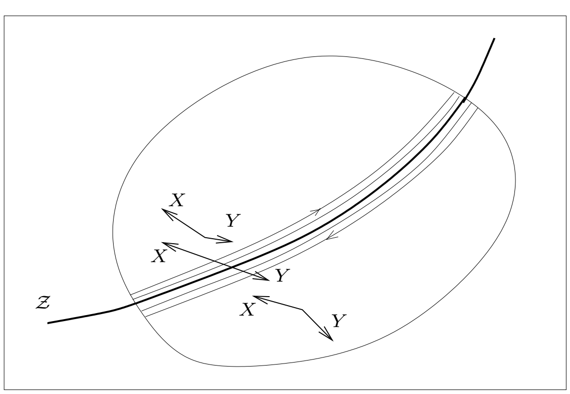

Fig. 1. A local foliation embedding $\mathcal{Z}$

The paper is organized as follows. In Section II, we recall the main definitions of stability in which we are interested, we introduce the convexified system, and we describe the topological structure of the set $\mathcal{Z}$. The main results are stated in Section III, where their robustness is also discussed. Finally, the proofs are sketched in Section IV.

II. Basic definitions and facts

A. Definitions of stability

Fix $n,m\in\mathbb{N}$ and consider the switched system

$$\dot q = f_u(q)\,,\qquad q\in\mathbb{R}^n\,,\qquad u\in U\subset\mathbb{R}^m\,, \tag{4}$$

where U is a compact subset of $\mathbb{R}^m$ and $(q,u)\mapsto f_u(q)$ is a $\mathcal{C}^\infty$ map from $\mathbb{R}^n\times U$ to $\mathbb{R}^n$. Assume that $f_u(0)=0$ for every $u\in U$. For every $\delta>0$, denote by $B_\delta\subset\mathbb{R}^n$ the ball of radius δ, centered at the origin. Set

$$\mathcal{U}=\{u:[0,\infty)\to U\mid u(.)\text{ measurable}\}\,.$$

For every $u(.)$ in $\mathcal{U}$ and every $p\in\mathbb{R}^n$, denote by $t\mapsto\gamma(p,u(.),t)$ the solution of (4) such that $\gamma(p,u(.),0)=p$. Notice that, in general, $t\mapsto\gamma(p,u(.),t)$ needs not to be defined for every $t\geq 0$, since the non-autonomous vector field $f_{u(t)}$ may not be complete. Denote by $\mathcal{T}(p,u(.))$ the maximal element of $(0,+\infty]$ such that $t\mapsto\gamma(p,u(.),t)$ is defined on $[0,\mathcal{T}(p,u(.)))$, and let

$$Supp(\gamma(p,u(.),.))=\gamma(p,u(.),[0,\mathcal{T}(p,u(.))))\,.$$

If $Supp(\gamma(p,u(.),.))$ is bounded, then $\mathcal{T}(p,u(.))=+\infty$.

Given $p\in\mathbb{R}^n$, the *accessible set from* p, denoted by $\mathcal{A}(p)$, is defined as

$$\mathcal{A}(p)=\cup_{u(.)\in\mathcal{U}}Supp(\gamma(p,u(.),.))\,.$$

Several notions of stability for the switched system (4) can be introduced.

Definition 1: We say that (4) is

- **unbounded** if there exist $p\in\mathbb{R}^n$ and $u(.)\in\mathcal{U}$ such that $\gamma(p,u(.),t)$ goes to infinity as t tends to $\mathcal{T}(p,u(.))$;
- **bounded** if, for every $K_1\subset\mathbb{R}^n$ compact, there exists $K_2\subset\mathbb{R}^n$ compact such that $\gamma(p,u(.),t)\in K_2$ for every $u(.)\in\mathcal{U}$, $t\geq 0$ and $p\in K_1$;
- **uniformly stable** at the origin if, for every $\delta>0$, there exists $\varepsilon>0$ such that $\mathcal{A}(p)\subset B_\delta$ for every $p\in B_\varepsilon$;
- **locally attractive** at the origin if there exists $\delta>0$ such that, for every $u(.)\in\mathcal{U}$ and every $p\in B_\delta$, $\gamma(p,u(.),t)$ converges to the origin as t goes to infinity;
- **globally attractive** at the origin if, for every $u(.)\in\mathcal{U}$ and every $p\in\mathbb{R}^n$, $\gamma(p,u(.),t)$ converges to the origin as t goes to infinity;
- **globally uniformly attractive** at the origin if, for every $\delta_1,\delta_2>0$, there exists $T>0$ such that $\gamma(p,u(.),T)\in B_{\delta_1}$ for every $u(.)\in\mathcal{U}$ and every $p\in B_{\delta_2}$;
- **globally uniformly stable** at the origin if it is bounded and uniformly stable at the origin;

- **locally asymptotically stable** at the origin if it is uniformly stable and locally attractive at the origin;
- **globally asymptotically stable** at the origin if it is uniformly stable and globally attractive at the origin;
- **globally uniformly asymptotically stable (GUAS)** at the origin if it is uniformly stable and globally uniformly attractive at the origin.

It has been showed by Angeli, Ingalls, Sontag, and Wang [4] that, thanks to the compactness of U, global asymptotic stability is equivalent to GUAS. Moreover, it is well known that, in the case in which all the vector fields f_u are linear, local and global properties are equivalent.

B. The convexified system

In this paper, we focus on the planar switched system

$$\dot{q} = u\,X(q) + (1-u)\,Y(q)\,, \quad q \in \mathbb{R}^2\,, \quad u \in \{0,1\}\,, \quad (5)$$

where X and Y denote two vector fields on $\mathbb{R}^2$, of class $\mathcal{C}^\infty$, such that $X(0) = Y(0) = 0$. We assume moreover that X and Y are globally asymptotically stable. Notice, in particular, that X and Y are forward complete.

A classical tool, in stability analysis, is the convexification of the set of admissible velocities. Such transformation does not change the closure of the accessible sets. Moreover, it was proved in [10] (see also [4, Proposition 7.2]) that, for every $p' \in \mathbb{R}^2$, every switching function $u' : [0,\infty) \to [0,1]$ and every positive continuous function r defined on $[0, \mathcal{T}(p', u'(.)))$, there exist $u(.) \in \mathcal{U}$ and $p \in \mathbb{R}^2$ such that

$$|\gamma(p, u(.), t) - \gamma(p', u'(.), t)| \leq r(t)$$

for every $t \in [0, \mathcal{T}(p', u'(.)))$. As a consequence, each of the notions introduced in Definition 1 holds for (5) if and only if it holds for the same system where $U = \{0,1\}$ is replaced by $[0,1]$.

In the following, to simplify proofs, we deal with the convexified system

$$\dot{q} = u\,X(q) + (1-u)\,Y(q)\,, \quad q \in \mathbb{R}^2\,, \quad u \in [0,1]\,. \quad (6)$$

Notations

When $u(.)$ is constantly equal to zero (respectively, one), we write $\gamma_Y(p,t)$ (respectively, $\gamma_X(p,t)$) for $\gamma(p, u(.), t)$. Given $p, p' \in \mathbb{R}^2$ and $u(.), u'(.)$ in $\mathcal{U}$, we say that $\gamma(p, u(.), .)$ and $\gamma(p', u'(.), .)$ *forwardly intersect* if $Supp(\gamma(p, u(.), .))$ and $Supp(\gamma(p', u'(.), .))$ have nonempty intersection.

C. The collinearity set of X and Y

A key object in order to detect stability properties of (6) turns out to be the set $\mathcal{Z}$ on which X and Y are parallel. We have that $\mathcal{Z} = Q^{-1}(0)$, where

$$Q(p) = det(X(p), Y(p))\,, \qquad p \in \mathbb{R}^2\,. \qquad (7)$$

In [6], the stability of the linear switched system (3) was studied by associating with every point of $\mathbb{R}^2$ a suitably defined "worst" trajectory passing through it, whose construction was based upon $\mathcal{Z}$. The global asymptotic stability of the linear switched system (3) was then proved to be equivalent to the convergence to the origin of every such worst trajectory. We recall that in the linear case, excepted for some degenerate situations, $\mathcal{Z}$ is either equal to $\{0\}$ or is made of two straight lines passing through the origin.

In the nonlinear case, the situation is more complex. Let us represent $\mathcal{Z}$ as

$$\mathcal{Z} = \{0\} \cup \bigcup_{\Gamma \in \mathcal{G}} \Gamma\,, \qquad (8)$$

where $\mathcal{G}$ is the set of all connected components of $\mathcal{Z} \setminus \{0\}$. Notice that $\mathcal{G}$ needs not, in general, to be countable. With a slight abuse of notation, we will refer to the elements of $\mathcal{G}$ as to the *components* of $\mathcal{Z}$.

Definition 2: Let Γ be a component of $\mathcal{Z}$ and fix $p \in \Gamma$. We say that Γ is *direct* (respectively, *inverse*) if $X(p)$ and $Y(p)$ have the same (respectively, opposite) direction.

Remark 3: Notice that the definition is independent of the choice of p, since neither X nor Y vanish along Γ.
An example of how $\mathcal{Z}$ can look like is represented in Figure 2.

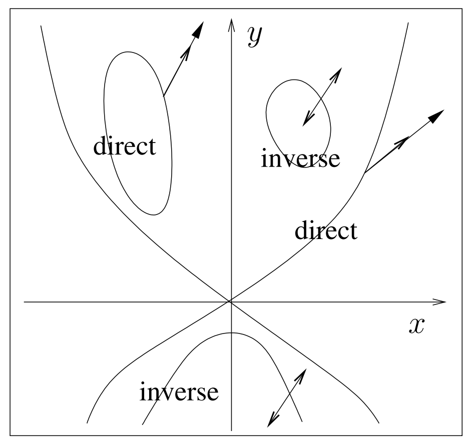

Fig. 2. The set $\mathcal{Z}$

Some of the results of this paper are obtained assuming that the set $\mathcal{Z}$ has suitable regularity properties, which are generic in the sense defined below.

A base to the *Withney topology* in $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ (the set of smooth vector fields on $\mathbb{R}^2$) can be defined, using the multi-index notation, as the family of subsets of $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ of the type

$$\mathcal{V}(k, f, r) = \left\{ g \;\middle|\; \left\| \frac{\partial^{|I|}(f-g)}{\partial x^I}(x) \right\| < r(x)\ \forall x,\ |I| \leq k \right\},$$

where k is a nonnegative integer, f belongs to $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and r is a positive continuous function defined on $\mathbb{R}^2$. Denote by GAS($\mathbb{R}^2$) the set of smooth vector fields on $\mathbb{R}^2$ which are globally asymptotically stable at the origin, and endow it with the topology induced by Withney's one. A *generic property* for (6) is a property which holds for an open dense subset of GAS($\mathbb{R}^2$)$\times$GAS($\mathbb{R}^2$), endowed with the product topology of GAS($\mathbb{R}^2$).

Lemma 4: For a generic pair of vector fields (X, Y), $\mathcal{Z} \setminus \{0\}$ is an embedded one-dimensional submanifold of $\mathbb{R}^2$. Moreover, $Q(p)$ changes sign while p crosses $\mathcal{Z} \setminus \{0\}$.
The lemma is a standard result in genericity theory. It follows from the fact that the condition

(G1) If $p \neq 0$ and $Q(p) = 0$, then $\nabla Q(p) \neq 0$,

is generic (see, for instance, [1]). When $\mathcal{Z} \setminus \{0\}$ is a manifold, we say that $p \in \mathcal{Z} \setminus \{0\}$ is a tangency point if $X(p)$ is tangent to $\mathcal{Z}$. Under condition (G1), $p \in \mathcal{Z} \setminus \{0\}$ is a tangency point if and only if $\nabla Q(p)$ and $X(p)$ (equivalently, $Y(p)$) are orthogonal.

Some of our results are obtained under additional generic conditions. One of these, namely,

(G2) The Hessian matrix of Q at the origin is non-degenerate,

ensures that $\mathcal{Z}$, in a neighborhood of the origin, is given either by $\{0\}$ or by the union of two transversal one-dimensional manifolds intersecting at the origin.

Under the generic conditions (G1) and (G2), the connected component of $\mathcal{Z}$ containing the origin looks like one of Figure 3.

A third generic condition which we will sometimes assume to hold is

(G3) If $p \neq 0$, $Q(p) = 0$, and $\nabla Q(p)$ is orthogonal to $X(p)$, then the second derivative of Q at p along X (equivalently, Y) is different from zero,

which, together with (G1), guarantees that the tangency points on $\mathcal{Z}$ are isolated.

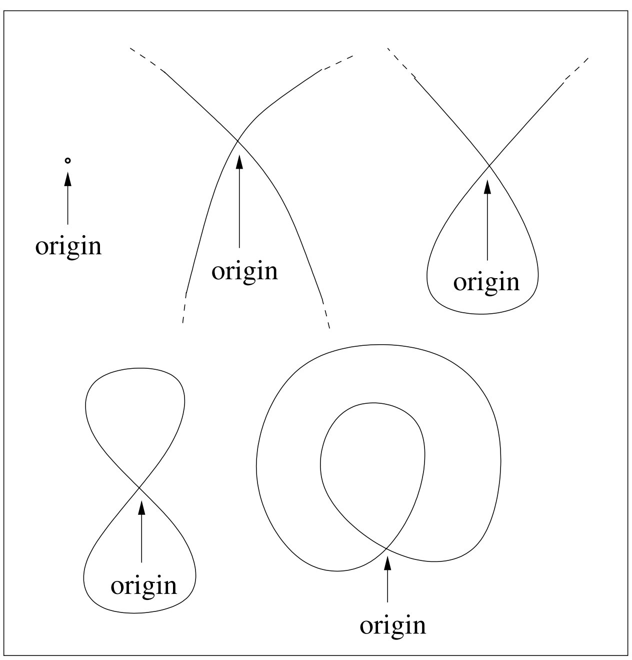

Fig. 3. The connected component of $\mathcal{Z}$ containing the origin

III. Statement of the Results

We organize our results in sufficient and necessary conditions with respect to the stability properties.

Notice that all such conditions are easily verified without any integration or construction of a Lyapunov function. Moreover, they are robust under small perturbations of the vector fields, as explained in Section III-C. Let us recall that X and Y are assumed to be globally asymptotically stable at the origin and that all the results given below, although stated for the case $u \in [0, 1]$, are also valid for the system where u varies in $\{0, 1\}$.

Before stating our main theorems, observe that classical results on linearization clearly imply the following.

Proposition 5: Assume that the eigenvalues of $A = \nabla X|_{p=0}$ and $B = \nabla Y|_{p=0}$ have strictly negative real part. Then (6) is locally asymptotically stable if and only if (3) is GUAS.

A. Sufficient conditions

The following theorem gives a simple sufficient condition for GUAS, which generalizes the analogous one already known for the linear system (3) (see [6], [14]).

Theorem 6: Assume that $\mathcal{Z} = \{0\}$. Then the switched system (6) is GUAS at the origin.

Remark 7: The proof of Theorem 6 naturally extends to the following case: if V is an open and simply connected subset of $\mathbb{R}^2$, if X and Y point inside V along its boundary, and if $\mathcal{Z} \cap V = \{0\}$, then (6) is uniformly asymptotically stable on V.
Under the generic assumptions (G1) and (G2), Theorem 6 can be generalized as follows.

Theorem 8: Assume that the generic conditions (G1) and (G2) hold. Assume, moreover, that the origin is isolated in $\mathcal{Z}$ and that $\mathcal{Z} \setminus \{0\}$ contains no tangency point. Then the switched system (6) is GUAS.
When $\mathcal{Z}$ is bounded, although different from $\{0\}$, some weaker version of Theorem 6 still holds.

Theorem 9: Assume that $\mathcal{Z}$ is compact. Then the switched system (6) is bounded.
As a direct consequence of Proposition 5 and Theorem 9, we have the following sufficient condition for global uniform stability.

Corollary 10: Let $\mathcal{Z}$ be compact, and the linearized switched system be non-degenerate and GUAS. Then the switched system (6) is globally uniformly stable.

Remark 11: The conclusion of Theorem 9 applies to the more general case where the points at which X and Y are globally asymptotically stable are allowed to be different.

Remark 12: The conclusion of Theorem 9 would not hold under the weaker hypothesis that X and Y are globally stable, instead of globally asymptotically stable.

B. Necessary conditions

The following proposition expresses the straightforward remark that the inverse components of $\mathcal{Z}$ constitute obstructions to the stability of (6). The reason is clear: if Γ is inverse and p belongs to Γ, then a constant switching function $u(.)$ exists such that $\gamma(p, u(.), t) = p$ for every $t \geq 0$.

Proposition 13: If $\mathcal{Z}$ has an inverse component, then the switched system (6) is not globally attractive.

The following theorem gives a necessary condition for boundedness, under generic conditions.

Theorem 14: Assume that the generic conditions (G1) and (G3) hold. If $\mathcal{Z}$ contains an unbounded inverse component, then the switched system (6) is unbounded.

Remark 15: In the non-generic case the statement of Theorem 14 is false. A counterexample can be found even in the linear case (see [6]).

C. Robustness

We say that a property satisfied by (X, Y) is robust if it still holds for small perturbations of the pair (X, Y), that is, if it holds for all the elements of a neighborhood of (X, Y) in GAS($\mathbb{R}^2$)×GAS($\mathbb{R}^2$). Such notion of robustness is also known as *structural stability*, an expression which we prefer to avoid, in order to prevent confusion with the many definitions of stability already introduced for (6).

Under the generic conditions (G1) and (G2), one can easily verify that the topology of the set $\mathcal{Z}$ does not change for small perturbations of X and Y. Moreover, fixed one component Γ of $\mathcal{Z}$, the fact that Γ is direct or inverse is robust. Similarly, if Γ is a component of $\mathcal{Z}$, which has not the origin in its closure, the absence of tangency points along Γ is robust. As a consequence, the conditions formulated by the theorems above are robust. More precisely:

Theorem 16: Under generic assumptions, if any of Theorems 6, 8, 9, 14, Corollary 10, or Proposition 13 applies to the pair (X, Y), then it applies in a neighborhood of (X, Y) in GAS($\mathbb{R}^2$)×GAS($\mathbb{R}^2$).

IV. Sketches of proofs

The complete proofs of the theorems stated in the previous section are given in [7]. We give here only sketches of them.

A. Theorem 6

The main point to be proved is the global attractivity of (2).

The first step, in order to prove global attractivity, is to show that every accessible set $\mathcal{A}(q)$ is bounded. In order to do this, let us consider two cases. If $\gamma_X(q, .)$ and $\gamma_Y(q, .)$ do not forwardly intersect, let us define

$$\gamma_{X,Y}(q, s) = \begin{cases} \gamma_X(q, \tan(s\pi)) & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \gamma_Y(q, \tan((1-s)\pi)) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Otherwise, if $\gamma_X(q, .)$ and $\gamma_Y(q, .)$ do forwardly intersect, let denote t the first positive time such that $\gamma_Y(q, t)$ is a point of the form $\gamma_X(q, \tau)$ $(\tau > 0)$ and define

$$\gamma_{X,Y}(q, s) = \begin{cases} \gamma_X(q, s) & \text{if } s \in [0, \tau], \\ \gamma_Y(q, t + \tau - s) & \text{if } s \in [\tau, t + \tau]. \end{cases}$$

In both cases, $\gamma_{X,Y}$ is a simple closed piecewise smooth curve. As a consequence, it separates the plane in two parts. Denote by $\mathcal{B}$ the bounded one. Using index theory, one can prove that $\mathcal{A}(q) \subset \bar{\mathcal{B}}$ (see [7]).

The second step consists in proving (through additional considerations on the structure of the accessible sets) that no

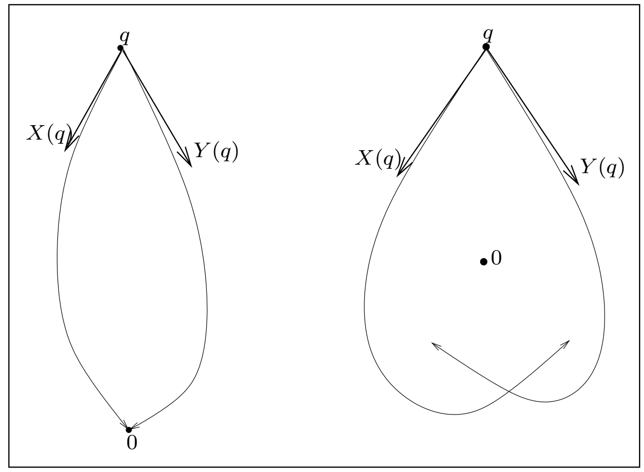

Fig. 4. The curve $\gamma_{X,Y}$ in the two cases

admissible curve has an accumulation point different from the origin. Together with the boundedness of the accessible sets, this proves that (2) is globally attractive.

The uniform stability is a direct consequence of the uniform stability of X and Y at the origin and of the description given above of the accessible sets.

Finally, it has been proved in [4] that global asymptotic stability implies global uniform asymptotic stability when the control set is compact, which allows to conclude.

B. Theorem 8

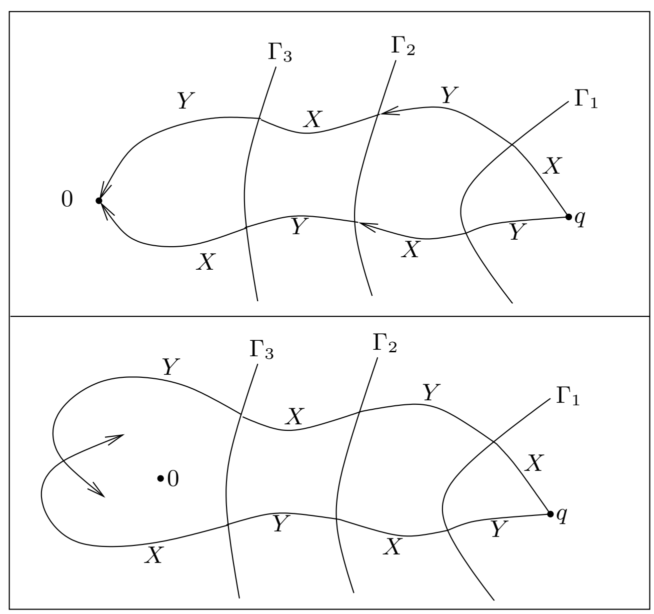

Fig. 5. The new curve $\gamma_{X,Y}$ in both cases

The proof of Theorem 8 is just an adaptation of the one of Theorem 6. The major difference consists in the characterization of the boundary of an accessible set $\mathcal{A}(q)$. Such boundary, which is given by the support of $\gamma_{X,Y}$ in the case $\mathcal{Z} = \{0\}$, is now obtained as the finite concatenation of pieces of curves $\gamma_X(p_i,.)$ and $\gamma_Y(q_i,.)$, where p_i and q_i are points of the plane belonging to $\mathcal{Z}$ (see Figure 5).This allows to prove that the accessible sets are bounded. The rest of the proof is almost unchanged.

C. Theorem 9

The main idea of the proof of Theorem 9 is to show that we can modify the vector fields X and Y on a compact set in such a way they satisfy the hypotheses of Theorem 6. The transformed system being GUAS, it is also bounded. This clearly implies that the original system is bounded.

D. Theorem 14

If (G1) holds, the components of $\mathcal{Z}$ are one-dimensional manifolds. In [7] we show that, if (G1) and (G3) hold, there exist admissible curves of (6) following the inverse components of $\mathcal{Z}$ in both directions. The result is proved using a local argument based on the normal forms for X and Y given by A. Davydov in [8]. Hence, if there exists at least one unbounded inverse component of $\mathcal{Z}$, then (6) admits a curve going to infinity.

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