Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Cycle Time Assignability and Feedback Design for Min-Max-Plus Systems

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Abstract— **A variety of problems arising in communication networks, computer networks, automated manufacturing plants, etc., can be described by min-max-plus system models. This paper investigates the cycle time assignability of general min-max-plus systems with min-max-plus input control functions, which are nonlinear extensions of the systems studied in recent years. The main results of the paper are the necessary and sufficient criterion for the cycle time assignability and the feedback design. The methods based on directed graph and max-algebra are constructive, and a numerical example is given to illustrate such methods.**

I. Introduction

There exist many discrete event systems with minimum and maximum constraints in real-world systems such as communication networks, computer networks, automated manufacturing plants, etc. (see, for example, [1], [2], [4], [17], [19], [20], [23], [25]). Such systems can be described by min-max-plus system models in which the operations minimization, maximization and addition are used, and are usually called min-max-plus systems, which are nonlinear extensions of the well-known linear max-plus system models (or max-plus systems) where only maximization and addition are used. Max-plus systems can be studied using linear methods based on max-plus algebra (see [1], [3], [7]–[9], [16]). There has been much research on min-max-plus systems in recent years. Some significant results have been obtained for autonomous systems, such as the existence and the calculation of a fixed point and a cycle time (see [6], [12]–[15], [20]–[22]; etc.). The control problems of min-max-plus systems have been studied lately. De Schutter and van den Boom investigated the model predictive control for min-max-plus systems (see [10]); Chen and Tao investigated the observability, reachability and cycle time assignment for min-max-plus systems with max-plus input and output control functions (see [5], [26]).

The stability, equilibrium states, cyclical behavior and asymptotic average delays of min-max-plus systems are of vital importance to system designers (see [2], [6], [14], [19], [20], etc.). The *cycle time* is the fundamental performance metric of min-max-plus systems and is defined as the limit vector $\lim_{k\to\infty} F^k(\mathbf{x})/k$, where $F(\mathbf{x})$ and $\mathbf{x}$ are the interior function and the state vector of the systems, respectively. In the applications of min-max-plus systems, the state vector $\mathbf{x}$ is often interpreted as a vector of occurrence times of certain events and the vector $F(\mathbf{x})$ as the times of next occurrence. Hence the limit vector above can be thought of as the vector of asymptotic average times to the next occurrence of the events:

$$\frac{(F^k(\mathbf{x})-F^{k-1}(\mathbf{x}))+\cdots+(F(\mathbf{x})-\mathbf{x})}{k}=\frac{F^k(\mathbf{x})-\mathbf{x}}{k},$$

which tends to $\lim_{k\to\infty} F^k(\mathbf{x})/k$ as $k \to \infty$. Cohen *et al.* studied the pole (cycle time) assignment of max-plus systems and established the assignment condition by an output feedback (see [7]). Tao and Chen gave an account of the cycle time assignment by a uniform state feedback for min-max-plus systems with max-plus input control functions (see [26]). The results obtained in the above works were used to analyze the stability of the systems. In the real-world non-autonomous min-max-plus systems, the input events with only maximum constraints correspond to max-plus input control functions. The input events, however, are with mixed constraints in the general case. This paper investigates general min-max-plus systems with min-max-plus input control functions, which are nonlinear extensions of the systems presented in [1], [5], [7] and [26], and focuses on the cycle time assignability with respect to the state feedback. The setting is chiefly the mathematics directed graph and max-plus algebra. The basic idea is the familiar system concept of reachability (see [7]), thought of as a graph property of distinguished max-plus projections. The directed graph setting rather quickly suggests new methods of attacking synthesis which are proved to be intuitive and

This work was supported by the Hundred Talents Program of the Chinese Academy of Sciences.

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economical. The necessary and sufficient criterion for the cycle time assignability with respect to the state feedback is established, and the design of such feedback is derived. A part of the results extends some work of [7] and [26].

The paper is organized as follows. Section 2 describes general min-max-plus system models. Section 3 introduces the reachability. Section 4 gives the max-plus projection representation of the closed-loop system. Section 5 investigates the cycle time assignability and the design of state feedback. Section 6 contains a numerical example. Finally, some conclusions are drawn in Section 7.

II. Min-Max-Plus System Models

Let us begin with some notations which are used through this paper. Let $\mathbb{R}$ be a set of all real numbers and $\mathbb{R}^n$ be an n-dimensional column vector set over $\mathbb{R}$. Vectors in $\mathbb{R}^n$ are denoted by $\mathbf{x}, \mathbf{x}^1$, etc. and x_i denotes the ith component of $\mathbf{x}$. The notation $\mathbf{x}^1 \leq \mathbf{x}^2$ denotes the usual partial order on $\mathbb{R}^n : \mathbf{x}^1 \leq \mathbf{x}^2 \iff x_i^1 \leq x_i^2$ for $1 \leq i \leq n$. It is convenient to use the infix operators $a \wedge b$ and $a \vee b$ to stand for minimum and maximum, respectively, *i.e.*, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Note that $+$ distributes over both $\wedge$ and $\vee$: $(a \wedge b) + c = a + c \wedge b + c, (a \vee b) + c = a + c \vee b + c$. In expressions like these it assumes that $+$ always has higher precedence than either $\wedge$ or $\vee$. The same symbols are also used for the corresponding operations on vectors. Since the ordering is the product ordering, it is easy to see that $(\mathbf{x}^1 \wedge \mathbf{x}^2)_i = x_i^1 \wedge x_i^2, (\mathbf{x}^1 \vee \mathbf{x}^2)_i = x_i^1 \vee x_i^2$ for $1 \leq i \leq n$.

A min-max-plus function of type $(n, 1)$ is denoted by $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^1$ which can be written as a term in the following grammar:

$$f := x_1, \cdots, x_n | f + a | f \wedge f | f \vee f, \tag{1}$$

where $x_1, \cdots, x_n$ are variables and $a \in \mathbb{R}$ is referred to as a parameter. The vertical bars separate the different ways in which terms can recursively be constructed. The simplest term is one of the n variables, x_i, thought of as the i-th component function. Given any term, a new one may be constructed by adding a; given two terms, a new one may be constructed by taking the minimum or the maximum. Only these rules may be used to build terms. For example, $(2 + x_3 \wedge x_1) \vee 1 + x_3$ is a min-max-plus function of type (3, 1) but neither $x_2 \wedge 2$ nor $x_2 \vee x_1 + x_3$ can be generated by (1).

A min-max-plus function of type (n, m) is denoted by $F(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$, such that each component $F_i(\mathbf{x})$ is a min-max-plus function of type $(n, 1)$. The set of min-max-plus functions of type (n, m) is denoted by $\mathrm{MM}(n, m)$. Let $f(\mathbf{x}) \in \mathrm{MM}(n, 1)$. If $f(\mathbf{x})$ can be represented by a term which uses $\vee$ but not $\wedge$, it is said to be max-plus. If $f(\mathbf{x})$ requires $\wedge$ but not $\vee$, it is min-plus. The same terminology extends to $F(\mathbf{x}) \in \mathrm{MM}(n, m)$ by asking that each component $F_i(\mathbf{x})$ has the property in question.

Now, recall some basic properties of min-max-plus functions. Let $F(\mathbf{x}) \in \mathrm{MM}(n, m)$. First, $F(\mathbf{x})$ is continuous. Second, $F(\mathbf{x})$ is monotone: $\mathbf{x}^1 \leq \mathbf{x}^2 \Longrightarrow F(\mathbf{x}^1) \leq F(\mathbf{x}^2)$. Third, $F(\mathbf{x})$ is homogeneous, in the sense that, for any $h \in \mathbb{R}$, $F(\mathbf{x} + h) = F(\mathbf{x}) + h$. Fourth, if $U \in \mathrm{MM}(n, m)$ and $V \in \mathrm{MM}(m, l)$, it is easy to see that $VU \in \mathrm{MM}(n, l)$. Fifth, let $F(\mathbf{x}) \in \mathrm{MM}(n, n)$, then $F(\mathbf{x})$ is non-expansive in the ℓ^∞ norm, *i.e.*, $\|F(\mathbf{x}^1) - F(\mathbf{x}^2)\| \leq \|\mathbf{x}^1 - \mathbf{x}^2\|, \forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$, where $\|\mathbf{x}\| = \max_{1 \leq i \leq n} |x_i|$, $|x_i|$ is the usual absolute value on real numbers.

A max-plus algebra is a structure consisting of the set $\mathbb{R} \cup \{-\infty\}$ together with two operators $\vee$ and +, denoted by $\mathbb{D}$. Here, $-\infty$ and 0 are the zero element and the identity element of $\mathbb{D}$, respectively. A detailed exposition can be found in [1] and [9].

The min-max-plus system is described using the state transition equation

$$\mathbf{x}(k+1) = F(\mathbf{x}(k)) \vee G(\mathbf{u}(k)), \ k = 0, 1, \cdots \tag{2}$$

where $\mathbf{x}(0) = \zeta \in \mathbb{R}^n, \mathbf{u}(0) = \eta \in \mathbb{R}^p$, $F(\mathbf{x}) \in \mathrm{MM}(n, n)$ and $G(\mathbf{u}) \in \mathrm{MM}(p, n)$ are interior and input functions, respectively, and $\mathbf{x}(k) = [x_1(k) \ \cdots \ x_n(k)]^\tau \in \mathbb{R}^n$ and $\mathbf{u}(k) = [u_1(k) \ \cdots \ u_p(k)]^\tau \in \mathbb{R}^p$ are state and input vectors, respectively. The $F(\mathbf{x})$ and $G(\mathbf{u})$ above are general min-max-plus functions. If $F(\mathbf{x})$ and $G(\mathbf{u})$ are max-plus functions, (2) is the model in [1] and [7]. If $F(\mathbf{x})$ is a min-max-plus function and $G(\mathbf{u})$ is a max-plus function, (2) is the model in [5] and [26]. Theoretically, the model (2) is the nonlinear extension of the models of [1], [5], [7] and [26]. Hence the results of this paper can be expected to include some results in [1], [5], [7] and [26] as special cases. The system described by model (2) is called the open-loop min-max-plus system with min-max-plus input control function and is denoted by S. In addition, the component $G_i(\mathbf{u}) = -\infty$ is allowed for the system S. If $G_i(\mathbf{u}), 1 \leq i \leq n$, are all $-\infty$, (2) is the autonomous min-max-plus system.

III. Reachability

Let $F(\mathbf{x}) \in \mathrm{MM}(n, m)$. $F(\mathbf{x})$ can be placed in a conjunctive normal form (see [14]):

$$F_i(\mathbf{x}) = f_i^1(\mathbf{x}) \wedge \cdots \wedge f_i^{l(i)}(\mathbf{x}), \quad 1 \leq i \leq m, \tag{3}$$

where the max-plus functions $f_i^{e_i}(\mathbf{x}) = a_{i1}^{e_i} + x_1 \vee \cdots \vee a_{in}^{e_i} + x_n, \ a_{ij}^{e_i} \in \mathbb{D}, \ 1 \leq e_i \leq l(i), \ l(i)$ is the number of max-plus functions of type $(n,1)$ in the component $F_i(\mathbf{x})$. $a_{ij}^{e_i}=-\infty$ merely indicates the absence of the variable x_j in $f_i^{e_i}(\mathbf{x})$. $[a_{i1}^{e_i}\ \cdots\ a_{in}^{e_i}]$ is said to be the coefficient row vector of $f_i^{e_i}(\mathbf{x})$ over $\mathbb{D}$ and $a_{ij}^{e_i}$ is called the coefficient of the jth variable x_j of $f_i^{e_i}(\mathbf{x})$. For example, by the distributivity of $\vee$ over $\wedge$, $(2+x_3\wedge x_1)\vee 1+x_3$ can be rewritten as the conjunctive normal form $(-\infty+x_1\vee-\infty+x_2\vee 2+x_3)\wedge(x_1\vee-\infty+x_2\vee 1+x_3)$, and $[-\infty\ \ -\infty\ \ 2]$ is the coefficient row vector corresponding to the preceding max-plus function. An $m\times n$ max-plus matrix A associating with $F(\mathbf{x})$ is constructed by taking the coefficient row vector of $f_i^{e_i}(\mathbf{x})$ as the ith row of A. The matrix A constructed in this way is called a max-plus projection of $F(\mathbf{x})$. The set of max-plus projections is the collection of all such matrices from a single conjunctive form for $F(\mathbf{x})$ such as (3) and is denoted by $P(F)$. $P(F)$ contains $\prod_{i=1}^{m} l(i)$ max-plus projections. Since the conjunctive normal form of each component $F_i(\mathbf{x})$ is unique up to re-ordering of the conjunctions, the set of max-plus projections of $F(\mathbf{x})$ is uniquely defined. In practice it is often more convenient to work with whatever set of projections is easiest to construct instead of doing the additional work necessary to find the set of normal projections.

Using (3) and the max-plus projections, $F(\mathbf{x})$ can be written as

$$F(\mathbf{x})=\wedge_{r\in\mathcal{I}}A_r\mathbf{x}, \tag{4}$$

where $A_r\in\mathbb{D}^{m\times n}$ are max-plus projections of $F(\mathbf{x})$, $A_r\mathbf{x}$ is the matrix product over $\mathbb{D}$, $\mathcal{I}$ is the finite index set of $P(F)$. (4) is called the max-plus projection representation of $F(\mathbf{x})$ and is used to calculate the cycle times and the globally optimal solutions of min-max-plus systems (see [12], [27]).

By the method stated above, the min-max-plus functions $G(\mathbf{u})$ of type (p,n) can also be written as

$$G(\mathbf{u})=\wedge_{s\in\mathcal{J}}B_s\mathbf{u}, \tag{5}$$

where $B_s\in\mathbb{D}^{n\times p}$, $B_s\mathbf{u}$ is the matrix product over $\mathbb{D}$, $\mathcal{J}$ is the finite index set of the max-plus projection set of $G(\mathbf{u})$. If $G_i(\mathbf{u})=-\infty$, the ith rows of all B_s are a zero row vector over $\mathbb{D}$. Using (4) and (5), (2) can be rewritten as

$$\mathbf{x}(k+1)=\wedge_{(r,s)\in\mathcal{I}\times\mathcal{J}}\left(A_r\mathbf{x}(k)\vee B_s\mathbf{u}(k)\right),\ k=0,1,\cdots \tag{6}$$

where $\mathcal{I}\times\mathcal{J}$ is the Cartesian product of $\mathcal{I}$ and $\mathcal{J}$. Correspondingly, S has the following open-loop max-plus projection systems $\mathbf{x}(k+1)=A_r\mathbf{x}(k)\vee B_s\mathbf{u}(k)$, $(r,s)\in\mathcal{I}\times\mathcal{J}$, and are denoted by $S_{(r,s)}$.

The system $S_{(r,s)}$ can be described by a directed graph. a_{ij}^r denotes the element of the matrix A_r in the ith row and the jth column. The precedence graph of A_r, denoted by $\mathcal{G}(A_r)$, is the directed graph with annotated edges which has the state nodes $x_1,\cdots,x_n$ and there exists an edge from x_j to x_i if and only if $a_{ij}^r\neq-\infty$, which has the annotation a_{ij}^r. In order to express $S_{(r,s)}$, the input nodes $u_1,\cdots,u_p$ to $\mathcal{G}(A_r)$ are added and joined to $x_1,\cdots,x_n$: there exists an edge from u_j to x_i with the annotation b_{ij}^s if and only if $b_{ij}^s\neq-\infty$, where b_{ij}^s is the element of B_s in the ith row and the jth column. The directed graph above is called the directed graph of $S_{(r,s)}$ and is denoted by $\mathcal{G}(S_{(r,s)})$. A path in this directed graph has the usual meaning of a chain of directed edges and a circuit is a path which starts and ends at the same node. A circuit is elementary if the nodes are all distinct. The weight of a path is the sum of the annotations on the edges in the path. The length of a path is the number of edges in the path.

Definition 1: If there is a path from an input node to the state node x_i in $\mathcal{G}(S_{(r,s)})$, then x_i is called the (r,s)-*reachable* state component of the system S. x_i is called $reachable$ if for all $(r,s)\in\mathcal{I}\times\mathcal{J}$, x_i is (r,s)-reachable. S is called reachable if all its state components are reachable.

The concept presented above includes the concepts of reachability in [1], [5], [7] and [26] as special cases. Clearly, $G_i(\mathbf{u})=-\infty$ implies that there does not exist any edge from an input node to the state node x_i in every $\mathcal{G}(S_{(r,s)})$. When $G_i(\mathbf{u})\neq-\infty$, the ith row of every max-plus projection of $G(\mathbf{u})$ is nonzero, $i.e.$, there exist some edges from input nodes to x_i in every $\mathcal{G}(S_{(r,s)})$ and hence x_i is reachable. Such a state node is said to be *directly reachable*.

IV. Direct Products of Max-Plus Projections

Suppose that the open-loop system (2) can freely be modified by setting the state feedback

$$\mathbf{u}(k)=K(\mathbf{x}(k)), \tag{7}$$

where $K(\mathbf{x})\in\mathrm{MM}(n,p)$ (here, $K_i(\mathbf{x})=-\infty$ is allowed), then (2) becomes the following closed-loop system

$$\mathbf{x}(k+1)=F(\mathbf{x}(k))\vee G(K(\mathbf{x}(k))),\ k=0,1,\cdots \tag{8}$$

where $\mathbf{x}(0)=\zeta\in\mathbb{R}^n$. The state feedback function in (7) is sometimes denoted by K, and (8) is denoted by $S(K)$.

It is clear that the mathematical setting of (8) is the min-max-plus function of type (n,n)

$$F(\mathbf{x})\vee G(K(\mathbf{x}))=\wedge_{(r,s)\in\mathcal{I}\times\mathcal{J}}\left(A_r\mathbf{x}\vee B_sK(\mathbf{x})\right). \tag{9}$$

Let us give the max-plus projection representation of $B_sK(\mathbf{x})$. Let $B_s=[B_s^1\ \cdots\ B_s^n]^\tau$, where $B_s^i, 1\leq i\leq n$ are the 1st, $\cdots$, nth row vectors of B_s, and $K(\mathbf{x})=\wedge_{d\in\mathcal{H}}\bar{K}_d\mathbf{x}$

be a max-plus projection representation of $K(\mathbf{x})$. Using max-plus projections of $K(\mathbf{x})$, it is constructed that $K_t := [\bar{K}_{d_1} \ \cdots \ \bar{K}_{d_n}]^\tau$, and $d_j \in \mathcal{H}, 1 \leq j \leq n$. The matrix K_t consists of some max-plus projections of $K(\mathbf{x})$, and is called the *direct product of max-plus projections* of $K(\mathbf{x})$. It is defined that the operation $B_s \odot K_t = [B_s^1 \bar{K}_{d_1} \ \cdots \ B_s^n \bar{K}_{d_n}]^\tau$, where $B_s^i \bar{K}_{d_i}, 1 \leq i \leq n$, are the matrix products over $\mathbb{D}$.

Lemma 1: $B_s K(\mathbf{x}) = \wedge_{t \in \mathcal{L}} (B_s \odot K_t)\mathbf{x}$, where $\mathcal{L}$ is the finite index set of the set of all direct products of max-plus projections of $K(\mathbf{x})$.

Proof: Let b_{ij}^s denote the element of the matrix B_s in the ith row and the jth column and let $K_i(\mathbf{x}) = k_i^1(\mathbf{x}) \wedge \cdots \wedge k_i^{l(i)}(\mathbf{x}), 1 \leq i \leq p$ be a conjunctive normal form corresponding to a max-plus projection representation of $K(\mathbf{x})$. Using the mutual distributivity of $\vee$ and $\wedge$,

$$\begin{aligned}
& [b_{i1}^s \ \cdots \ b_{ip}^s][K_1(\mathbf{x}) \ \cdots \ K_p(\mathbf{x})]^\tau \\
= \ & b_{i1}^s + K_1(\mathbf{x}) \vee \cdots \vee b_{ip}^s + K_p(\mathbf{x}) \\
= \ & (b_{i1}^s + k_1^1(\mathbf{x}) \wedge \cdots \wedge b_{i1}^s + k_1^{l(1)}(\mathbf{x})) \vee \cdots \\
& \vee (b_{ip}^s + k_p^1(\mathbf{x}) \wedge \cdots \wedge b_{ip}^s + k_p^{l(p)}(\mathbf{x})) \\
= \ & (b_{i1}^s + k_1^1(\mathbf{x}) \vee \cdots \vee b_{ip}^s + k_p^1(\mathbf{x})) \wedge \cdots \\
& \wedge (b_{i1}^s + k_1^{l(1)}(\mathbf{x}) \vee \cdots \vee b_{ip}^s + k_p^{l(p)}(\mathbf{x})) \\
= \ & [b_{i1}^s \ \cdots \ b_{ip}^s] \begin{bmatrix} k_{11}^1 & \cdots & k_{1n}^1 \\ \vdots & \ddots & \vdots \\ k_{p1}^1 & \cdots & k_{pn}^1 \end{bmatrix} \mathbf{x} \wedge \cdots \\
& \wedge [b_{i1}^s \ \cdots \ b_{ip}^s] \begin{bmatrix} k_{11}^{l(1)} & \cdots & k_{1n}^{l(1)} \\ \vdots & \ddots & \vdots \\ k_{p1}^{l(p)} & \cdots & k_{pn}^{l(p)} \end{bmatrix} \mathbf{x}. \qquad (10)
\end{aligned}$$

The result follows by the definitions of max-plus projection and its direct products. The proof is completed.

Using Lemma 1, (9) can be rewritten as

$$F(\mathbf{x}) \vee G(K(\mathbf{x})) = \wedge_{(r,s,t) \in \mathcal{I} \times \mathcal{J} \times \mathcal{L}} (A_r \vee (B_s \odot K_t))\mathbf{x}, \qquad (11)$$

where $\mathcal{I} \times \mathcal{J} \times \mathcal{L}$ is the Cartesian product of $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$. $A_r \vee (B_s \odot K_t)$ is called the max-plus projection of $F(\mathbf{x}) \vee G(K(\mathbf{x}))$, and the corresponding system is called the closed-loop max-plus projection system of $S(K)$ and denoted by $S_{(r,s)}(K_t)$. It merits attention that if $\bar{K}_{d_1} = \cdots = \bar{K}_{d_n} = \bar{K}_{d_0}$, $B_s \odot K_t = B_s \bar{K}_{d_0}$, where $B_s \bar{K}_{d_0}$ is the matrix product over $\mathbb{D}$. Hence, (11) includes the closed-loop system modified by means of a max-plus state feedback as a special case. From (11), the following result is derived immediately.

Theorem 1: $\{A_r \vee (B_s \odot K_t) | (r,s,t) \in \mathcal{I} \times \mathcal{J} \times \mathcal{L}\}$ is the set of max-plus projections of $F(\mathbf{x}) \vee G(K(\mathbf{x}))$.

Let us see the directed graph of the closed-loop max-plus projection system $S_{(r,s)}(K_t)$. In $\mathcal{G}(S_{(r,s)})$, an edge from the state node x_j to the input node u_i with the annotation k_{ij}^t is drawn if and only if k_{ij}^t is a nonzero element at position (i,j) of a direct product factor of K_t, and the edge $x_j u_i$ is called the t-feedback edge. Such a graph is called the directed graph of $S_{(r,s)}(K_t)$ and is denoted by $\mathcal{G}(S_{(r,s)}(K_t))$. In graph theory, $\mathcal{G}(S_{(r,s)}(K_t))$ is a directed pseudograph without any self-loop with feedback edges. The precedence graph of $A_r \vee (B_s \odot K_t)$, denoted by $\mathcal{G}(A_r \vee (B_s \odot K_t))$, is the directed graph with annotated edges which has the state nodes $x_1, \cdots, x_n$ and an edge from x_j to x_i with the annotation a_{ij}^r if and only if $a_{ij}^r \neq -\infty$, or an edge from x_j to x_i with the annotation $(B_s \odot K_t)_{ij}$ if and only if $(B_s \odot K_t)_{ij} \neq -\infty$. It can be seen that the directed graph $\mathcal{G}(A_r \vee (B_s \odot K_t))$ is different from the directed graph $\mathcal{G}(S_{(r,s)}(K_t))$.

V. Cycle Time Assignability and Feedback Design

Let $F(\mathbf{x}) \in \mathrm{MM}(n,n)$. $F^k(\mathbf{x})$ is defined as $F^0(\mathbf{x}) = \mathbf{x}, F^k(\mathbf{x}) = F(F^{k-1}(\mathbf{x}))$. It is easy to see that $F^k(\mathbf{x}) \in \mathrm{MM}(n,n)$, for any $k \geq 0$. The *cycle time vector* of the function $F(\mathbf{x})$, denoted by $\chi(F)$, is defined as $\chi(F) = \lim_{k \to \infty} F^{k+1}(\mathbf{x})/(k+1)$. $\chi(F)$ exists and is independent of the initial vector $\mathbf{x}$ (see [12]). The cycle time vectors of the systems $S_{(r,s)}$ and S are denoted by $\mu(S_{(r,s)})$ and $\chi(S)$, respectively. Since the interior functions of $S_{(r,s)}$ and S are $A_r\mathbf{x}$ and $F(\mathbf{x})$, respectively, $\mu(S_{(r,s)})$=$\mu(A_r)$ $\chi(S)$=$\chi(F)$. It is clear that the performance metric of the closed-loop system $S(K)$ depends on the parameters of its state feedback function. By Corollary 1 of [12], the cycle times $\chi(S(K))$ exist and are equal to $\chi(F(\mathbf{x}) \vee G(K(\mathbf{x})))$ for all values of $K(\mathbf{x})$. Let $\mathbf{k}$ be the real vector consisting of all distinct parameters of $K(\mathbf{x})$ and $Z(\mathbf{x}, \mathbf{k}) = F(\mathbf{x}) \vee G(K(\mathbf{x}))$. The notation $Z(\mathbf{x}, \mathbf{k})$ is used to show the dependency of the value of $F(\mathbf{x}) \vee G(K(\mathbf{x}))$ on parameters of $K(\mathbf{x})$. It can be seen from non-expansiveness that $\chi(Z(\mathbf{x}, \mathbf{k}))$ are independent of the initial vector $\mathbf{x}$.

Definition 2: If there exists a state feedback (7) such that the tth component of $\chi(S(K))$ does not have any upper bound with respect to parameters of $K(\mathbf{x})$, then the tth component of the cycle time of S is said to be *assignable* by a state feedback. The cycle time of S is said to be assignable if all its cycle time components are assignable.

Theorem 2: The cycle time of the system S is assignable if and only if S is reachable.

Before proceeding to the proof we separate off the following lemmas which will serve as the steps of the proof.

Lemma 2: $\chi(Z(\mathbf{x},\mathbf{k}))$ is a monotone increasing and continuous function with respect to $\mathbf{k}$.

Proof: It is direct to verify that if $\mathbf{k}_1 \geq \mathbf{k}_2$, then $K_1(\mathbf{x}) \geq K_2(\mathbf{x})$ for any $\mathbf{x}$. It follows from monotonicity that $G(K_1(\mathbf{x})) \geq G(K_2(\mathbf{x}))$. Hence $F(\mathbf{x}) \vee G(K_1(\mathbf{x})) \geq F(\mathbf{x}) \vee G(K_2(\mathbf{x}))$, *i.e.*, $Z(\mathbf{x},\mathbf{k}_1) \geq Z(\mathbf{x},\mathbf{k}_2)$. By monotonicity, $Z(Z(\mathbf{x},\mathbf{k}_1),\mathbf{k}_1) \geq Z(Z(\mathbf{x},\mathbf{k}_2),\mathbf{k}_1) \geq Z(Z(\mathbf{x},\mathbf{k}_2),\mathbf{k}_2)$. By induction, $Z^{k+1}(\mathbf{x},\mathbf{k}_1) \geq Z^{k+1}(\mathbf{x},\mathbf{k}_2)$ for any $k \geq 0$. Dividing both sides with $k+1$ and taking limits gives $\chi(Z(\mathbf{x},\mathbf{k}_1)) \geq \chi(Z(\mathbf{x},\mathbf{k}_2))$, *i.e.*, $\chi(Z(\mathbf{x},\mathbf{k}))$ is monotone increasing. It is clear that $\chi(Z(\mathbf{x},\mathbf{k}))$ is a continuous function with respect to $\mathbf{k}$. The proof is completed.

Definition 3[15]: If A is an $n\times n$ matrix over $\mathbb{D}$, let $\mu(A)$ be the vector such that $\mu_i(A) = \max\{w(c)/l(c)|c$ is a circuit in $\mathcal{G}(A)$ upstream from node $i\}$, where $w(c)$ and $l(c)$ are the weight and length of a circuit c, respectively. $\mu(A)$ is called the vector of *maximum cycle means* of A.

Lemma 3[12]: Let $F(\mathbf{x}) \in \mathrm{MM}(n,n)$ and be given by (4). Then $\chi(F) = \wedge_{r\in\mathcal{I}}\mu(A_r)$.

Lemma 4: $\chi(S(K)) = \wedge_{(r,s,t)\in\mathcal{I}\times\mathcal{J}\times\mathcal{L}}\, \mu(A_r \vee (B_s \odot K_t))$ for all values of $\mathbf{k}$.

Proof: The result follows immediately from Theorem 1 and Lemma 3. The proof is completed.

Let us now return to Theorem 2.

Proof of Theorem 2:

Necessity Assume that S is not reachable. Without loss of generality, assume that the state node x_1 is not reachable, $i.e.$, there exists at least an $S_{(r_0,s_0)}$ such that there does not exist any path from an input node to x_1 in $\mathcal{G}(S_{(r_0,s_0)})$. Assume $S(K)$ is a closed-loop system modified using any state feedback function $K(\mathbf{x})$. Since there must exist an input node in any circuit with feedback edges, all circuits in $\mathcal{G}(S_{(r_0,s_0)}(K_t))$ upstream from x_1 do not contain any feedback edge. Hence $\mu_1(A_{r_0} \vee (B_{s_0} \odot K_t))$ is independent of parameters of $K(\mathbf{x})$ and is equal to $\mu_1(A_{r_0})$. It follows from Lemma 4 that $\chi_1(S(K)) \leq \wedge_{t\in\mathcal{L}}\mu_1(A_{r_0} \vee (B_{s_0} \odot K_t)) = \mu_1(A_{r_0})$. Hence $\chi_1(S(K))$ has an upper bound for all values of parameters of $K(\mathbf{x})$. This contradicts the cycle time assignability of S, and hence proves the necessity.

Sufficiency Since S is a reachable system, without loss of generality, assume $G_1(\mathbf{u}) \neq -\infty$, *i.e.*, x_1 is a directly reachable state note. By constructing the max-plus state feedback $\mathbf{u}(k) = K\mathbf{x}(k)$, where $K \in \mathbb{D}^{p\times n}$ in which all elements of the 1st column are nonzero and the elements of the other columns are all zero, the closed-loop system $S(K)$ is obtained. It can be said with certainty that the cycle time of the system S is assignable by the K above. In fact, $G_1(\mathbf{u}) \neq -\infty$ implies that the 1st rows of max-plus projections B_s are all nonzero. Let us consider the closed-loop max-plus projection $S_{(r,s)}(K)$. Assume that $b^s_{1j} \neq -\infty$. Then there exists the circuit $x_1u_jx_1$ upstream from x_1 in $\mathcal{G}(S_{(r,s)}(K))$. The state node x_1 may have the other upstream circuits with the feedback edges $x_1u_j, 1 \leq j \leq p$. Hence $\mu_1(A_r \vee B_sK)$ is a non-constant function with respect to nonzero elements of K. It follows from Lemma 2 that $\mu_1(A_r \vee B_sK) \to +\infty$, as $k_{j1} \to +\infty$ for $1 \leq j \leq p$. Since $\chi_1(S(K))$ is built from $\mu_1(A_r \vee B_sK)$, where $(r,s) \in \mathcal{I}\times\mathcal{J}$, by application of finitely many $\wedge$, $\chi_1(S(K)) \to +\infty$, as $k_{j1} \to +\infty$ for $1 \leq j \leq p$, $i.e.$, $\chi_1(S(K))$ does not have any upper bound with respect to nonzero elements of K. On the other hand, for all $(r,s) \in \mathcal{I}\times\mathcal{J}$, it follows from the structure of K and the reachability of S that any circuit upstream from x_1 in $\mathcal{G}(S_{(r,s)}(K))$ is a circuit upstream from $x_i, 2 \leq i \leq n$. Hence, $\chi_i(S(K)) \to +\infty, 2 \leq i \leq n$, as $k_{j1} \to +\infty$ for $1 \leq j \leq p$. The proof is completed.

It can be seen from the proof of Theorem 2 that the cycle time assignability of a reachable min-max-plus system can be achieved by the max-plus state feedback. In fact, the proof of sufficiency of Theorem 2 constructs such a feedback. In general, such a feedback is not unique (see Example 1 in the next section). In addition, it should be pointed that the constructed state feedback differs from one of [28] in structure.

This definition of assignability is a weak one. The results above can be applied to stabilize the systems. Because of its complexity and the limitation of space for this paper the corresponding results will be discussed in future papers.

VI. Numerical Example

The following example illustrates how the methods work in practice.

Example 1: The non-autonomous dynamical min-max-plus system $\hat{S}$ is determined by the interior function $\hat{F}(\mathbf{x})$: $\hat{F}_1(\mathbf{x}) = 1+x_1, \hat{F}_2(\mathbf{x}) = 2+x_3 \wedge (x_1 \vee 1+x_3), \hat{F}_3(\mathbf{x}) = 3+x_2$ and the input function $\hat{G}(\mathbf{u}) : \hat{G}_1(\mathbf{u}) = 3+u_2, \hat{G}_2(\mathbf{u}) = -\infty, \hat{G}_3(\mathbf{u}) = 1+u_1 \wedge 4+u_2$. Here, both $\hat{F}(\mathbf{x})$ and $\hat{G}(\mathbf{u})$ are min-max-plus functions. $\hat{F}(\mathbf{x})$ has two max-plus projections:

$$\hat{A}_1 = \begin{bmatrix} 1 & -\infty & -\infty \\ -\infty & -\infty & 2 \\ -\infty & 3 & -\infty \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 1 & -\infty & -\infty \\ 0 & -\infty & 1 \\ -\infty & 3 & -\infty \end{bmatrix}$$

$\hat{G}(\mathbf{u})$ has two max-plus projections:

$$\hat{B}_1 = \begin{bmatrix} -\infty & 3 \\ -\infty & -\infty \\ 1 & -\infty \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -\infty & 3 \\ -\infty & -\infty \\ -\infty & 4 \end{bmatrix}.$$

Constructing the state feedback matrix

$$\hat{K} = \begin{bmatrix} -\infty & -\infty & k_{13} \\ -\infty & -\infty & k_{23} \end{bmatrix},$$

obtains the closed-loop system $\hat{S}(\hat{K})$. By a direct calculation, $\chi_i(\hat{S}(\hat{K})) \to +\infty,\ 1 \leq i \leq 3$, as $k_{j3} \to +\infty$ for $j = 1, 2$, $i.e.$, the cycle time of $\hat{S}$ is assignable by the state feedback $\hat{K}$. The state feedback used to the assignment is also

$$\tilde{K} = \begin{bmatrix} -\infty & k_{12} & -\infty \\ -\infty & k_{22} & -\infty \end{bmatrix}.$$

In addition, let us use the result of the paper to analyze the stability of $\hat{S}$. A min-max-plus system is said to be stable if each component of its cycle time has the same value (see [6], [15], [20]). By Definition 3 and Lemma 3, the open-loop cycle time $\chi(\hat{S}) = [1\ 2\ 2]^\tau$ (For min-max-plus systems an efficient algorithm for computing cycle times was presented in [12]). This means that $\hat{S}$ is not stable. By a simple calculation, if $k_{23} \geq k_{13} \geq 3$, $\hat{S}(\hat{K})$ is stable.

VII. Conclusions

This paper has studied the cycle time assignability of min-max-plus systems with min-max-plus input control functions. The systems considered are more general than those considered earlier by Baccelli *et al.* (see [1], [5], [7], [26]). The algebraic and graphic type necessary and sufficient criterion for the cycle time assignability with respect to the state feedback has been established, and the design for the state feedback has been presented. It has also been pointed out that the state feedback used to the cycle time assignment can be set by the max-plus function. The max-plus projection representation based on directed graph and max-algebra are used to analyze the reachability and construct the state feedback. The proposed methods are constructive in nature.

VIII. Acknowledgments

The authors would like to thank the reviewers for their valuable comments and suggestions.

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