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GPC-LPV: a predictive LPV controller based on BMIs†

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***Abstract*— In this paper the authors present a predictive linear parameter varying (LPV) controller based on the GPC controller [1]–[3], for nonlinear systems. The resulting controller is denoted as GPC-LPV. This one has the same structure as a general LPV controller [4]–[7], which has a lineal fractional dependence on the process signal measurements. Therefore, this controller has the ability of modifying its dynamics depending on measurements leading to a possibly nonlinear controller. That controller is designed in two steps. First, for a given steady state point is obtained a linear GPC using a local model of the nonlinear system around that operating point. And second, using bilinear matrix inequalities (BMIs) the remaining matrices of GPC-LPV are selected in order to achieve some closed loop properties: stability in some operation zone, norm bounding of some input/output channels, maximum settling time, maximum overshoot, etc. This methodology of design can be applied to nonlinear systems which can be embedded in a LPV system using differential inclusion techniques. Finally, the GPC-LPV is applied to the nonlinear model of a liquid-gas separation process.**

I. INTRODUCTION

The generalized predictive controller (GPC) originally was developed by Clarke [1], [2]. This linear controller is a particular case of model based predictive controllers, which uses a CARIMA (controlled autoregressive integral moving average) model for the process. GPC has some interesting properties [3]:

- It can be applied to unstable and nonmininum-phase processes.
- It can be used as and adaptive controller.
- It has a more complex noise model than dynamic matrix control (DMC) [8] and Identification Command controller (IDCOM) [9].

Moreover, it has been validated in a wide spectrum of real-life applications [10].

Starting from this point, the authors have developed a reformulation of this controller in state space [11], since the GPC of Clarke was designed using transfer functions. The result is a GPC composed by a full rank observer and a state feedback controller, which gives an output feedback GPC controller [12]:

$$\begin{aligned} \boldsymbol{x}^c(k+1) &= \boldsymbol{A}^c\boldsymbol{x}^c(k) + \boldsymbol{B}_r^c\boldsymbol{r}(k) + \boldsymbol{B}_y^c\boldsymbol{y}(k),\\ \boldsymbol{u}(k) &= \boldsymbol{C}^c\boldsymbol{x}^c(k) + \boldsymbol{D}_r^c\boldsymbol{r}(k) + \boldsymbol{D}_y^c\boldsymbol{y}(k), \end{aligned} \tag{1}$$

being $\boldsymbol{y}$ the output vector of size n, $\boldsymbol{r}$ the reference vector, $\boldsymbol{u}$ the control action vector of size m, and $\boldsymbol{x}^c$ the controller state vector of size n_c.

A. Matrix inequalities

A linear matrix inequality (LMI) is an expression of the form [13]:

$$\boldsymbol{F}(\boldsymbol{x}) \triangleq \boldsymbol{F}_0 + \sum_{i=1}^{m} x_i\boldsymbol{F}_i > 0, \tag{2}$$

where $\boldsymbol{x}\in\mathbb{R}^m$ is the unknowns vector and the symmetric matrices $\boldsymbol{F}_i = \boldsymbol{F}_i^T \in \mathbb{R}^{n\times n}$, $i = 0,\ldots,m$ are given. The inequality symbol $>$ means that $\boldsymbol{F}(\boldsymbol{x})$ is a positive definite matrix. By definition, the previous LMI is strict, although it is possible to consider non-strict LMIs using $\geq$ instead of $>$.

A bilinear matrix inequality is a generalization of a LMI incorporating products between unknowns:

$$\boldsymbol{F}(\boldsymbol{x}) \triangleq \boldsymbol{F}_0 + \sum_{i=1}^{m} x_i\boldsymbol{F}_i + \sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \boldsymbol{F}_{i,j} > 0. \tag{3}$$

Following these lines, a nonlinear matrix inequality (NMI) is matrix inequality where the dependence with respect to unknowns is a general nonlinear function. A special case that frequently occurs in practice consists of a polynomial dependence, which gives polynomial matrix inequalities (PMI).

B. LPV controllers

For last years many authors [4]–[7], [14]–[17] have been developed linear parameter varying (LPV) controllers for nonlinear systems. The key idea consists to to modify the controller matrices to adapt the controller to the nonlinear system depending on signal measurements. The most general dependence of controller matrices with respect to measurements is linear fractional (LFR) [4], and in particular for discrete-time systems the controller structure is [18]:

$$\begin{pmatrix} \boldsymbol{x}^c(k+1)\\ \boldsymbol{y}_\Delta^c(k)\\ \boldsymbol{u}(k)\end{pmatrix} = \underbrace{\left(\begin{array}{c|ccc} \boldsymbol{A}^c & \boldsymbol{B}_\Delta^c & \boldsymbol{B}_r^c & \boldsymbol{B}_y^c\\ \hline \boldsymbol{C}_\Delta^c & \boldsymbol{D}_\Delta^c & \boldsymbol{D}_{\Delta,r}^c & \boldsymbol{D}_{\Delta,y}^c\\ \boldsymbol{C}^c & \boldsymbol{D}_{u,\Delta}^c & \boldsymbol{D}_r^c & \boldsymbol{D}_y^c \end{array}\right)}_{\boldsymbol{K}(z)} \begin{pmatrix} \boldsymbol{x}^c(k)\\ \boldsymbol{u}_\Delta^c(k)\\ \boldsymbol{r}(k)\\ \boldsymbol{y}(k)\end{pmatrix},$$
$$\boldsymbol{u}_\Delta^c(k) = \Delta_m^c(k)\boldsymbol{y}_\Delta^c(k), \tag{4}$$

$\Delta_m^c(k)$ is a matrix which affinely depends on signal measurements. In Fig. 1 this structure is represented, which is essentially composed of an upper linear fractional transformation between a linear time invariant controller and the time varying matrix Δ_m^c.

The synthesis of such controllers is based on solving a feasibility problem with LMIs and/or BMIs [13], or based on

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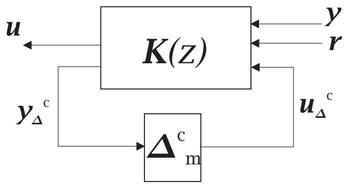

Fig. 1. Structure of a general LPV controller.

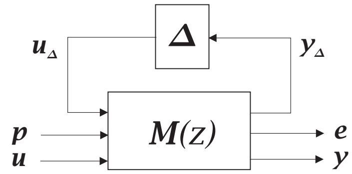

Fig. 2. Structure of the linear time varying model.

solving a linear optimization subject to LMIs and/or BMIs. The problems with LMIs are always convex and can be efficiently solved in polynomial time using, for example, interior points algorithms [13]. As opposite, the problems with BMIs are nonconvex, and there do not exist algorithms to solve them in polynomial time, and depending on the particular structure problem neither in nonpolynomial time.

C. LPV models for nonlinear systems

All the references presented in the previous section require a LPV model for the nonlinear system in order to design the LPV controller. Usually the available information about a certain nonlinear system is a nonlinear model. Therefore, the first step for these methods is to obtain a LPV model whose dynamical trajectories contain the nonlinear model ones, using techniques of linear differential inclusion [13]. The key idea used in differential inclusion consists on replacing the nonlinear part of the system model by an expression which has a linear fractional (LFR) dependence with respect to the signals present in this nonlinear part [4], [13]. The result of this operation is a linear time varying model which depends LFR on that signals, as Fig. 2 shows. In particular, its mathematical representation is:

$$\begin{pmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{y}_\Delta(k) \\ \boldsymbol{e}(k) \\ \boldsymbol{y}(k) \end{pmatrix} = \underbrace{\left(\begin{array}{c|ccc} \boldsymbol{A} & \boldsymbol{B}_\Delta & \boldsymbol{B} & \boldsymbol{B}_u \\ \hline \boldsymbol{C}_\Delta & \boldsymbol{D}_\Delta & \boldsymbol{D}_{\Delta,p} & \boldsymbol{D}_{\Delta,u} \\ \boldsymbol{C} & \boldsymbol{D}_{e,\Delta} & \boldsymbol{D} & \boldsymbol{D}_{e,u} \\ \boldsymbol{C}_y & \boldsymbol{D}_{y,\Delta} & \boldsymbol{D}_{y,p} & \boldsymbol{D}_{y,u} \end{array}\right)}_{\boldsymbol{M}(z)} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{u}_\Delta(k) \\ \boldsymbol{p}(k) \\ \boldsymbol{u}(k) \end{pmatrix},$$

$$\boldsymbol{u}_\Delta(k) = \Delta(k)\boldsymbol{y}_\Delta(k), \tag{5}$$

where Δ depends affinely on some system signals, $\boldsymbol{p}$ is a vector containing any input signal different from control actions, and $\boldsymbol{e}$ is a vector containing all the output signals which can give a system performance measure.

This linear time varying system can be viewed as a linear parameter varying one since the signals present in Δ can be interpreted as parameters that are time varying. Therefore, this is a LPV model for the nonlinear system.

In general, not all the signals present in Δ will be measurable, and so the LPV controllers designed for this LPV model only will can use measured ones Δ_m:

$$\Delta = \begin{pmatrix} \Delta_m & \mathbf{0} \\ \mathbf{0} & \Delta_{nm} \end{pmatrix}, \tag{6}$$

and so, Δ_m^c only will depend on Δ_m.

In these cases the designed LPV controllers are called robust, since, at least, they must stabilize the LPV model with time varying parameters that cannot be measured on line.

II. GPC-LPV

The GPC-LPV controller is a LPV controller based on the linear state space GPC (1) presented in section 1. The key idea consists of adding to a designed GPC a linear fractional dependence with respect the signals present in the matrix Δ_m, and therefore, this gives the LPV controller (4) (Fig. 3), but in this particular case the matrices $\boldsymbol{A}^c$, $\boldsymbol{B}_r^c$, $\boldsymbol{B}_y^c$, $\boldsymbol{C}^c$, $\boldsymbol{D}_r^c$ and $\boldsymbol{D}_y^c$ are known since the linear GPC is designed in a first step. The remaining matrices must be designed in a second step: $\boldsymbol{B}_\Delta^c$, $\boldsymbol{C}_\Delta^c$, $\boldsymbol{D}_\Delta^c$, $\boldsymbol{D}_{\Delta,r}^c$, $\boldsymbol{D}_{\Delta,y}^c$ and $\boldsymbol{D}_{u,\Delta}^c$, which can be referred as delta matrices.

The initial linear GPC is designed by using a linear local model of the LPV model around an operating point, which belongs to the nonlinear system operation zone. This local model is obtained from LPV model assuming the signals of matrix Δ a constant and equal to the signal values at that operating point. The LTI GPC has a number of integrators equal to the number of output controlled signals.

The second step of the design consists of obtaining the delta matrices. However, there is a initial problem which must be solved: the resulting GPC-LPV may not have, in general, the integral behaviour of LTI GPC. The state matrix of GPC-LPV (4) is:

$$\boldsymbol{A}^c + \boldsymbol{B}_\Delta^c \Delta_m^c \left(\boldsymbol{I} - \boldsymbol{D}_\Delta^c \Delta_m^c\right)^{-1} \boldsymbol{C}_\Delta^c. \tag{7}$$

It must be assured that this matrix has exactly a number of eigenvalues at one equal to the number of controlled outputs. By design, $\boldsymbol{A}^c$ satisfies this condition, and so there exists a

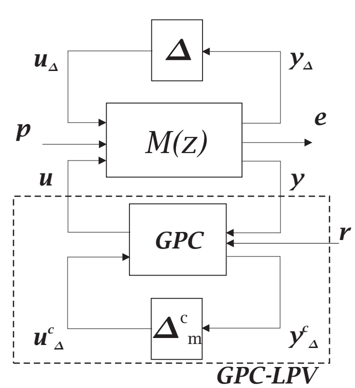

Fig. 3. Structure of GPC-LPV

linear state transformation $\boldsymbol{T}$ such that:

$$\boldsymbol{T}^{-1}\boldsymbol{A}^c\boldsymbol{T} = \begin{pmatrix} \boldsymbol{A}^{c\prime} & \boldsymbol{Z} \\ \boldsymbol{0} & \boldsymbol{I}_n \end{pmatrix}, \tag{8}$$

using this transformation in (7) the number of integrators will be exactly n if:

$$\begin{pmatrix} \boldsymbol{A}^{c\prime} & \boldsymbol{Z} \\ \boldsymbol{0} & \boldsymbol{I}_n \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_\Delta^{c\,\prime} \\ \boldsymbol{0}_{n\times n_{\Delta_m^c}} \end{pmatrix} \Delta_m^c \left(\boldsymbol{I} - \boldsymbol{D}_\Delta^c \Delta_m^c\right)^{-1} \boldsymbol{C}_\Delta^{c\,\prime}, \tag{9}$$

where $n_{\Delta_m^c}$ is the size of Δ_m^c, and some of the delta matrices have changed to new values as a result of this linear transformation: $\boldsymbol{B}_\Delta^{c\,\prime}$ and $\boldsymbol{C}_\Delta^{c\,\prime}$. Besides, some matrices of LTI GPC have also changed by this transformation: $\boldsymbol{B}_r^{c\prime}$, $\boldsymbol{B}_y^{c\prime}$ and $\boldsymbol{C}^{c\prime}$.

The design of delta matrices is done solving feasibility problems with LMIs and/or BMIs, or optimizing linear functions subject to LMIs and/or BMIs. These matrix inequalities were obtained starting from previous results found in the literature, usually for continuous-time systems, and adapting them to the particular case of GPC-LPV.

A. Matrix inequalities conditions for robust stability

The main result which enables the most part of matrix inequalities obtained in the last years is the Lyapunov condition of stability. If it exists a positive definite matrix $\boldsymbol{Q}$ such that:

$$\boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} - \boldsymbol{Q} < 0, \tag{10}$$

then the linear autonomous system $\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k)$ is asymptotically stable. This condition is used in [7], [13], [19]–[22]. The main difference between these results consists of the dependence with respect to matrix Δ: affine, quadratic or LFR. In this work the authors have used the most general dependence, that is to say, LFR.

Following, mainly, the ideas of [7], [22] it is possible to obtain a set of LMIs and a PMI which provides a sufficient condition for the robust stability of the closed loop formed by GPC-LPV and the LPV model (5):

$$\begin{pmatrix} * \\ * \\ \hline * \\ * \end{pmatrix}^T \left(\begin{array}{cc|c} -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & \boldsymbol{V} \end{array}\right) \left(\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{CL} & \boldsymbol{B}_{CL,\Delta} \\ \hline \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_{CL,\Delta} & \boldsymbol{D}_{CL,\Delta} \end{array}\right) < 0, \tag{11}$$

$$\boldsymbol{A}_{CL} = \begin{pmatrix} \boldsymbol{A} + \boldsymbol{B}_y\boldsymbol{D}_y^c\boldsymbol{C}_u & \boldsymbol{B}_u\boldsymbol{C}^{c\prime} \\ \boldsymbol{B}_y^{c\prime}\boldsymbol{C}_y & \begin{pmatrix} \boldsymbol{A}^{c\prime} & \boldsymbol{Z} \\ \boldsymbol{0} & \boldsymbol{I}_n \end{pmatrix} \end{pmatrix},$$

$$\boldsymbol{B}_{CL,\Delta} = \begin{pmatrix} \boldsymbol{B}_\Delta + \boldsymbol{B}_u\boldsymbol{D}_y^c\boldsymbol{D}_{y,\Delta} & \boldsymbol{B}_u\boldsymbol{D}_{u,\Delta}^c \\ \boldsymbol{B}_y^{c\prime}\boldsymbol{D}_{y,\Delta} & \begin{pmatrix} \boldsymbol{B}_\Delta^{c\,\prime} \\ \boldsymbol{0}_{n\times n_{\Delta_m^c}} \end{pmatrix} \end{pmatrix},$$

$$\boldsymbol{C}_{CL,\Delta} = \begin{pmatrix} \boldsymbol{C}_\Delta + \boldsymbol{D}_{\Delta,u}\boldsymbol{D}_y^c\boldsymbol{C}_y & \boldsymbol{D}_{\Delta,u}\boldsymbol{C}^{c\prime} \\ \boldsymbol{D}_{\Delta,y}^c\boldsymbol{C}_y & \boldsymbol{C}_\Delta^{c\,\prime} \end{pmatrix},$$

$$\boldsymbol{D}_{CL,\Delta} = \begin{pmatrix} \boldsymbol{D}_\Delta + \boldsymbol{D}_{\Delta,u}\boldsymbol{D}_y^c\boldsymbol{D}_{y,\Delta} & \boldsymbol{D}_{\Delta,u}\boldsymbol{D}_{u,\Delta}^c \\ \boldsymbol{D}_{\Delta,y}^c\boldsymbol{D}_{y,\Delta} & \boldsymbol{D}_\Delta^c \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta} \\ \boldsymbol{I} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} \begin{pmatrix} \overline{\Delta} \\ \boldsymbol{I} \end{pmatrix} > 0,\ \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix},\ \boldsymbol{V}_{11} < 0,$$

$$\forall \overline{\Delta}\ ,\ \overline{\Delta} = \operatorname{diag}(\Delta, \Delta_m^c). \tag{12}$$

Equation (11) is a PMI since there are products of three variables. This condition is sufficient due to the characteristic complexity of LFR dependence [22], and by the use of a constant Lyapunov matrix $\boldsymbol{Q}$, which does not depend on matrix $\overline{\Delta}$. It is possible to override the second limitation by using a parameter dependent Lyapunov matrix, although this extreme provides a PMI much more complex, and so with more computational complexity.

The selection of matrix Δ_m^c in terms of Δ_m can be made arbitrarily complex by taking a LFR dependence:

$$\Delta_m^c = \Delta_M^c + \Delta_L^c\Delta_m\left(\boldsymbol{I} - \Delta_D^c\Delta_m\right)\Delta_R^c, \tag{13}$$

where Δ_M^c, Δ_L^c, Δ_D^c and Δ_R^c are unknown matrices. As a particular case, it is possible to take $\Delta_m^c = \Delta_m$ providing a simpler and more conservative condition.

PMI (11) can be recast as a BMI by using Schur complement lemma [13] and adding the condition $\boldsymbol{V}_{22} > 0$:

$$\begin{pmatrix} \boldsymbol{M}^T \left(\begin{array}{cc|c} -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & \boldsymbol{V} \end{array}\right) \boldsymbol{M} - -\boldsymbol{M}_1^T \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{22} \end{pmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_1^T \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{22} \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{22} \end{pmatrix} \boldsymbol{M}_1 & -\begin{pmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{22} \end{pmatrix} \end{pmatrix} < 0,$$

$$\boldsymbol{M} = \left(\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{CL} & \boldsymbol{B}_{CL,\Delta} \\ \hline \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_{CL,\Delta} & \boldsymbol{D}_{CL,\Delta} \end{array}\right)\ ,\ \boldsymbol{M}_1 = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{B}_{CL,\Delta} \\ \boldsymbol{C}_{CL,\Delta} & \boldsymbol{D}_{CL,\Delta} \end{pmatrix}. \tag{14}$$

B. Matrix inequalities conditions for norm bounding

In the literature there are also conditions to ensure bounds for different norms: ∞, 2 and 1. For ∞-norm it is applied, for example, the *real lemma* [23], for 2-norm the grammians can be used [24], and for 1-norm the star norm (*-norm) which is an upper bound [25], [26] can be recast as matrix inequalities.

Following the same lines as previous subsection the authors have developed BMIs for these three norms by using LFR dependence. For example, a sufficient condition that ensures that ∞-norm of channel p/e is bounded by $\gamma > 0$ is:

$$\begin{pmatrix} \boldsymbol{L}^T \left(\begin{array}{cc|cc|c} -\boldsymbol{Q} & \boldsymbol{0} & \multicolumn{2}{c|}{\boldsymbol{0}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & & & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & -\gamma^2\boldsymbol{I}_m & \boldsymbol{0} & \boldsymbol{0} \\ & & \boldsymbol{0} & \boldsymbol{I}_r & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & \multicolumn{2}{c|}{\boldsymbol{0}} & \boldsymbol{V} \end{array}\right) \boldsymbol{L}^T - -\boldsymbol{L}_1^T \left(\begin{array}{c|c|c} \boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{V}_{22} \end{array}\right) \boldsymbol{L}_1 & \boldsymbol{L}_1^T \left(\begin{array}{c|c|c} \boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{V}_{22} \end{array}\right) \\ \left(\begin{array}{c|c|c} \boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{V}_{22} \end{array}\right) \boldsymbol{L}_1 & -\left(\begin{array}{c|c|c} \boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{V}_{22} \end{array}\right) \end{pmatrix} < 0,$$

$$
\boldsymbol{L}=\left(\begin{array}{ccc} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_{CL} & \boldsymbol{B}_{CL,p} & \boldsymbol{B}_{CL,\Delta} \\ \hline \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C}_{CL,e} & \boldsymbol{D}_{CL,pe} & \boldsymbol{D}_{CL,e\Delta} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}_{CL,\Delta} & \boldsymbol{D}_{CL,\Delta p} & \boldsymbol{D}_{CL,\Delta} \end{array}\right), \quad \boldsymbol{L}_1=\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}_{CL,\Delta} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_{CL,e\Delta} \\ \boldsymbol{C}_{CL,\Delta} & \boldsymbol{D}_{CL,\Delta p} & \boldsymbol{D}_{CL,\Delta} \end{pmatrix},
$$

$$
\boldsymbol{B}_{CL,p}=\begin{pmatrix} \boldsymbol{B}+\boldsymbol{B}_u\boldsymbol{D}_y^c\boldsymbol{D}_{y,p} \\ \boldsymbol{B}_y^{c\prime}\boldsymbol{D}_{y,p} \end{pmatrix}, \quad \boldsymbol{C}_{CL,e}=\begin{pmatrix} \boldsymbol{C}+\boldsymbol{D}_{e,u}\boldsymbol{D}_y^c\boldsymbol{C}_y & \boldsymbol{D}_{e,u}\boldsymbol{C}^{c\prime} \end{pmatrix},
$$

$$
\boldsymbol{D}_{CL,pe}=\boldsymbol{D}+\boldsymbol{D}_{e,u}\boldsymbol{D}_y^c\boldsymbol{D}_{y,p},
$$

$$
\boldsymbol{D}_{CL,e\Delta}=\boldsymbol{D}_{e,\Delta}+\boldsymbol{D}_{e,u}\boldsymbol{D}_y^c\boldsymbol{D}_{y,\Delta}, \quad \boldsymbol{D}_{CL,\Delta p}=\begin{pmatrix} \boldsymbol{D}_{\Delta,p}+\boldsymbol{D}_{\Delta,u}\boldsymbol{D}_y^c\boldsymbol{D}_{y,p} \\ \boldsymbol{D}_{\Delta,y}^c\boldsymbol{D}_{y,p} \end{pmatrix}. \qquad (15)
$$

C. Matrix inequalities conditions for closed loop specifications

Specifications such as settling time, overshoot, etc, can be guaranteed by using sufficient conditions based on BMIs and LMIs [23], [27]. Basically, the procedure is the same as in previous sections, although in this case the matrix inequalities are developed by employing closed loop pole clustering techniques. For example, Lyapunov stability condition for discrete-time systems imposes pole clustering on the open unit disk. Pole clustering in other complex plane subsets guarantees other specifications, for example:

- Maximum settling time: disk centered at origin with radius smaller than one.
- Maximum transient oscillation frequency w_p: sector with vertex at the origin and angle $w_p \cdot T$, being T the sampling period.

For these and other more complex subsets BMIs and/or LMIs sufficient conditions for pole clustering of closed loop poles are developed. They are omitted due to limited space.

D. Other matrix inequalities conditions

It is possible to obtain sufficient conditions in order to guarantee other properties:

- Time domain constraint satisfaction: saturation of actuators, safety limits in some signals, etc.
- Generalized 2-norm [24].
- etc.

E. Numerical resolution of problems with BMIs

As previously stated, for problems with LMIs there are efficient algorithms that obtain the solution in polynomial time. However, the problems with BMIs [28]:

- In the actual literature of robust control based on matrix inequalities they have a great importance [29], [30].
- They can be NP-hard problems [31].
- They are nonconvex and so may be exist local solutions that can be considered as suboptimal one.
- There do not exist, in general, algorithms which can obtain in polynomial time their global solution.
- There exist algorithms based on branch and bound techniques which can solve problems of small size (low number of variables and small BMIs) in exponential time [29], [30], [32], [33].

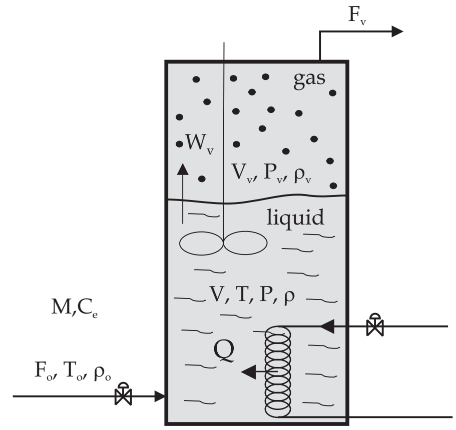

Fig. 4. Liquid-gas separation process.

- By other side, there exist algorithms that obtain only local solutions but they can solve problems of medium and large size [34]–[36].

The algorithm proposed in [36] has been implemented in the commercial software *PENBMI* from *PENOPT*[1]. *PENBMI* has been used in this work to obtain local solutions to the problems with BMIs. The main reason to use this software is that the BMI problems presented in this work have large size. In particular this software can be used in Matlab through the free toolbox *YALMIP*[2].

III. APPLICATION EXAMPLE

In this section the previous design methodology of GPC-LPV controller will be applied to a liquid-gas separation process (Fig. 4), extensively used in petrol engineering to vaporize liquable gases, which in this example consists of liquid propane. The system nonlinear model has been taken from [37], [38]:

$$
\dot{V}_L = F_o - \frac{K}{\rho}\left(\frac{e^{A_1/T+A_2}}{T}\cdot T - \frac{RT}{M}\cdot\rho_v\right),
$$

$$
\dot{T} = \frac{1}{V_L}\left[F_o(T_o-T)+\frac{Q}{\rho C_p} - \frac{K\lambda_v}{\rho C_p}\left(\frac{e^{A_1/T+A_2}}{T}\cdot T - \frac{RT}{M}\cdot\rho_v\right)\right],
$$

$$
\dot{\rho}_v = \frac{1}{V-V_L}\left[\rho_v(F_o-F_v) + K\left(1-\frac{\rho_v}{\rho}\right)\left(\frac{e^{A_1/T+A_2}}{T}\cdot T - \frac{RT}{M}\cdot\rho_v\right)\right], \quad (16)
$$

In this state space model the state variables are: T (^{o}C) gas temperature equal to liquid temperature, V_L (m^3) liquid volume and ρ_v (Kg/m^3) gas density. Input signals: F_o (m^3/s) liquid propane flow, Q (Kcal/s) heat power that provides the

[1] www.penopt.com
[2] http://control.ee.ethz.ch/ joloef/yalmip.php

intercooler and F_v (m^3/s) the upper extraction flow, which is constant and equal to 0.105 m^3/s. This model supposes that liquid density is constant and equal to $\rho = 500\ Kg/m^3$. The remaining signals are considered to be constant and their values and the model constant values are shown in table I.

Next step is to sample this nonlinear model (16) by using a ZOH (zero order hold) with sampling period T_s taking a first order approximation of matrix exponential [39]:

$$\boldsymbol{A}_d = e^{\boldsymbol{A}\cdot T_s} \approx \boldsymbol{I} + \boldsymbol{A}\cdot T_s, \tag{17}$$

since with $T_s = 0.5$ s the approximate discrete matrix is a good estimate of exact one. With such approximation the discrete-time model is:

$$\begin{pmatrix} V_L(k+1) \\ T(k+1) \\ \rho_v(k+1) \end{pmatrix} = \boldsymbol{A}_d \begin{pmatrix} V_L(k) \\ T(k) \\ \rho_v(k) \end{pmatrix} + \boldsymbol{B}_d \begin{pmatrix} F_o(k) \\ Q(k) \end{pmatrix},$$

$$\boldsymbol{A}_d = \boldsymbol{T}_s \begin{pmatrix} \frac{1}{T_s} & -\frac{K}{\rho}\frac{e^{A_1/T+A_2}}{T} & \frac{KRT}{\rho M} \\ 0 & \frac{1}{T_s} - \frac{K\lambda_v}{\rho C_p}\frac{e^{A_1/T+A_2}}{T}\frac{1}{V_L} & \frac{K\lambda_v RT}{\rho C_p M}\frac{1}{V_L} \\ 0 & \psi\frac{e^{A_1/T+A_2}}{T} & \frac{1}{T_s} - \frac{1}{V-V_L}F_v - \psi\frac{RT}{M} \end{pmatrix},$$

$$\psi = \frac{1}{V-V_L}K\left(1 - \frac{\rho_v}{\rho}\right),$$

$$\boldsymbol{B}_d = \begin{pmatrix} T_s & 0 \\ \frac{1}{V_L}(T_o - T)\cdot T_s & \frac{1}{\rho C_p V_L}\cdot T_s \\ \frac{\rho_v}{V-V_L}\cdot T_s & 0 \end{pmatrix}. \tag{18}$$

As it can be seen, $\boldsymbol{A}_d$ and $\boldsymbol{B}_d$ matrices depend nonlinearly on state variables, so all of them must be included in time varying parameters such that a LPV model can be obtained:

$$\delta_1 = 1/V_L, \quad \delta_2 = T - T_o, \quad \delta_3 = \rho_v. \tag{19}$$

This selection of time varying parameters is justified by the form how the LPV model is obtained [4], [40]. The obtaining of this LPV model is omitted due to its large extension.
By other side, as it is present a exponential dependence with respect temperature in $\boldsymbol{A}_d$ it is necessary to use a Taylor series of degree 2 in order to obtain a LFR dependence:

$$e^{A_1/T+A_2} \approx a_0 + a_1\cdot(T-T_o) + a_2\cdot(T-T_o)^2 \tag{20}$$

TABLE I

LIQUID-GAS SEPARATOR PHYSICS PARAMETERS

Name	Description	Value
V	Separator volume	3.14 m^3
T_o	input flow temperature	323 K
λ_v	vaporization heat	75 Kcal/Kg
C_p	Liquid heat capacity at constant pressure	0.6 Kcal/(Kg· K)
A_1		-2359
A_2		10.165
R	Perfect gases constant	2 Kcal/(K· Kmol)
M	Propane molecular mass	44 Kg/Kmol
K	Vaporization constant	$2.7554\cdot 10^{-5}$ Kg/(s· Pa)

with this series a good adjust is obtained along the temperature operation range. Finally, taking V_L and T as measured output signals the resulting LPV model is:

$$\begin{pmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{y}_\Delta \\ \boldsymbol{y} \end{pmatrix} = \left(\begin{array}{c|cc} \boldsymbol{A} & \boldsymbol{B}_\Delta & \boldsymbol{B}_u \\ \hline \boldsymbol{C}_\Delta & \boldsymbol{D}_\Delta & \boldsymbol{D}_{\Delta,u} \\ \boldsymbol{C}_y & \boldsymbol{D}_{y,\Delta} & \boldsymbol{D}_{y,u} \end{array}\right) \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{u}_\Delta \\ \boldsymbol{u}(k) \end{pmatrix}, \ \boldsymbol{u}_\Delta = \Delta(k)\boldsymbol{y}_\Delta,$$

$$\Delta(k) = \mathrm{Diag}\left(\delta_1(k)\boldsymbol{I}_2,\, \delta_2(k),\, \delta_3(k),\, \delta_2(k)\boldsymbol{I}_4\right), \tag{21}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & -\frac{Ka_0T_s}{\rho T_o} & \frac{KRT_sT_o}{\rho M} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{B}_\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{KRT_s}{\rho M} & \frac{KT_s}{\rho T_o} & -\frac{Ka_2T_s}{\rho T_o} & -\frac{Ka_1T_s}{\rho T_o} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{B}_u = \begin{pmatrix} T_s & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, C_\Delta = \begin{pmatrix} 0 & -\frac{K\lambda_v a_0 T_s}{\rho C_p T_o} & \frac{K\lambda_v RT_sT_o}{\rho C_p M} \\ 0 & \frac{Ka_0T_s}{T_o} & -F_vT_s - \frac{KRT_sT_o}{M} \\ 0 & 0 & 0 \\ 0 & -\frac{Ka_0T_s}{\rho T_o} & \frac{KRT_sT_o}{\rho M} \\ 0 & 0 & 1 \\ 0 & a_0/T_o & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\boldsymbol{D}_\Delta = \begin{pmatrix} & 0 & 1 & 0 & \frac{K\lambda_v RT_s}{\rho C_p M} & \frac{KT_s\lambda_v}{\rho T_o C_p} & -\frac{K\lambda_v a_2 T_s}{\rho C_p T_o} & -\frac{K\lambda_v a_1 T_s}{\rho C_p T_o} \\ 0 & V & 0 & 1 & -\frac{KRT_s}{M} & -\frac{KT_s}{T_o} & \frac{KT_sa_2}{T_o} & \frac{KT_sa_1}{T_o} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{KRT_s}{\rho M} & \frac{KT_s}{\rho T_o} & -\frac{Ka_2T_s}{\rho T_o} & -\frac{Ka_1T_s}{\rho T_o} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/T_o & a_2/T_o & a_1/T_o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{D}_{\Delta,u} = \begin{pmatrix} 0 & \frac{T_s}{\rho C_p} \\ 0 & 0 \\ -T_s & 0 \\ T_s & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \boldsymbol{C}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\boldsymbol{D}_{y,\Delta} = \boldsymbol{0}_{2\times 8}, \quad \boldsymbol{D}_{y,u} = \boldsymbol{0}_2. \tag{22}$$

The operating point for the liquid-gas separator is $T_{eq} = T_o = 323$ K and $V_{L,eq} = 1.57\ m^3$. Following this specification a GPC-LPV will be designed to ensure robust stability around this operating point under the condition that temperature, liquid volume and gas density lie in the ranges: $300K \le T \le 350K$, $0.3V \le V_L \le 0.7V$ and $20Kg/m^3 \le \rho_v \le 30Kg/m^3$. The real system only has sensors to measure temperature and liquid volume, and so the GPC-LPV only will beadle to use both measures in Δ_m^c:

$$\Delta_m^c(k) = f(\delta_1(k), \delta_2(k)). \tag{23}$$

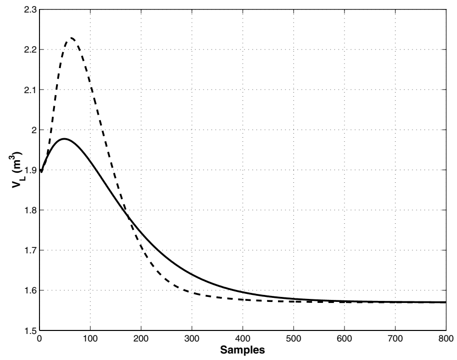

Fig. 5. Liquid volume. Continuous line: LTI GPC. Dashed line: GPC-LPV.

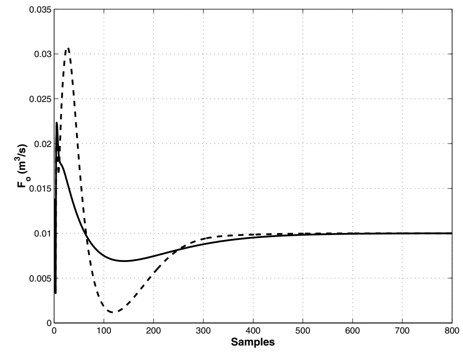

Fig. 7. Liquid propane flow. Continuous line: LTI GPC. Dashed line: GPC-LPV.

In particular, the easiest dependence was used:

$$\Delta_m^c(k) = \begin{pmatrix} \delta_1(k) & 0 \\ 0 & \delta_2(k) \end{pmatrix}. \tag{24}$$

As it is known, in a first step a LTI GPC is designed by using the linear local model corresponding to the operating point of liquid-gas separator. In this design the selected parameters were:

- $N_1 = 1$, $N_2 = 120$ $N_u = 1$.
- $\boldsymbol{Q}_i = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\boldsymbol{R}_i = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.01 \end{pmatrix}$ $\forall$ i.
- Observer poles. For first output: 0.4, 0.5, 0.7 and 0.8. For second output: 0.2, 0.4 and 0.7.

These parameters were adjusted manually after different experiments over the separator, which guarantee closed loop stability inside the operating ranges, by using LMIs conditions.

In Fig. 5, 6, 7 and 8 temperature, liquid volume, F_o flow and heat power are represented when the LTI GPC controls the separator, under the assumption that this one starts from the equilibrium point given by $T = 340$ K and $V_L = 1.9$ m^3.

In a second step, the delta matrices of GPC-LPV are designed under the condition that ∞-norm of channel $\boldsymbol{r}/\boldsymbol{u}$ is samller than LTI GPC one. This design is based on the BMI (15). The optimization with PENBMI took around 30 minutes in a PENTIUM IV at 2.8 Ghz with 512 MB of RAM under Windows XP. After the calculation, the LTI GPC provides a ∞-norm of 514.5597 whereas the GPC-LPV 392.8146. Delta matrices corresponding to GPC-LPV designed are:

$$\boldsymbol{B}_\Delta^{c\,\prime} = \begin{pmatrix} -1.7537\cdot 10^{-2} & 3.0914\cdot 10^{-3} \\ 1.6954\cdot 10^{-4} & -6.6281\cdot 10^{-5} \\ -1.8793\cdot 10^{-5} & 3.9026\cdot 10^{-6} \\ -5.1626\cdot 10^{-5} & 4.7361\cdot 10^{-6} \\ -1.2388\cdot 10^{-5} & -7.6908\cdot 10^{-8} \\ 4.1258\cdot 10^{-7} & -1.9465\cdot 10^{-7} \\ -5.2850\cdot 10^{-7} & 6.9737\cdot 10^{-8} \end{pmatrix},$$

$$\boldsymbol{C}_\Delta^{c\,\prime T} = \begin{pmatrix} 2.5626 & -0.5310 \\ 20.1832 & -21.4447 \\ -72.2428 & -181.1467 \\ 118.1280 & 353.9015 \\ -54.0962 & -365.9777 \\ -111.2757 & -68.8352 \\ -14.0149 & -57.4931 \\ -11.8806 & -42.3621 \\ 211.3161 & 36.0944 \end{pmatrix},$$

$$\boldsymbol{D}_\Delta^c = \begin{pmatrix} -0.4583 & 0.0298 \\ 0.0573 & 0.0185 \end{pmatrix},$$

$$\boldsymbol{D}_{\Delta,y}^c = \begin{pmatrix} 125.5063 & -26.0878 \\ 300.6801 & -3.8997 \end{pmatrix},$$

$$\boldsymbol{D}_{\Delta,r}^c = \begin{pmatrix} -55.2494 & -11.6330 \\ -3.1525 & 0.8039 \end{pmatrix} \quad ; \quad \boldsymbol{D}_{u,\Delta}^c = \boldsymbol{0}_2. \tag{25}$$

In the aforementioned figures the results obtained with GPC-LPV are also represented. Basically, the GPC-LPV pro-

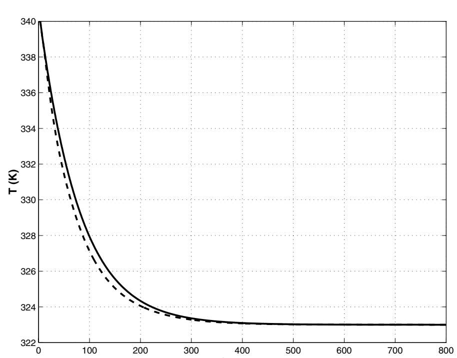

Fig. 6. Temperature. Continuous line: LTI GPC. Dashed line: GPC-LPV.

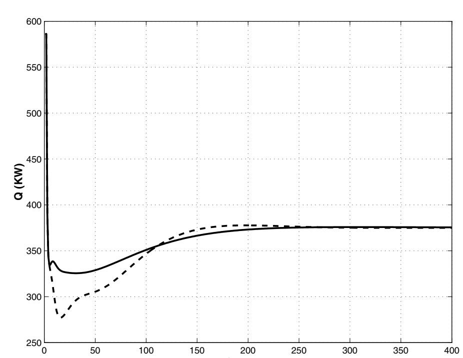

Fig. 8. Heat power. Continuous line: LTI GPC. Dashed line: GPC-LPV.

vides a faster closed loop response and a bigger undershoot in temperature.

IV. CONCLUSIONS

- The GPC-LPV controller is presented as a LPV controller designed in two steps.
- In first step a LTI GPC is designed by using a local model of the nonlinear system.
- In a second step delta matrices are selected in order to satisfy a set of LMIs and/or BMIs, which guarantees: robust stability, norm bounding, closed loop specifications, etc.
- The set of LMIs and/or BMIs is obtained starting from the results of analyzed literature, and working with the matrix inequalities in this particular case.
- Along this work the dependence with respect to Δ matrix is always LFR.
- In most cases BMIs are obtained from PMIs by applying *schur lema*.
- For the numerical computation of solutions, the commercial software *PENBMI* is used to obtain local solutions.
- The design methodology of GPC-LPV is applied to a highly nonlinear model of a liquid-gas separator process.

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