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A State Bounding Observer for Uncertain Non-linear Continuous-time Systems based on Zonotopes

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Abstract—**A state bounding observer aims at computing some domains which are guaranteed to contain the set of states that are consistent both with the uncertain model and with the uncertain measurements. In this paper, the estimated domains are represented by zonotopes. A zonotope is a particular polytope defined as the linear image of a unit interval vector (i.e. unit hypercube). Some results about the validated integration of ordinary differential equations are used to guarantee the inclusion of sampling errors. The main loop of the observation algorithm consists of a one step prediction with a limitation of the domain complexity and a correction using the measurements. The observer is applied to a Lotka-Volterra predator-prey model.**

I. INTRODUCTION

THIS paper proposes an algorithm with polynomial complexity estimating the state vector of an uncertain non-linear continuous-time system described by (1),

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)) & v(t) \in [-1;+1]^r \\ y(t) = Cx(t) + Du(t) + Fw(t) & w(t) \in [-1;+1]^m \end{cases} \quad (1)$$

where f is a non linear function of the state x, the known input u which is assumed to be held by a zero order hold, and the unknown but bounded input v. The uncertainty on the measurement y is described by the bounded input w. Modeling uncertainties in a deterministic (bounded error) rather than in a stochastic context is motivated by the fact that assumptions about the distribution of the random variables may be difficult to validate. Set-membership approaches consist in computing the state estimation as a compact set enclosing all the states that are consistent both with the uncertain model and the uncertain measurements. Since [21], many authors have worked on different domain representations such as ellipsoids [5], [6], parallelotopes [2], boxes and paving methods [8]. Following the work of Kühn [10]-[12] showing that zonotopes may be a suitable alternative to control the wrapping effect [13], a correction step has been introduced in [3] and [1] to obtain a state bounding observer based on zonotopes. A zonotope is a particular polytope defined as the linear image of a unit hypercube. Noticing that a unit hypercube can also be viewed as a unit box or as a vector of unit intervals ($\square$=

[-1;+1]) underlines the links between zonotopes and interval analysis [14], [8], [7] that will be used together in this paper. The motivation for this is explained in relation with the two steps often characterizing a state bounding observer: a one step prediction and a correction using the measurements (as in Kalman filter). The literature on bounded error state estimation has been focusing on discrete time systems until the recent work of Jaulin [9]. In [9], interval analysis for Ordinary Differential Equations (ODE) is used in conjunction with consistency techniques [4]. Unfortunately, the proposed algorithm has an exponential complexity in the state space dimension (due to domain partitioning also called bisection). Then, it was shown in [17] and [18] that using a high order Taylor expansion in an interval based validated integration of ODE ([19], [15], [16]) made bisections unnecessary to obtain good predictions. However, bisections are still used in the correction step, resulting in an observation algorithm that still has an exponential complexity and that deals with domains always having the shape of a box. Based on a restricted measurement equation, the aim of this paper is to develop a state bounding algorithm having a polynomial complexity. It is based on the results about the validated integration of ODE to find zonotope enclosures of the solution sets induced by a model like (1).

The paper is organized as follows: interval computation and zonotopes are introduced in section II. Some results about the validated integration of ODE are combined with a representation of domains by zonotopes to obtain a one step prediction algorithm in section III. Then, section IV recalls the principle of bounded error state observation and describes a possible correction algorithm. The resulting state bounding observer is applied to a Lotka-Volterra predator-prey model in section V, before the paper conclusion.

II. INTERVALS AND ZONOTOPES

A. Interval computation

The name of a variable x in brackets, $[x]$, will denote a domain of possible values for x: $x \in [x]$. A (scalar) interval $[x] = [\underline{x}, \overline{x}]$ is a closed and connected subset of $\mathbb{R}$. The empty square $\square$ will denotes the unit interval: $\square = [-1;+1]$. $\mathbb{IR}$ is the set of all real intervals of $\mathbb{R}$. If $\bullet$ denotes any one of the operations $+$, $-$, $\times$, $\div$ on real numbers x and y, then the corresponding operation on intervals $[x]$ and $[y]$ is: $[x] \bullet [y] = \{x \bullet y \mid x \in [x], y \in [y]\}$. An interval vector of $\mathbb{R}^n$ can be

viewed as an (aligned) box. $\mathbb{IR}^n$ is the set of all the boxes of $\mathbb{R}^n$. The interval function $[f](.) : \mathbb{IR}^n \to \mathbb{IR}^m$ is an inclusion function of $f : \mathbb{R}^n \to \mathbb{R}^m$ if $\forall \ [x] \in \mathbb{IR}^n$, $f([x]) \subset [f]([x])$. Inclusion functions for standard functions like sqrt, exp, sin,… as well as matrix operations with interval elements can be easily computed using a usual interval package. In this work, the Intlab toolbox has been used [20]. It is entirely written in Matlab and it is free for academic use. A direct way to compute a box enclosing the image of a box by a function f is now described and discussed. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function such that an algorithm $\{f\}$ based on the evaluation of elementary operators and functions for which an interval extension is available: the image $y \in \mathbb{R}^m$ of $x \in \mathbb{R}^n$ by f can be computed as $y=\{f\}(x)$. Let $[x] \in \mathbb{IR}^n$ be a box. A box $[y] \in \mathbb{IR}^m$ enclosing $f([x])$ can be computed as $[y]=\{f\}([x])$ where elementary operators and functions in the algorithm $\{f\}$ are evaluated using the interval arithmetic. $\{f\}(.)$ evaluated over intervals is thus an inclusion function for f. However, each occurrence of the same (interval) variable is implicitly assumed to be independent, leading thus to a conservative approximation of the image domain. This is known as the dependency problem. For example: let f be the square function: $y = x^2$. An algorithm to evaluate f is $\{f\}=x\times x$. The image of $[x]=[-1;+1]$ by f is $f([x])=[0;+1]$ whereas $\{f\}([x])=[x]\times[x]=[-1;+1]$. Even if the approximation may be very pessimistic, $f([x]) \subset \{f\}([x])$ is guaranteed. In order to limit the conservatism related to the computation of the image of a box, several approaches exist (constraint propagation, factorization, for instance). Another approach consists in characterizing the dependency between the variables using more complex domains than boxes. In this case, care has to be taken to develop algorithms of reasonable (i.e. polynomial) complexity. This motivates the use of zonotopes.

B. Zonotope: definition and properties

Zonotopes are a special class of convex polytopes. More precisely, a p-zonotope $[z]$ in $\mathbb{R}^n$ is the linear image of a p-dimensional unit box (unit hypercube) in $\mathbb{R}^n$:

$$[z] = c + R.\square^p = c + \blacklozenge R, \quad R \in \mathbb{R}^{n\times p} \tag{2}$$
$$c = c_1+\ldots+c_p, \quad R = [\, r_1\ldots r_p \,] \tag{3}$$

c is the zonotope center and the unary operator $\blacklozenge$ over the real matrices denotes the linear image of a unit box of appropriate dimension. The Minkowski sum of two sets, $[x]$ and $[y]$, is the set defined as $[x] + [y] = \{x + y \mid x \in [x], y \in [y]\}$. The zonotope $[z]$ can also be defined as the Minkowski sum of p straight line segments in $\mathbb{R}^n$. The ith segment, $i=1\ldots p$, is then $[S_i] = c_i + r_i.\square$ where $c_i \in \mathbb{R}^n$ stands for the segment center, and $r_i \in \mathbb{R}^n$ defines both the direction and the radius of $[S_i]$ (3). Some links exist between zonotopes and other domain representations [10]. In particular, if the matrix R is square and diagonal, then $\blacklozenge R$ is an (aligned) box (like $\blacksquare R$ in Fig. 1). Following the previous definitions, some

properties of centered zonotopes will be stated: (*a*) If R_1 and R_2 are identical up to a permutation of their column vectors, then $\blacklozenge R_1 = \blacklozenge R_2$. (*b*) The (Minkowski) sum of two centered zonotopes in $\mathbb{R}^n$ can be computed by a matrix concatenation: $\blacklozenge R_1 + \blacklozenge R_2 = \blacklozenge [R_1\ R_2]$. (*c*) The image of a centered zonotope by a linear mapping L can be computed by a standard matrix product: $L(\blacklozenge R) = \blacklozenge(LR)$. (*d*) A centered zonotope is a convex set with central symmetry and satisfies $\square(\blacklozenge R) = [-1;+1](\blacklozenge R) = (\blacklozenge R)$. (*e*) The smallest centered interval vector containing $\blacklozenge R$ is called the interval hull (Fig. 1) of $\blacklozenge R$ and denoted $\blacksquare R$. It can be computed from the 1-norm of the lines of $R \in \mathbb{R}^{n\times p}$ (4). rs stands for "row sum" and $rs(R)$ is a diagonal matrix:

$$\blacksquare R = \blacklozenge rs(R), \quad rs(R)_{ii} = \sum_{j=1}^{p}\left|R_{ij}\right| \tag{4}$$
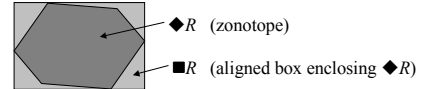

Fig. 1. Interval hull of a zonotope and related notations

C. A reduction operator for zonotopes

The purpose of the reduction operator is to enclose a given zonotope by a zonotope of reduced complexity [3], [11]. Let $\blacklozenge R$ be a centered zonotope in $\mathbb{R}^n$ generated by p segments : $R \in \mathbb{R}^{n\times p}$. The reduction operator $Red_d(.)$ associates to $\blacklozenge R$ the zonotope $Red_d(\blacklozenge R)$ satisfying $\blacklozenge R \subseteq Red_d(\blacklozenge R)$ and generated by at most nd segments. The parameter d results from a tradeoff between a limited complexity (d low) and enough precision (d high) to further control the wrapping effect. The segments generating $\blacklozenge R$ (i.e. the columns of R) are first sorted on decreasing Euclidian norm (5). Those of lower size are then out-bounded by their interval hull (6):

$$\blacklozenge R = \blacklozenge[r_1 \ldots r_i \ldots r_p], \quad \|r_i\| \geq \|r_{i+1}\| \tag{5}$$
$$\textbf{If} \ \ p\leq nd \ \ \textbf{Then} \ \ Red_d(\blacklozenge R) = \blacklozenge R \ \ \textbf{Else} \tag{6}$$
$$Red_d(\blacklozenge R) = \blacklozenge[r_1 \ldots r_{nd-n}\ N], \quad N = rs([r_{nd-n+1} \ldots r_p])$$

Proof that $\blacklozenge R \subseteq Red_d(\blacklozenge R)$: The case $p\leq nd$ being trivial, let assume $p>nd$. $\blacklozenge R = \blacklozenge[r_1 \ \ldots \ r_p] = \blacklozenge[r_1 \ \ldots \ r_{nd-n}] + \blacklozenge[r_{nd-n+1} \ldots r_p]$. Moreover, $\blacklozenge[r_{nd-n+1} \ldots r_p] \subseteq \blacksquare[r_{nd-n+1} \ldots r_p] = \blacklozenge N$ where $N \in \mathbb{R}^{n\times n}$ (4). Thus, $\blacklozenge R \subseteq \blacklozenge[r_1 \ldots r_{nd-n}] + \blacklozenge N = \blacklozenge[r_1 \ldots r_{nd-n}\ N] = Red_d(\blacklozenge R)$, what completes the proof.

D. Image of a zonotope by a non-linear function

The image of a zonotope by a non-linear function is not, in general, a zonotope. The purpose of this paragraph is to find an enclosure of the image of a zonotope $[z]=c+\blacklozenge R$ ($R \in \mathbb{R}^{n\times p}$) by a non-linear function f. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $\{f\}$ be an algorithm used to evaluate f. A box inclusion of $f([z])$ can be computed using direct interval extensions of standard operators and functions as:

$$f(c+\blacklozenge R) \subset \{f\}(c+\blacksquare R) \in \mathbb{IR}^m \tag{7}$$

As discussed in the paragraph *A*, such a box inclusion may be too conservative in practice. However, assuming some regularity conditions on *f*, it can be used to bound the remainder of a Taylor expansion of *f*. Moreover, as the linear image of a zonotope by a linear application is a zonotope, a zonotope inclusion of $f([z])$ can be computed using a first order Taylor expansion of *f*. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a C^2 function,

$$f(c+\delta) = f(c) + \mathbf{L}f(c).\delta + \mathbf{R_1}f(c,\delta) \qquad (8)$$

$\mathbf{L}f(c)$ is the jacobian of *f* evaluated at point *c*. $\mathbf{L}$ stands for "linear" and $\mathbf{L}f_{i,j} = \partial f_i/\partial x_j$. $\mathbf{R_1}f(c,\delta)$ is the remainder of the first order Taylor expansion of *f* around *c* at point $c+\delta$. Let f_q : $\mathbb{R}^n \to \mathbb{R}$ denote the qth component of f ($q=1\dots m$). Then,

$$\forall \delta \in \blacklozenge R, \ \exists \gamma \in (\square\delta) \subset \blacklozenge R, \qquad (9)$$
$$f_q(c+\delta) = f_q(c) + \mathbf{L}f_q(c).\delta + \delta^T.\mathbf{Q}f_q(c+\gamma).\delta$$

$\mathbf{Q}f_q(c+\gamma)$ is an upper triangular matrix representing half the Hessian of f_q at point $c+\gamma$. $\mathbf{Q}$ stands for "quadratic". Let $\mathbf{Q}f_{q,i,j}(c+\gamma)$ denote the element at line *i* and column *j* of $\mathbf{Q}f_q(c+\gamma)$: $\mathbf{Q}f_{q,i,j} = (1/2)\partial^2 f_q/\partial x_i^2$ if $i=j$, $\mathbf{Q}f_{q,i,j} = \partial^2 f_q/\partial x_i \partial x_j$ if $i<j$, $\mathbf{Q}f_{q,i,j} = 0$ if $i>j$. Compared to the usual definition of the Hessian, the upper triangular form of $\mathbf{Q}f_q$ avoids the multi-occurrence of non diagonal terms.

$$\forall \delta \in \blacklozenge R, \ f_q(c+\delta) \in f_q(c) + \mathbf{L}f_q(c).\delta + \delta^T.[Q_q].\delta \qquad (10)$$
$$[Q_q] = [Qf_q(c,R)] = \{\mathbf{Q}f_q\}(c+\blacksquare R)$$

where $[Q_q]$ is a simplified notation for $[Qf_q(c,R)]$, an interval matrix computed using interval arithmetic. The matrices $[Q_q]$, $q=1\dots m$, will be used to compute a zonotope enclosing the remainder $\mathbf{R_1}f(c,\delta)$. From the definition of a zonotope, $\forall \delta \in \blacklozenge R$, $\exists s \in \square^p$, $\delta = R.s$. From (8) and (10), it can be deduced that $\mathbf{R_1}f(c,\delta) \in [\dots s^T.[\underline{Q_q}].s \dots]^T$, $q=1\dots m$, where $[\underline{Q_q}] = R^T.[Q_q].R$ is an interval matrix:

$$[\underline{Q_q}] = R^T.[Q_q].R = c\underline{Q_q} + r\underline{Q_q} \otimes \square^{p \times p} \qquad (11)$$

where $c\underline{Q_q} \in \mathbb{R}^{p \times p}$ (resp. $r\underline{Q_q} \in \mathbb{R}^{p \times p}$) contains the centers (resp. the radii) of each element of the interval matrix $[\underline{Q_q}]$. $\otimes$ is the Hadamard (i.e. element by element) product and $\square^{p \times p}$ is a $p \times p$ interval matrix each element of which is $\square = [-1;+1]$. From (9), (10) and (11), it follows that $R^T.\mathbf{Q}f_q(c+\gamma).R \in [\underline{Q_q}]$ and, consequently:

$$\forall \delta \in \blacklozenge R, \exists s \in \square^p, \exists \alpha_q \in \square^{p \times p}, q=1\dots m, \qquad (12)$$
$$\delta = R.s, \ \mathbf{R_1}f(c,\delta) = [\dots s^T.(c\underline{Q_q} + r\underline{Q_q} \otimes \alpha_q).s \dots]^T$$

$$\mathbf{R_1}f_q(c,\delta) = s^T.(c\underline{Q_q} + r\underline{Q_q} \otimes \alpha_q).s \qquad (13)$$
$$= \sum_i c\underline{Q}_{q,i,i} s_i^2 + \sum_{i<j}(c\underline{Q}_{q,i,j} + c\underline{Q}_{q,j,i})s_i s_j +$$

$$\sum_{(i,j)} r\underline{Q}_{q,i,j} \alpha_{q,i,j} s_i s_j$$

As $\forall i$, $s_i \in \square$, therefore $s_i^2 \in [0;+1] = 1/2 + \square/2$ and $s_i s_j \in \square$. Moreover, $\forall (q,i,j)$, $\alpha_{q,i,j} \in \square$, therefore, $\alpha_{q,i,j} s_i s_j \in \square$. Each term $\alpha_{q,i,j} s_i s_j$ appears just one time in the expression of $\mathbf{R_1}f(c,\delta)$ (whereas each term s_i^2 or $s_i s_j$ may influence several $\mathbf{R_1}f_q(c,\delta)$). Assuming the terms s_i^2, $s_i s_j$, $\alpha_{q,i,j} s_i s_j$ are all independent (i.e. losing the information about the non linear dependencies that can not be represented by a zonotope), a zonotope enclosing $\mathbf{R_1}f(c,\blacklozenge R)$ is:

$$\mathbf{R_1}f(c,\blacklozenge R) \subset c_R + \blacklozenge Z_R \qquad (14)$$

$c_R = trace(c\underline{Q_q})/2$
$Z_R = [Z_Q \ Z_H]$
Z_Q = matrix the qth ($q=1\dots m$) line of which is:
$$Z_{Q,q} = [\cdots \underset{\forall i}{c\underline{Q}_{q,i,i}/2} \ \cdots \mid \cdots \ \underset{\forall i<j}{(c\underline{Q}_{q,i,j} + c\underline{Q}_{q,j,i})} \ \cdots]$$
$Z_H = diag(v_H)$, $v_H = [\dots v_{H,q} \dots]^T$, $q=1\dots m$
$$v_{H,q} = \sum_{(i,j) \in [1;p] \times [1;p]} \left| r\underline{Q}_{q,i,j} \right|$$

where $diag(.)$ transforms a vector into a diagonal matrix (or reciprocally), $Z_Q \in \mathbb{R}^{m \times (p(p+1)/2)}$ (resp. $Z_H \in \mathbb{R}^{m \times m}$) defines a zonotope enclosing the quadratic (resp. higher order) terms. The enclosure of $f([z]) = f(c+\blacklozenge R)$ by a zonotope is then directly deduced from (8):

$$f(c+\blacklozenge R) \subset (f(c)+c_R) + \blacklozenge[\mathbf{L}f(c)R \ Z_R] \qquad (15)$$

As a zonotope is the *linear* image of an hypercube, the proposed inclusion method makes quadratic and higher order terms disappear at the price of an outer approximation of the resulting domain. However, the linear dependencies that may exist due to the presence of similar quadratic terms in several functions f_q is still taken into account in the proposed inclusion method. It can also be noticed that the method could be generalized to take into account the linear dependencies between the non linear terms until an arbitrary order in the Taylor expansion of *f*. In this paragraph, a box inclusion (7) and a zonotope inclusion (15) of a static non-linear function *f* have been studied. This will be used to deal with non-linear dynamic systems in the following.

Remark: (an enclosure of) the image of several zonotopes by a non-linear function *f* can be computed from the image of a single zonotope: $f(c_1+\blacklozenge R_1, \ c_2+\blacklozenge R_2) = f(c+\blacklozenge R)$ where $c=[c_1;c_2]$ and $R=[R_1 \ 0; 0 \ R_2]$).

III. VALIDATED INTEGRATION OF ODE AND ZONOTOPES

This section deals with the validated integration of Ordinary Differential Equations (ODE) having the form of the state equation in (1). The aim of this section is to enclose

(by a zonotope) the set of possible states which results from the one step integration of the continuous state equation with an initial state belonging to a given zonotope. Enclosing the sampling errors to get a validated one step prediction is thus necessary. Notice that the word "validated" must be commented: the algorithm that is developed hereafter guarantees a true inclusion provided the operations over the real numbers are rigorously exact, what is not true in practice due to numeric rounding errors. Therefore, the operations over the real numbers should be replaced by their interval counterpart and the computed domains should be inflated to contain the numeric errors in order to get a true "validated" method. Validated methods to solve ODE necessitates two steps: firstly, proving existence, uniqueness and giving an a priori solution (paragraph *A*), secondly, tightening the a priori solution using, for instance, a Taylor expansion (paragraph *B*). This will be used to compute a one step prediction based on zonotopes (paragraph *0*).

A. Inclusion of the trajectories on one integration step

A result about the validated integration of ODE is first recalled. It is based on the Picard-Lindelöf operator and the fixed point theorem [19] and has already been used in [17], [9], for instance.

Theorem 1. Given the initial value problem $\dot{x}(t) = f_{uv}(x(t),t)$, $x(t_0) \in [x(t_0)]$ where f_{uv} is C^1 with respect to x and continuous on $[t_0;t_1]$. Let $T=t_1-t_0$ be the step size. Let $[w]$ be a set such that: $[x(t_0)]+[0;T]f_{uv}([w],[t_0;t_1]) \subset [w]$. Then, $x(t)$ exists on $[t_0;t_1]$ and $\forall t \in [t_0;t_1]$, $x(t) \in [w]$.

Corollary 1. Let us consider a system modeled by the following uncertain non-linear continuous time state equation: $\dot{x}(t) = f(x(t),u(t),v(t))$, $x(t_0) \in [x(t_0)]$, $v(t) \in \Box^r$ where f is C^1 with respect to x and continuous. Let T be the sampling time and $t_1=t_0+T$. $u(t)$ is assumed to be held by a zero order hold ($\forall t \in [t_0;t_1]$, $u(t)=u(t_0)$) and $v(t)$ is assumed to be continuous on $[t_0;t_1]$. Let $[w]$ and W be two sets such that: $[x(t_0)]+[0;T][f]([w]) \subset [w] \subset W$, where $[f](.)$ is an inclusion function of $f(.,u(t_0),\Box^r)$ valid on W (i.e. $f([w],u(t_0),\Box^r) \subset [f]([w])$ provided $[w] \subset W$). Then, $x(t)$ exists on $[t_0;t_1]$ and $\forall t \in [t_0;t_1]$, $x(t) \in [w]$.

Proof: Let $[w]$ be a set satisfying the conditions in the corollary 1. As $[w] \subset W$, the inclusion function $[f](.)$ is valid. Therefore, $[x(t_0)]+[0;T]f([w],u(t_0),\Box^r) \subset [w]$. Let v be the true realization of the state uncertainty (which is assumed to be continuous and bounded). Let $f_{uv}(x,t) = f(x,u(t_0),v(t))$. As $v([t_0;t_1]) \subset \Box^r$, then $[x(t_0)]+[0;T]f_{uv}([w],[t_0;t_1]) \subset [w]$ and the corollary 1 results from the theorem 1.

A usual way to obtain a set $[w]$ consists in inflating $[x(t_0)]$ by iterations until it satisfies the conditions of the theorem 1. A solution exists provided the step size T is sufficiently small [19]. In the following, the sampling time T is assumed to be constant and sufficiently small. However, it will be possible

to detect if T is too large. The following of this paragraph is dedicated to the direct computation (with no iteration) of a zonotope $[w]$ satisfying the corollary 1 for a particular class of inclusion functions $[f](.)$.

Theorem 2. Let $[x(t_0)] = c+\blacklozenge R$, $R \in \mathbb{R}^{n \times p}$ ($p > n$). Let $W = c+\alpha_{max}\blacklozenge R$ where $\alpha_{max} \in \mathbb{R}$ is a given constant. Let $[f](x) = A.x+b+E\Box^r$ be an inclusion function of $f(x,u(t_0),\Box^r)$ valid for any $[x]=c+\alpha\blacklozenge R$, $0 \le \alpha \le \alpha_{max}$. $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $E \in \mathbb{R}^{n \times r}$. R^+ is the pseudo-inverse of R and I an identity matrix of appropriate dimensions. δ, M_0 and M_1 being defined as follows:

$$\delta = (c+(T/2)b)-(I-(T/2)A)c \qquad (16)$$
$$M_0 = R^+AR \qquad (17)$$
$$M_1 = [\; R^+\delta \quad I \quad (T/2)R^+(Ac+b) \quad TR^+E \;] \qquad (18)$$

if $\exists \alpha \in \mathbb{R}$ such that $(\blacklozenge M_1)/\alpha+T(\blacklozenge M_0) \subset \Box^p$ and $0 \le \alpha \le \alpha_{max}$, **then** $[w] = c+\alpha\blacklozenge R$ satisfies the conditions in Corollary 1: $[x(t_0)]+[0;T][f]([w]) \subset [w] \subset W$.

Proof: Assuming $(\blacklozenge M_1)/\alpha+T(\blacklozenge M_0) \subset \Box^p$, it can be deduced that: $(R^+\delta\Box + [s_0] + (T/2)R^+(Ac+b)[\sigma] + TR^+E[\underline{v}])/\alpha + TR^+AR[s_1] \subset [s_2]$, where $[s_0]=[s_1]=[s_2]=\Box^p$, $[\sigma]=\Box$, $[\underline{v}]=\Box^r$. Considering the image by the linear application αR on both sides of the operator $\subset$, it comes ($RR^+=I$, T is scalar and $1 \in \Box$): $\delta + R[s_0] + (T/2)(Ac+b)[\sigma] + T(A(\alpha R)[s_1] + E[\underline{v}]) \subset \alpha R[s_2]$. As $(1+[\sigma])/2 \subset \Box$ and as any centered zonotope satisfies $\Box(\blacklozenge Z) = (\blacklozenge Z)$, $\delta + R[s_0] + (T/2)(Ac+b)[\sigma] + (T/2)(1+[\sigma])(A(\alpha R)[s_1] + E[\underline{v}]) \subset \alpha R[s_2]$. Substituting (16) for δ in the previous expression and after simple re-arrangements, it comes: $R[s_0] + (T/2)(1+[\sigma]) (A(c+\alpha R[s_1]) + b + E[\underline{v}]) \subset \alpha R[s_2]$. As $(T/2)(1+[\sigma]) = [0;T]$ and after a translation of vector c, we have: $[x(t_0)]+[0;T][f]([w]) \subset [w]$ where $[x(t_0)] = c+\blacklozenge R$ and $[w]=c+\alpha\blacklozenge R$. Moreover, $0 \le \alpha \le \alpha_{max}$ guarantees that $[w] \subset W$.

Theorem 3. Let $b_0 = diag(rs(M_0)) \in \mathbb{R}^p$ and $b_1 = diag(rs(M_1)) \in \mathbb{R}^p$. $b_{0,i}$ (resp. $b_{1,i}$) is thus a scalar representing half the size of the box $\blacksquare M_0$ (resp. $\blacksquare M_1$) along the ith dimension. Let $a \in \mathbb{R}^p$ be such that $a_i = b_{1,i}/(1-T.b_{0,i})$, $i=1\ldots p$.

If $max(b_0)T<1$ **Then** $\forall \varepsilon > 0$, $\alpha=max(a).(1+\varepsilon)$ satisfies $(\blacklozenge M_1)/\alpha+T(\blacklozenge M_0) \subset \Box^p$.

Proof. $(\blacksquare M_1)/\alpha+T(\blacksquare M_0) \subset \Box^p$ implies $(\blacklozenge M_1)/\alpha+T(\blacklozenge M_0) \subset \Box^p$. The condition $max(b_0)T<1$ guarantees that $T(\blacksquare M_0) \subset \Box^p$. It is then possible to find a $\alpha \ge 0$ sufficiently large so that $(\blacksquare M_1)/\alpha+T(\blacksquare M_0) \subset \Box^p$ is verified. A projection along the ith dimension leads to $b_{1,i}/\alpha+T.b_{0,i}<1$ i.e. $\alpha > a_i$. As the sets are aligned boxes, $\alpha > max(a)$ suffices to guarantee that $(\blacksquare M_1)/\alpha + T(\blacksquare M_0) \subset \Box^p$. In practice, the tolerance ε is chosen small (it could be chosen null if numeric rounding errors were rigorously taken into account).

In conjunction with the theorem 2, the theorem 3 provides a constructive way to compute a set $[w]$ satisfying the conditions of corollary 1 and consequently enclosing all the possible trajectories between t_0 and t_1. It is to be underlined that two easily computable conditions have to be fulfilled.

The first one, $max(b_0)T<1$, checks whether the sampling time is sufficiently small. The second one, $\alpha \leq \alpha_{max}$, checks whether the function $[f](.)$ enclosing the non-linear vector field $f(.)$ is used in its domain of validity. It is also interesting to notice that the tradeoff between the step size T and the integration errors (quantified by the inflation ratio α) is put into evidence by the condition $(\blacklozenge M_1)/\alpha + T(\blacklozenge M_0) \subset \square^p$ (The smaller T is, the smaller α can be). In the following, T will be constant and the two above mentioned conditions are computed on-line to check the validity of the results.

B. A Taylor expansion to tighten the a priori solution

Let us consider again a system modeled by the following state equation: $\dot{x}(t) = f(x(t),u(t),v(t))$, $x(t_0) \in [x(t_0)]$, $v(t) \in \square^r$ where f is at least C^1 with respect to x and continuous. T is the sampling time, $t_1 = t_0 + T$ and $u(t)$ is held by a zero order hold as in Corollary 1. The set $[w]$ computed in paragraph A is an often too conservative enclosure of $[x(t_1)]$. However, as $[w]$ encloses the trajectory between t_0 and t_1, a Taylor expansion can be used to compute a tighter domain $[x(t_1)]$. $[w]$ is then used to enclose the remainder term. f is now assumed to be $C^{(d+2)}$ where d is the order of the Taylor expansion used to compute $[x(t_1)]$. Two cases are studied:

Case 1: f does not depend on v. A Taylor expansion of $x(t)$ with an arbitrary order d gives:

$$\exists \tau \in [t_0;t_1], \quad x(t_1) = F_d(x(t_0),u(t_0),x(\tau)) \quad (19)$$

$$= x(t_0) + \sum_{i=0}^{d-1} f^{\{i\}}(x(t_0),u(t_0)).T^{i+1} + f^{\{d\}}(x(\tau),u(t_0)).T^{d+1}$$

where $\quad f^{\{0\}} = f \quad$ and $\quad f^{\{i\}} = (1/i).(\partial f^{\{i-1\}}/\partial x).f$

As $x(t_0) \in [x(t_0)]$ and $x(\tau) \in [w]$, it comes:

$$x(t_1) \in F_d([x(t_0)], u(t_0), [w]) \quad (20)$$

According to (20) and provided $[x(t_0)]$ is a zonotope, a domain $[x(t_1)]$ enclosing $x(t_1)$ can be computed as the (enclosure of) the image of a zonotope by the non-linear application F_d, the expression of which is known (19). The results of the paragraph II.D can be used to do this. When computing $[x(t_1)]$, $[w]$ only appears in the remainder term of (19) in which a factor $(1/d!)$ is present. This would plead in favor of high order Taylor expansions if the computations would not become tedious as the order d increases.

Case 2: f depends on v. A zero order expansion of $x(t)$ is:

$$\exists t \in [t_0;t_1], x(t_1) = x(t_0) + T.f(x(t),u(t_0),v(t)) \quad (21)$$

A higher order expansion of $x(t)$ is not directly possible in this case because the unknown time derivative of v would appear. However, $f(x(t),u(t_0),v(t))$ can be expanded with respect to x:

$$\exists \tau \in [t_0;t], \quad f(x(t),u(t_0),v(t)) = f^{\{0\}}(x(t_0),u(t_0),v(t)) + \dots \quad (22)$$
$$f^{\{1\}}(x(\tau),u(t_0),v(t),v(\tau)).(t-t_0)$$

$$f^{\{0\}}(\dots) = f(x(t_0),u(t_0),v(t)) \quad (23)$$
$$f^{\{1\}}(\dots) = (\partial f/\partial x)(x(\tau),u(t_0),v(t)).f(x(\tau),u(t_0),v(\tau)) \quad (24)$$

As $t \in [t_0;t_1]$, $(t-t_0) \in [0;T]$. Moreover, as $x(t_0) \in [x(t_0)]$, $v(t) \in \square^r$, $v(\tau) \in \square^r$, and $x(\tau) \in [w]$, a domain enclosing $x(t_1)$ is deduced from the equations (21) to (24) :

$$x(t_1) \in [x(t_0)] + T.f[[x(t_0)],u(t_0),\square^r] + \quad (25)$$
$$[0;T^2].f^{\{1\}}([w],u(t_0),\square^r,\square^r)$$

Considering zonotopes as domains, $[x(t_0)] = c + \blacklozenge R$, $R \in \mathbb{R}^{n \times p}$ and $[w] = c + \alpha \blacklozenge R$. Moreover, let denote $c_0 + \blacklozenge R_0$ a zonotope enclosing $([x(t_0)],u(t_0),\square^r)$ and $c_w + \blacklozenge R_w$ a zonotope enclosing $([w],u(t_0),\square^r,\square^r)$ (see the last remark in II.D). Then, $x(t_1) \in [I \ 0](c_0 + \blacklozenge R_0) + T.f(c_0 + \blacklozenge R_0) + [0;T^2].f^{\{1\}}(c_w + \blacklozenge R_w)$ and the results of paragraph II.D can be applied to obtain a zonotope enclosure of $[x(t_1)]$. $f(c_0 + \blacklozenge R_0) \subset (f(c_0) + c_{R0}) + \blacklozenge[\mathbf{L}f(c_0)R_0 \ Z_{R0}]$ (15) where $\mathbf{R_1}f(c_0, \blacklozenge R_0) \subset c_{R0} + \blacklozenge Z_{R0}$ (14). Similarly, $f^{\{1\}}(c_w + \blacklozenge R_w) \subset (f^{\{1\}}(c_w) + c_{Rw}) + \blacklozenge[\mathbf{L}f^{\{1\}}(c_w)R_w \ Z_{Rw}]$ (15) where $\mathbf{R_1}f^{\{1\}}(c_w, \blacklozenge R_w) \subset c_{Rw} + \blacklozenge Z_{Rw}$ (14). The resulting enclosure of $[x(t_1)]$ is:

$$x(t_1) \in [x(t_1)] = c_1 + \blacklozenge R_1 \text{ where:} \quad (26)$$
$$c_1 = [I \ 0]c_0 + T(f(c_0) + c_{R0}) + T^2(f^{\{1\}}(c_w) + c_{Rw})/2$$
$$R_1 = [([I \ 0] + T\mathbf{L}f(c_0))R_0 \ TZ_{R0} \ T^2(f^{\{1\}}(c_w) + c_{Rw})/2$$
$$T^2\mathbf{L}f^{\{1\}}(c_w)R_w \ T^2 Z_{Rw}]$$

Remark: In (26), the relation $[0;1](c + \blacklozenge Z) \subset c/2 + \blacklozenge[c/2 \ Z]$ has been applied to $[0;T^2].f^{\{1\}}(c_w + \blacklozenge R_w)$. Proof: $[0;1]c = c/2 + (c/2)\square$. $[0;1]\blacklozenge Z \subset \blacklozenge Z$. $[0;1](c + \blacklozenge Z) \subset [0;1]c + [0;1]\blacklozenge Z \subset c/2 + (c/2)\square + \blacklozenge Z = c/2 + \blacklozenge[c/2 \ Z]$.

C. Practical computation of the one step prediction

The aim of this paragraph is to describe a one step prediction algorithm based on the previous results. The function f is assumed to be C^3 with respect to x and continuous. Provided numeric rounding errors can be neglected, the computation of a zonotope $[x(t_1)]$ such that $x(t_1) \in [x(t_1)]$, $t_1 = t_0 + T$, from the zonotope $[x(t_0)] = c + \blacklozenge R$ such that $x(t_0) \in [x(t_0)]$ can be summarized as follows:

Step 1: Choose $\alpha_{max} > 0$ (The choice is validated a posteriori) and set $W = c + \blacklozenge(\alpha_{max}R)$.

Step 2: Compute a zonotope enclosure of $f(W,u(t_0),\square^r)$:
(15): $f(W,u(t_0),\square^r) \subset (f(c) + c_W) + \blacklozenge[\mathbf{L}f(c)\alpha_{max}R \ Z_W]$
(14): $\mathbf{R_1}f(c, \alpha_{max}\blacklozenge R) \subset c_W + \blacklozenge Z_W$

Step 3: Choose $A = \mathbf{L}f(c)$, $b = (f(c) - A.c + c_W)$, $E = Z_W$.
$[f](x) = A.x + b + E\square^r$ is a valid inclusion function of $f(x,u(t_0),\square^r)$ for any $[x] = c + \alpha \blacklozenge R$, $0 \leq \alpha \leq \alpha_{max}$. Proof: As $(\alpha \blacklozenge R) \subset (\alpha_{max}\blacklozenge R)$, $\mathbf{R_1}f(c, \alpha \blacklozenge R) \subset \mathbf{R_1}f(c, \alpha_{max}\blacklozenge R) \subset c_W + \blacklozenge Z_W$. From (8), $\forall \delta \in \alpha \blacklozenge R$, $f(c + \delta) \in f(c) + \mathbf{L}f(c).\delta + c_W + \blacklozenge Z_W = (f(c) - \mathbf{L}f(c).c + c_W) + \mathbf{L}f(c).(c + \delta) + \blacklozenge Z_W = [f](c + \delta)$. Therefore, $f(c + \alpha \blacklozenge R) \subset [f](c + \alpha \blacklozenge R)$, $0 \leq \alpha \leq \alpha_{max}$.

Step 4: Compute M_0 with (17) and M_1 with (18) and (16).

Step 5: Application of theorem 3: Compute b_0, b_1. If $max(b_0)T<1$ Then the sampling time T is not sufficiently small: stop and display an error message Else compute α with a small ε (for instance, $\varepsilon = 10^{-5}$).

Step 6: Application of theorem 2 and corollary 1. If $\alpha \leq \alpha_{max}$ Then $[w]=c+\alpha \blacklozenge R$ is such that $\forall t \in [t_0;t_1]$, $x(t) \in [w]$ Else stop and display an error message (it would also be possible to choose $\alpha_{max}>\alpha$ and loop to Step 1…).

Step 7: Compute two zonotope enclosures: $f(c+\blacklozenge R, u(t_0)$, $\square^r) \subset (f(c_0)+c_{R0}) + \blacklozenge[\mathbf{L}f(c_0)R_0 \quad Z_{R0}]$ and $f^{\{1\}}(c+\alpha \blacklozenge R, u(t_0)$, $\square^r) \subset (f^{\{1\}}(c_w)+c_{Rw}) + \blacklozenge[\mathbf{L}f^{\{1\}}(c_w)R_w \quad Z_{Rw}]$ using (15). Notice that a formal calculus software may be very helpful to implement the following functions: $\mathbf{L}f(.)$, $\{\mathbf{Q}f_q\}(.)$, $\mathbf{L}f^{\{1\}}(.)$, $\{\mathbf{Q}f^{\{1\}}_q\}(.)$, $q=1…n$.

Step 8: Compute $[x(t_1)] = c_1+\blacklozenge R_1$ according to (26).

Step 9: Compute $[x(t_1)] = c_1+\blacklozenge Red_\bullet(R_1)$ in order to control the domain complexity (5), (6). The zonotope resulting from the step 8 is generated by more segments than $[x(t_0)]$ (R_1 has more columns than R). Therefore, the complexity of zonotopes would increase at each sample time if no reduction step was applied to limit the number of generator segments.

IV. FROM PREDICTION TO OBSERVATION

A. The principle of bounded error state observation

In order to detail the algorithm steps, some notations are first introduced. Due to sampling (with period T), a discrete time scale k is introduced. It is related to the continuous time scale by $t = kT$. $[x_{k/k'}]$ denotes an outer approximation of the domain of possible states at time k resulting from the state space model of the system and the measurements until time k' (time k' included). When $k=k'$, the notation is simplified as $[x_k]=[x_{k/k}]$. $[x_k|y_k]$ denotes the domain of possible states at time k resulting from the measurements at time k, y_k, and the measurement equation only. The principle of bounded error state estimation is based on the following recurrence: After initialization, assuming $[x_{k-1}]$ is known, a one step prediction is used to compute $[x_{k/k-1}]$ from the state equation. Then, the correction step consists in computing $[x_k]$ as an outer approximation of the intersection between $[x_{k/k-1}]$ and $[x_k|y_k]$.

TABLE I
ALGORITHM: MAIN LOOP OF THE STATE BOUNDING OBSERVER

Initialization ($[x_0] \leftarrow$ domain such that $x(t=0) \in [x_0]$)
For $k=1$ to k_{max}
$\quad [x_{k/k-1}] \leftarrow$ Prediction($[x_{k-1}]$)
$\quad [x_k] \quad \leftarrow$ Correction($[x_{k/k-1}], y_k$)

B. Correction using measurements

The correction step aims at taking into account the information provided by the measurements at time k. Ideally, it would consist in computing the exact intersection between $[x_{k/k-1}] = c_{k/k-1}+\blacklozenge R_{k/k-1}$ and $[x_k|y_k]$. As rapidly computing the exact intersection of two zonotopes is not trivial (moreover,

it is not a zonotope), a zonotope only including the searched intersection will be computed. One of the specifications of the approach detailed in [3] was to reduce the correction step to standard and rather simple vector or matrix operations. The correction algorithm can be summarized as follows:

$$y_{k/k-1} = C.c_{k/k-1} + D.u_k \tag{27}$$

$$\eta = F^{-1}(y_k - y_{k/k-1}), \quad M = F^{-1}CR_{k/k-1}$$

$$M = U.S.V^T = \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix} \tag{28}$$

$$c^{inter} + \blacklozenge R^{inter} = (S_1^{-1}U_1^T\eta - \blacksquare S_1^{-1}U_1^T) \cap (\blacksquare V_1^T) \tag{29}$$

$$[s] = (V_1 c^{inter}) + \blacklozenge[V_1 R^{inter} \quad V_0 V_0^T] \tag{30}$$

$$[x_k] = c_{k/k-1} + R_{k/k-1}[s] \tag{31}$$

By analogy with Kalman filtering, η corresponds to an innovation term (27). A singular value decomposition of the matrix M is computed (28). The aim is to decompose the space generating $[x_{k/k-1}]$ into two orthogonal subspaces: the first one, generated by V_0, is not influenced by η (i.e. by the measurement at time k), contrary to the second one, generated by V_1. The intersection takes place in this second subspace using interval hulls as set approximations (29). The zonotope $[x_k]$ resulting from the intersection is finally computed using (30) and (31). Its center does not only depend on the innovation but also on the dynamic evolution of the computed domains.

V. APPLICATION

A. The Lotka-Volterra predator-prey model

The observation algorithm has been applied to an extended Lotka-Volterra predator-prey model:

$$\begin{cases} \dot{x}_1 = ax_1 - bx_1x_2 \\ \dot{x}_2 = -cx_2 + dx_1x_2 \\ \dot{b} = 0 \\ \dot{d} = 0 \end{cases}, \quad y = x_1 + e\square \tag{32}$$

x_2 (resp. x_1) represents the predator (resp. prey) population size. a and c are the relative birth rates and are assumed to be known ($a=1$, $c=1$). The parameters b and d represent the interaction between the two populations. As they are not always exactly known, a state extension is used (32). The initial state set then allows to represent both the uncertainty about the initial populations and their interactions. The estimation of the population sizes will be studied in two cases: with no measurement or when the prey population is measured with some uncertainty ($e=1.5$).

The measurements used to feed the observation algorithm are obtained by simulating the model with $x_1(0)=x_2(0)=50$, $b=0.01$, $d=0.02$, and by adding a uniform random noise on x_1

to obtain y.

B. Simulation results with correction (Observer)

The observation algorithm in Table I has been implemented using the one step prediction algorithm described in paragraph III.C and the correction algorithm described in paragraph IV.B. The one step prediction is called with $t_0=(k-1)T$ and $[x(t_0)]=[x_{k-1}]$ and the return value is $[x_{k/k-1}]=[x(t_1)]$. All the simulation results have been obtained with $\alpha_{max}=3$ (and no iterative adaptation), $\varepsilon = 10^{-5}$ and $Red_5(.)$ as reduction operator.

The state bounding observer has been applied to large initial uncertainties on the population sizes. Two cases have been studied depending on whether the parameters b and d are almost known (Fig. 2a) or not (Fig. 2b). The differences in the size of the domains thus illustrate the impact of the parametric uncertainties. Whereas the large initial uncertainties would make the domains diverge very quickly if no correction was applied, the measurement of the prey population, even if uncertain, allows to estimate the predator population with reasonable precision, especially when the parameters b and d are well known.

(x_1, x_2) (x_1, x_2)

$x_1(t)$ $x_2(t)$ $x_1(t)$ $x_2(t)$

$T = 0.01$, $k_{max} = 2000$ $T = 0.01$, $k_{max} = 2000$
$[x_1(0)]\times[x_2(0)]=[10; 60]\times[30; 120]$ $[x_1(0)]\times[x_2(0)]=[10; 60]\times[30; 120]$
$[b(0)]=0.01\pm10^{-8}$ $[b(0)]=[0.01-2\%; 0.01+1\%]$
$[d(0)]=0.02\pm10^{-8}$ $[d(0)]=[0.02-1\%; 0.02+2\%]$
(a) almost no uncertainty on b and d (b) uncertainty on b and d
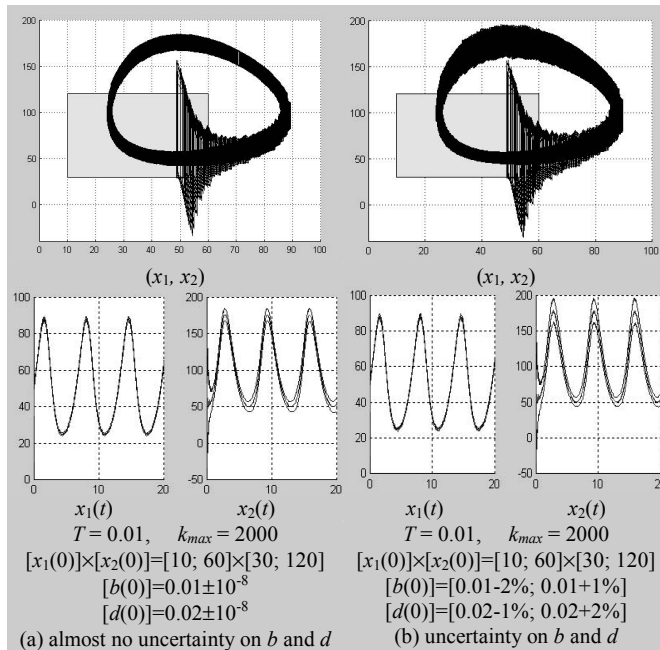
Fig. 2. Simulation results for the state bounding observer.

VI. Conclusion

The main contribution of this paper is to combine zonotopes with results from the validated integration of ODE. Combining the resulting one step prediction with a correction step provides a state bounding observer for a large class of uncertain non-linear continuous-time systems (bounded error context). The resulting observation algorithm has a polynomial complexity which is an advantage other previous results in the literature for an application to high dimensional systems. However, the correction step may be improved as it still relies on interval hulls as set

approximation (after a suitable change of coordinates). Further work should thus primarily focus on that step of the zonotope based state bounding observer.

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