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Stochastic Passivity and its Application in Adaptive Control

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Abstract— **A positive-real like lemma for finite dimensional linear time invariant uncertain systems with state multiplicative noise is derived. The system uncertainties are assumed to be of either polytopic or Markov jump type. Passivity conditions for both cases are derived in terms of linear matrix inequalities. The results are used to the design a direct adaptive controller for a tracking system.**

I. Introduction

Adaptive controllers provide a possible alternative to fixed compensators when large parameter uncertainty is encountered in the model that describes the system. Often, the conditions for closed-loop stability when using adaptive controllers include a strict passivity requirement of the controlled plant. For example, when using the direct adaptive control method [1], which is referred to as Simplified Adaptive Control (SAC), the passivity of the plant guarantees the robust stability of the closed-loop. The SAC applies a tracking error gain which is simply adapted by using proportional and integral versions of the squared tracking error. In fact, the relaxed condition of almost passivity, requiring the plant to be stabilizable and passive via static output-feedback gain, suffices in many cases. A similar situation is encountered when controlling uncertain plants with a class of neural network controllers (NNC) (see [2] and [3]). Also there, the plant is required to be almost passive to ensure closed loop stability. In some applications the system uncertainties can be fixed but unknown but other cases may as well involve stochastic uncertainties. Two such cases of stochastic uncertainties are those described by state-multiplicative noise (e.g. [4]) or Markov jump systems (e.g. [5]). In the latter case, the system matrices are piecewise constant in time and they jump according to system modes that are determined by a Markov chain. In the former case, the system matrices are corrupted by white noise while their deterministic components lie in a convex polytope. The case where these deterministic components do not involve uncertainties has been considered in [6]. There, a stochastic passivity definition, closely related to ours, has been used to prove the closed-loop stochastic stability of a static output-feedback controller for a class of nonlinear plants. This class includes, as a special case, linear systems with state-multiplicative noise. It should be noted that when the system matrices do not involve uncertainties (besides the multiplicative noise) constant gain controllers can be used as in [6]. However, when these uncertainties are significant, direct adaptive controllers maybe useful.

In the present paper, the concept of stochastic passivity is, therefore, generalized to the above mentioned two types of stochastic systems, both including state multiplicative noise and which differ in the uncertainty model of state space matrices : one defined by a convex polytope and the other described by Markov chains. The new stochastic passivity conditions for the different cases that are considered are expressed in form of Linear Matrix Inequalities (LMIs) in Sections 2 - 4 to allow efficient solutions. In Section 5 it is shown that these stochastic passivity conditions ensure closed-loop stochastic stability when applying a class of SAC. Section 6 brings a numerical example for the control of a target tracking system with Markov jump uncertainties.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). The entries of the vector $h\in\mathcal{R}^n$ are denoted by $h_i, i=1,2,...,r$, the inequality $h<0$ is interpreted as $h_1<0, h_2<0, ...h_r<0$. Let $\eta(t)$ be a stochastic process defined on a given probability space $(\bar{\Omega},\mathcal{F},\mathcal{P})$. Expectation is denoted by $E\{\cdot\}$ and the conditional expectation of x given the event $\eta(t)=i$ is denoted by $E[x|\eta(t)=i]$. We also say that $w(t)\in\tilde{L}^2([0,\infty),\mathcal{R}^q)$ when a q-dimensional $w(t)\in L_2, t\geq 0$ is measurable with respect to $\mathcal{F}_t$, where $\mathcal{F}_t$ is defined to be the smallest $\sigma-$ algebra $\mathcal{F}_t\in\mathcal{F}$ containing all sets $\mathcal{M}\in\mathcal{F}$ with respect to which all random vectors e.g. $\gamma(s), s\in[0,\ t]$ are measurable. Also, considering $\eta(t)\in\bar{\Omega}, t\geq 0$ which is independent of $\gamma(t)$, where $\eta(t)$ attains only values in on $\mathcal{D}=\{1,2,...,r\}$, then $\mathcal{F}_t$ will denote the smallest $\sigma-$ algebra $\mathcal{F}_t\in\mathcal{F}$ containing all sets $\mathcal{M}\in\mathcal{F}$ with respect to which both $\gamma(s),\eta(s), s\in[0,\ t]$ are measurable. In such a case, we will still retain, for simplicity, the notation $w(t)\in\tilde{L}^2([0,\infty),\mathcal{R}^q)$ when $w(t)\in L_2, t\geq 0$ is measurable on this new $\mathcal{F}_t$.

II. Problem Formulation and Preliminaries

Consider the following time-invariant linear system with state-dependent noise:

$$
\begin{aligned}
dx_t &= (Ax_t+Bw_t)dt + Dx_t d\beta_t + Gw_t d\sigma_t,\\
dy_t &= (Cx_t+D_{21}w_t)dt
\end{aligned}
\tag{1}
$$

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defined on the filtered probability space $(\bar{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a family of σ-algebras generated by the Wiener process:

$$\gamma_t = col\{\beta_t,\ \sigma_t\} \tag{2}$$

where x_t is the $\mathcal{R}^n$- valued solution to (1); $x(0) = 0$, y_t is the $\mathcal{R}^q$-valued observation and $w \in \tilde{L}^2([0,\ \infty); \mathcal{R}^q)$. The stochastic processes β_t and σ_t are zero-mean scalar standard Wiener processes defined on the probability space $(\bar{\Omega}, \mathcal{F}, \mathcal{P})$.

It is desired to verify whether the the following passivity like condition

$$J = E\{\int_0^\infty y_\tau^T w_\tau d\tau\} \geq 0, \quad \forall\ w_t \in \tilde{L}^2([0,\ \infty)) \text{ over } \bar{\Omega} \tag{3}$$

is satisfied for two types of uncertainties:

- Polytopic uncertainties (denoted by PU): The matrices $A,\ B,\ C,\ D_{21},\ D$ and G are unknown constant matrices that lie in the following uncertainty polytope:

$$\Omega \stackrel{\Delta}{=} \sum_{i=1}^N \tau_i(A_i,\ B_i,\ C_i,\ D_{21,i},\ D_i,\ G_i);\ \tau_i \geq 0,\ \sum_{i=1}^N \tau_i = 1. \tag{4}$$

- Markov jump type uncertainties (denoted by MU): The matrices A, B, C, D_{21}, D and G are piecewise constant matrices of appropriate dimensions whose entries depend upon the mode $\eta(t) \in [1, r]$ of the system, where r is a positive integer denoting the number of possible models between which the system parameters can jump. It is assumed that $\eta(t), t \geq 0$ is a right continuous homogeneous Markov chain on $\mathcal{D} = \{1, 2, ..., r\}$, independent of $\gamma(t), t \geq 0$, with a probability transition matrix $\Pi(t) = e^{Qt}$; $Q = [q_{ij}]$; $q_{ij} \geq 0$ if $i \neq j; \sum_{j=1}^r q_{ij} = 0$; $i = 1, 2, .., r$ consequently leading also to $q_{ii} \leq 0$.
  Given the initial condition $\eta(0) = i$, at each time instant t, the mode may maintain its current state or jump to another mode $i \neq j$.

We denote by L the infinitesimal generator of the stochastic differential equation (1). Choosing the Lyapunov function and the supply rate to be, respectively,

$$V(x_t) = x_t^T P x_t, \text{ and } S(x_t, w_t) = 2y_t^T w_t, \tag{5a,b}$$

where P is a positive-definite constant matrix in $\mathcal{R}^{n\times n}$, in the case of PU, and a function, $V(x, \eta) = x_t^T P(\eta(t))x_t$, of $\eta(t)$, in the case of MU, we find the following results:

Lemma 1: (Polytopic Uncertainties)
i. The system (1) is globally asymptotically stable in probability if for $w_t \equiv 0$ and for all $x \in \mathcal{R}^n$ the following holds over the polytope Ω.

$$LV(x) < 0. \tag{6}$$

ii. If the system (1) is stable in probability over Ω, then a sufficient condition for (3) to hold is:

$$LV(x) \leq S(x, w)\ \forall x \in \mathcal{R}^n,\ w \in \tilde{L}_2([0,\ \infty), \mathcal{R}^q) \text{ over } \Omega \tag{7}$$

Proof: Part i. is well known (see, e.g. [7], [6] and [8]). To prove Part ii. (which has been adopted, in fact, as alternative definition to passivity in [6]) we first realize that

$$dx_t = f(x,t)dt + g(x,t)d\gamma_t \tag{8}$$

where γ_t is the standard Wiener process of (2) and where $f(x,t) = Ax_t + Bw_t$, $g(x,t) = \begin{bmatrix} Dx_t & Gw_t \end{bmatrix}$ and $d\gamma_t = col\{d\beta_t,\ d\sigma_t\}$. We then consider:

$$\begin{aligned} LV(x_t) &= f^T \tfrac{\partial}{\partial x} V(x_t) + \tfrac{1}{2} trace\{gg^T \tfrac{\partial^2}{\partial^2 x} V(x_t)\} \\ &= V_x(x_t)(Ax_t + Bw_t) \\ &+ \tfrac{1}{2}\{x_t^T D^T V_{xx}(x_t) D x_t + w_t^T G^T V_{xx}(x_t) G w_t\} \end{aligned}$$

By Ito formula

$$V(x_t) = V(x_0) + \int_0^t LV(x_s)ds + \int_0^t V_x(x_s)Dx_s d\beta_s + \int_0^t V_x(x_s)Gw_s d\sigma_s \tag{9}$$

and since $x(0) = 0$ we find that

$$E\{\int_0^t LV(x_s)ds\} = E\{V(x_t)\} \geq 0, \quad \forall t \geq 0.$$

If (7) is satisfied, the results of (3) readily follows. $\bigtriangledown$

In the case of Markov jumps in (8), f and g are given by $f(x,t) = A(\eta(t))x_t + B(\eta(t))w_t, g(x,t) = \begin{bmatrix} Dx_t & Gw_t \end{bmatrix}$ where, as described before, $A(\eta(t))$ attains the values of $A_1, A_2, ..., A_r$ and a generalized version of Lemma 1 can be readily obtained for this case, by noting that the infinitesimal generator M associated with $V(x, \eta)$ is then given [9] by

$$\mathbf{M}\mathbf{V}(x_t, \eta) = (Q + diag\{(x^T A_1^T + w^T B_1^T)\tfrac{\partial}{\partial x},\ (x^T A_2^T + w^T B_2^T)\tfrac{\partial}{\partial x},\ ..., (x^T A_r^T + w^T B_r^T)\tfrac{\partial}{\partial x}\} + \tfrac{1}{2} gg^T \tfrac{\partial^2}{\partial^2 x})\mathbf{V}(\mathbf{x_t}, \eta) \tag{10}$$

where $\mathbf{V}(x_t, \eta) \stackrel{\Delta}{=} \begin{bmatrix} V(x,1) & V(x,2) & ... & V(x,r) \end{bmatrix}^T$. We have also used the notation:
$\mathbf{M}\mathbf{V}(x_t, \eta) \stackrel{\Delta}{=} \begin{bmatrix} MV(x,1) & MV(x,2) & ... & MV(x,r) \end{bmatrix}^T$.

Lemma 2: (Markov Jump Uncertainties)
i. The system (1) is globally asymptotically stable in probability if for $w_t \equiv 0$ and for all $x \in \mathcal{R}^n$ the following holds:

$$E\{MV(x, \eta)|\eta(0)\} < 0. \tag{11}$$

ii. If the system (1) is stable in probability over $\bar{\Omega}$, then a sufficient condition for (3) to hold is:

$$E\{MV(x,\eta)|\eta(0)\} \leq E\{S(x,w)|\eta(0)\} \quad \forall x \in \mathcal{R}^n,\ w \in \mathcal{R}^q \quad \text{over } \Omega \tag{12}$$

Proof: Part i. is again well known (see, e.g. [9] and [5]). Part ii. is proved by considering the stochastic Lyapunov function

$$V(x_t, \eta_t) = x_t^T P(\eta(t))x_t. \tag{13}$$

Applying the infinitesimal generator M associated on $V(x, \eta)$ of (10) we obtain [9] that

$$MV(x_t,i) = \sum_{j=1}^{r} q_{ij}V(x,j) + (x^TA_i^T + w^TB_i^T)\frac{\partial V(x,i)}{\partial x}$$

$$+\frac{1}{2}trace\{x_t^TD_i^TV_{xx}(x_t,i)D_ix_t + w_t^TG_i^TV_{xx}(x_t,i)G_iw_t\} \quad (14)$$

However, from the Ito type formula ([11] and [5]), we have:

$$E\{V(x_t,\eta(t))|\eta(0)\} = E\{V(x_0,\eta(0))|\eta(0)\}$$
$$+E\{\int_0^t MV(x_s,\eta(s))ds + \int_0^t V_x(x_s,\eta_s)Dx_sd\beta_s$$
$$+\int_0^t V_x(x_s,\eta_s)Gw_sd\sigma_s|\eta(0)\}$$

Since $x(0) = 0$ we then find that for all $t \geq 0$

$$E\{\int_0^t MV(x_s,\eta_s)ds|\eta_0\} = E\{V(x_t,\eta(t))|\eta(0)\} \geq 0.$$

If (12) is satisfied the results of (3) readily follows. $\bigtriangledown$

III. LMI Based Results for Polytopic Type Uncertainties

For the case of polytopic type uncertainties we see that for the specific choice of (5a)

$$LV(x_t) = x_t^TP(Ax_t + Bw_t) + (Ax_t + Bw_t)^TPx_t$$
$$+x_t^TD^TPDx_t + w_t^TG^TPGw_t \quad (15)$$

and the condition of (7) is thus

$$\begin{bmatrix} x_t^T & w_t^T \end{bmatrix} \left[\begin{bmatrix} -PA - A^TP - D^TPD & -PB \\ -B^TP & -G^TPG \end{bmatrix} + \begin{bmatrix} 0 & C^T \\ C & D_{21} + D_{21}^T \end{bmatrix}\right] \begin{bmatrix} x_t \\ w_t \end{bmatrix} \geq 0 \quad (16)$$

The existence of $0 < P \in \mathcal{R}^{n\times n}$ that satisfies:

$$\begin{bmatrix} PA + A^TP + D^TPD & PB - C^T \\ * & -D_{21} - D_{21}^T + G^TPG \end{bmatrix} < 0 \quad (17)$$

over Ω would thus ensure that (7) is satisfied. Since the requirement of (6) is also satisfied by the first block on the diagonal in (17), stability is also ensured.

Although (17) is also affine in the decision variable P, it is not affine in the matrices D and G. Affinity is obtained by applying Schur's complements formula. The following is then achieved.

Theorem 1: The system (1) is stable (in probability) over $\bar{\Omega}$ and (3) is satisfied over the uncertainty polytope Ω, if there exists $0 < P \in \mathcal{R}^{n\times n}$ that satisfies the following LMIs.

$$\begin{bmatrix} PA_i + A_i^TP & PB_i - C_i^T & D_i^TP & 0 \\ * & -D_{21,i} - D_{21,i}^T & 0 & G_i^TP \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0, \ i = 1,..., N \quad (18)$$

The latter result may turn out to be conservative since it applies the same decision variable P to all the vertices of Ω. Realizing that (18) can be written as

$$\bar{P}\bar{A}_i + \bar{A}_i^T\bar{P} < 0, \quad i = 1, ..., N$$

where

$$\bar{P} = diag\{P,\ I_q,\ P,\ P\} \quad \text{and} \quad \bar{A}_i = \begin{bmatrix} A_i & B_i & 0 & 0 \\ -C_i & -D_{21,i} & 0 & 0 \\ D_i & 0 & -\frac{1}{2}I_n & 0 \\ 0 & G_i & 0 & -\frac{1}{2}I_n \end{bmatrix} \quad (19a\text{-}c)$$

Applying then the result of [10], and defining $\bar{n} = 3n + q$, the following is obtained.

Corollary 1: The system (1) is stable (in probability) over Ω and (3) is satisfied if there exist $0 < P_i \in \mathcal{R}^{n\times n}$, G and H in $\mathcal{R}^{\bar{n}\times\bar{n}}$ that satisfy the following LMIs.

$$\begin{bmatrix} G^T\bar{A}_i + \bar{A}_i^TG & G^T - diag\{P_i,\ I,\ P_i,\ P_i\} - \bar{A}_i^TH \\ * & -H - H^T \end{bmatrix} < 0, \quad i = 1, ..., N. \quad (20)$$

The above produced conditions for the stability in probability of the system. We next inquire what are the conditions for the exponential mean-square stability of the system. The following result is standard:

Lemma 3: ([7])Assume there exists a positive function $V(x,t) \in C^{2,1}$, with $V(0,t) = 0$. Then the system (1) is globally exponentially stable if for $w = 0$ there are positive numbers k_1, k_2, k_3 such that the following hold.

$$k_1||x||^2 \leq V(x,t) \leq k_2||x||^2$$
$$LV(x,t) \leq -k_3||x||^2, \quad \forall t \geq 0 \quad \forall x \in \mathcal{R}^n \quad (21)$$

In our case, as $V(x) = x^TPx$, the first condition is satisfied as long as $P > 0$. To satisfy the second condition we require $LV(x) \leq -\epsilon||x||^2$ for some $\epsilon > 0$ over Ω. Obviously, in terms of LMI, this sufficient condition would be: The system (1) (with $w = 0$) is exponentially stable in the mean square sense if there is $P > 0$ and $\epsilon > 0$ such that

$$PA + A^TP + D^TPD + \epsilon I < 0, \quad \text{over } \Omega. \quad (22)$$

It is clear then that (22) is satisfied for small enough ϵ if there exists a solution $0 < P \in \mathcal{R}^{n\times n}$ to (17).

IV. LMI Based Results for Markov Jump Type Uncertainties

For the case of polytopic type uncertainties we see that for the specific choice of (13) we get from (14) that

$$MV(x_t,i) = x_t^TP_i(A_ix_t + B_iw_t) + (A_ix_t + B_iw_t)^TP_ix_t$$
$$+x_t^TD_i^TP_iD_ix_t + w_t^TG_i^TP_iG_iw_t + \textstyle\sum_{j=1}^{r} q_{ij}x^TP_jx \quad (23)$$

following the lines of the derivation that led to (17) and (18), $MV(x_t,i) < S$ and subsequently (12) read:

$$\begin{bmatrix} P_iA_i + A_i^TP_i + \sum_{j=1}^{r} q_{ij}P_j + D_i^TP_iD_i & P_iB_i - C_i^T \\ * & -D_{21,i} - D_{21,i}^T + G_i^TP_iG_i \end{bmatrix} < 0, \quad i = 1, 2, ..., r \quad (24)$$

V. Application to Simplified Adaptive Control

Consider the following system

$$dx_t = (Ax_t + Bu_t))dt + Dx_t d\beta, \quad dy_t = Cx_t dt \quad \text{(25a,b)}$$

where the matrices $A,\ B,\ C,\ D$ are again unknown constant matrices that lie in the following uncertainty polytope:

$$\hat{\Omega} \triangleq \sum_{i=1}^{N} \tau_i (A_i,\ B_i,\ C_i,\ D_i);\ \tau_i \geq 0,\ \sum_{i=1}^{N} \tau_i = 1. \quad (26)$$

This system should be regulated using a direct adaptive controller [1] of the type:

$$u_t = -Ky_t \quad \text{where} \quad \dot{K} = y_t y_t^T \quad \text{(27a,b)}$$

In the context of deterministic systems, such a controller has been known to stabilize the plant and result in a finite gain matrix K if the plant is Almost Passive (AP). In our stochastic context we may conjecture that stochastic stability of this direct adaptive controller (which usually referred to as SAC) will be guaranteed by the stochastic version of the AP property. Namely, the existence of a constant output-feedback matrix K_e which using the control signal

$$u = u' - K_e x \quad (28)$$

makes the transference relating u' and y stochastically passive. To see that this conjecture is true, we first substitute (27a,b) in (25) and get the following closed-loop system :

$$dx_t = (Ax_t - BKCx_t)dt + Dx_t d\beta \quad (29)$$

Defining $\bar{A} = A - BK_eC$ and $\bar{K} = K - K_e$ the closed-loop system equations are given by :

$$dx_t = (\bar{A}x_t - B\bar{K}Cx_t))dt + Dx_t d\beta \quad (30)$$

and

$$\dot{\bar{K}} = Cx_t x_t^T C^T \quad (31)$$

In the sequel, we choose for simplicity to deal with Single-Input-Single-Output systems and define the augmented state vector $\bar{x} = col\{x,\ \bar{K}\} \triangleq col\{\bar{x}_1,\ \bar{x}_2\}$, and choose the Lyapunov function candidate (see [1]) $V(\bar{x}_t) = \bar{x}_t^T \bar{P} \bar{x}_t$ where $\bar{P} = diag\{P,\ 1\}$. We note that $\bar{x}$ satisfies (8) with $\bar{x}$ replacing x and where:

$$f(\bar{x}) = \begin{bmatrix} \bar{A}\bar{x}_1 - B\bar{x}_2 C\bar{x}_1 \\ \bar{x}_1^T C^T C \bar{x}_1 \end{bmatrix},\ g(\bar{x}) = \begin{bmatrix} D\bar{x}_1 \\ 0 \end{bmatrix} \text{ and } d\gamma_t = d\beta_t. \quad \text{(32a-c)}$$

By Lemma 1, the closed-loop system (30) will be stochastically stable if $LV < 0$. However,

$$LV(\bar{x}_t) = f^T \frac{\partial V(\bar{x}_t)}{\partial \bar{x}} + \frac{1}{2} trace\{gg^T \frac{\partial^2 V(\bar{x}_t)}{\partial^2 \bar{x}}\} \quad (33)$$

Substituting (32a,b) into the last equation, we readily find that:

$$LV(\bar{x}_t) = \bar{x_1}^T [\bar{A}^T P + P\bar{A}] \bar{x_1} - \zeta^T [B^T P - C] \bar{x}_1$$
$$- \bar{x}_1^T [PB - C^T] \zeta + \bar{x_1}^T D^T P D \bar{x_1}$$

where the last term is obtained from the second term in (33) and where $\zeta \triangleq \bar{x}_2 C \bar{x}_1$. Therefore, a sufficient condition for $LV < 0$ is

$$\bar{A}^T P + P\bar{A} + D^T P D < 0, \quad PB = C^T \quad (34)$$

which by a limiting argument shows that (17) is equivalent to the requirement that (25) can be stabilized and made stochastically passive using (28). Namely, that (25) is almost stochastically passive.

A LMI version of (34), for the polytope Ω is then the following.

$$\begin{bmatrix} P\bar{A}_i + \bar{A}_i^T P & PB_i - C_i^T & D_i^T P \\ B_i^T P - C_i & -\epsilon I & 0 \\ PD_i & 0 & -P \end{bmatrix} < 0, \quad i = 1, ..., N \quad (35)$$

for small enough $\epsilon > 0$, where $\bar{A}_i = A_i - B_i K_e C_i$. We note that when the polytopic uncertainties are replaced by Markov jump type uncertainties, the system of (25) now reads:

$$\begin{aligned} dx_t &= (A(\eta(t))x_t + B(\eta(t))u_t)dt + Dx_t d\beta, \\ dy_t &= C(\eta(t))x_t \end{aligned} \quad (36)$$

In such a case, the derivations in the previous section are repeated with $MV < 0$ replacing $LV < 0$. Following then the lines of Section 2 above for calculating MV we readily obtain the following.

$$\begin{bmatrix} P_i\bar{A}_i + \bar{A}_i^T P_i + {}_{j=1}^{r} q_{ij} P_j + D_i^T P_i D_i & P_i B_i - C_i^T \\ * & -\epsilon I \end{bmatrix} < 0,$$
$$i = 1, 2, ..., r \quad (37)$$

for small enough $\epsilon > 0$.

Note that the system is strictly passive if (35) and (37) hold for $K_e = 0$. If these inequalities hold only for $K_e \neq 0$, the system is AP.

In the above, the K_e was a constant gain matrix. However, the above arguments still hold if K_e is time-varying gain that is independent of x.

VI. Numerical Example

We bring a numerical example from the field of motion control. We consider the following system

$$\tfrac{dx_1}{dt} = x_2,\ \tfrac{dx_2}{dt} = 5(-x_2 + u),\ z = x_1 + x_2$$

where it is desired to achieve a regulation of $z(t)$ using a RADAR measurement of the position x_1 and the velocity x_2. A simplified adaptive control is suggested for this control task. Note that while the transference relating u and x_1 is not passive, the one relating u and z is passive (see [12] for a similar idea where the actual controlled variable is chosen as close as possible to the desired control variable under a passivity constraint). Motivated by the fact that in many measurement systems, such as RADAR, achieving good accuracy simultaneously in both position and velocity are conflicting (see [13] for the so called ambiguity functions describing this phenomenon), we suggest the following stochastic controller:

$$u(t) = -Ky$$

where

$$y(t) = z(t) + C(\eta(t))v(t), \; z = C(\eta(t))x \; \text{ and } x = col\{x_1, \; x_2\}$$

and where $v(t)$ is a first order Markov processes vector having a covariance $R_v = diag\{\rho_1^2, \rho_2^2\}$ with a correlation time of 1 second and where K obeys the simplified adaptation law of (27). We consider four cases:

- Low Position Noise Controller (LPN) : $\rho_1 = 0.1$, $\rho_2 = 10$, $C = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$.
- Low Velocity Noise Controller (LVN): $\rho_1 = 10$, $\rho_2 = 0.1$, $C = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$.
- Low Position and Velocity Noise Controller (LPVN): $\rho_1 = 0.1$, $\rho_2 = 0.1$ where $C = \left[\begin{array}{cc} 1 & 1 \end{array} \right]$. Namely this is an idealized controller which violates the condition $\rho_1 \rho_2 = 1$.
- Markov Jump Controller (MJC): $C(\eta(t))$ attains the values of $C_1 = \left[\begin{array}{cc} 1 & 0.2 \end{array} \right]$, for $\rho_1(\eta) = 0.1$, and $C_2 = \left[\begin{array}{cc} 0.2 & 1 \end{array} \right]$, for $\rho_1(\eta) = 10$, ($\rho_1 \rho_2 = 1$, with $\Pi(t) = e^{Qt}$ and where the corresponding infinitesimal matrix is $Q = \left[\begin{array}{cc} -\alpha & \alpha \\ \beta & -\beta \end{array} \right]$ where $\alpha = \beta = 0.5$. It follows that $\Pi(t) = e^{Qt} = (\alpha t + \beta t)^{-1} \left[\begin{array}{cc} \beta t + \mu(t)\alpha t & \alpha t - \mu(t)\alpha t \\ \beta t - \mu(t)\beta t & \alpha t + \mu(t)\beta t \end{array} \right]$ where $\mu(t) = e^{-(\alpha+\beta)t}$.

We note that the LPN and LVN correspond to two extreme cases in which the RADAR is configured to low position error at the expense of high velocity error and vice versa. Note also that in these cases $\rho_1 \rho_2$ is kept constant in order to represent the ambiguity between accurate position and velocity measurements. The LPVN case corresponds to and idealized controller which violates the condition $\rho_1 \rho_2 = 1$ and assumes that both position and velocity can be measured accurately and is considered to represent a desired regulation. The MJC case, however, requires configuring the RADAR so that it jumps between the low position error and low velocity error modes according the transition matrix Π and where during each mode the measurement with the low error is utilized for feedback with higher gain (i.e. 1 comparing to 0.2). Note that the MJC does not violate the $\rho_1 \rho_2 = 1$ condition. By the results of Section 4, the closed-loop stability is ensured if $\dot{x} = Ax + Bu, y = C(\eta(t))x$ is passive. For the deterministic cases of LPN, LVN and LPVN the passivity condition can be verified by noting that the phase of $C(sI - A)^{-1}B$ does not exceed 90 degrees or by solving (35). For the MJC case, although both $C_1(sI - A)^{-1}B$ and $C_2(sI - A)^{-1}B$ are passive by their phase, the passivity of the Markov Jump System due to switchings between C_1 and C_2 should be verified using the above LMIs. Indeed, invoking the results of (24), we obtain the following positive definite matrices:

$$P_1 = \left[\begin{array}{cc} 4.9997 & 0.1877 \\ 0.1877 & 0.0456 \end{array} \right] \quad \text{and} \quad P_2 = \left[\begin{array}{cc} 9.871 & 0.0408 \\ 0.0408 & 0.2041 \end{array} \right].$$

The simulation results are depicted in Fig. 1-4 : The position x_1 and velocity x_2 are depicted in Fig. 1 and Fig. 2, respectively, whereas the control signal is depicted in Fig. 3. The adaptive gain K is depicted in Fig. 4. The results in Fig. 1-2 show that in terms of regulation quality of x_1 and x_2 the MJC using switched controls (see Fig. 3) nearly recovers the idealized results of LPVN. The adaptive gain of MJC is larger than that of the LPVN but is still smaller than those of LPN and LVN (see Fig. 4).

VII. Conclusions

The concept of passivity has been generalized for a class of stochastic systems of practical significance. The passivity conditions in the form of Linear Matrix Inequalities can be efficiently solved using state of the art LMI solvers (e.g. [14]). It was shown that a relaxed version of the stochastic passivity conditions, namely the almost stochastic passivity, guarantees closed-loop stochastic stability.

A numerical example from the field of target tracking was considered. By alternately measuring position and velocity, a Markov jump controller was applied to a fixed tracking system to alleviate the ambiguity restrictions which apply to simultaneous and accurate measurement of both position and velocity. The results show that such a Markov controller is closer to the ideal controller neglecting the ambiguity problem.

The case of controlling stochastic plants which are not stochastically passive but are instead almost stochastically passive, requires finding a static output feedback controller which can not be purely expressed by LMIs. This topic is left for a future research.

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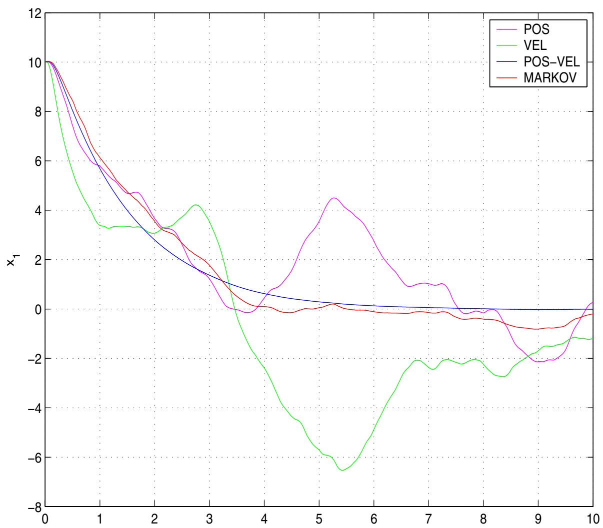

Fig. 1. Position

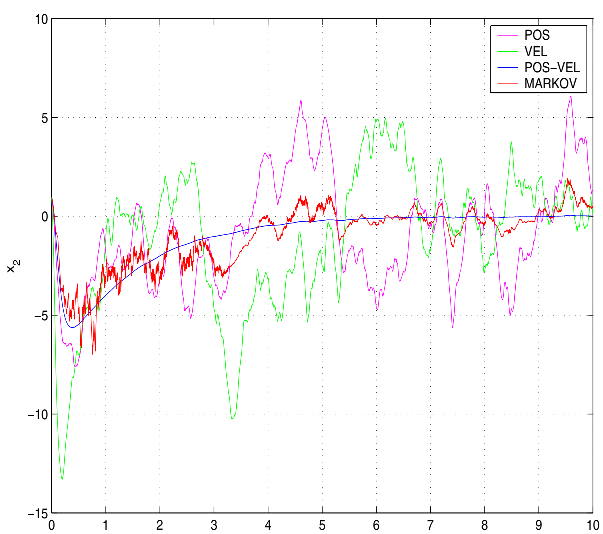

Fig. 2. Velocity

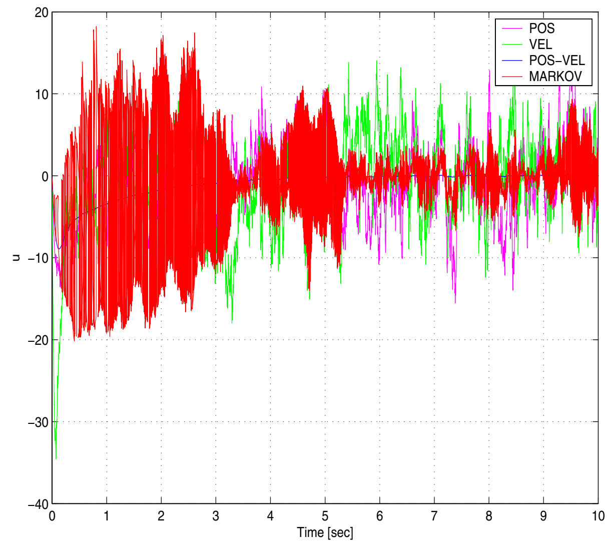

Fig. 3. Velocity Command

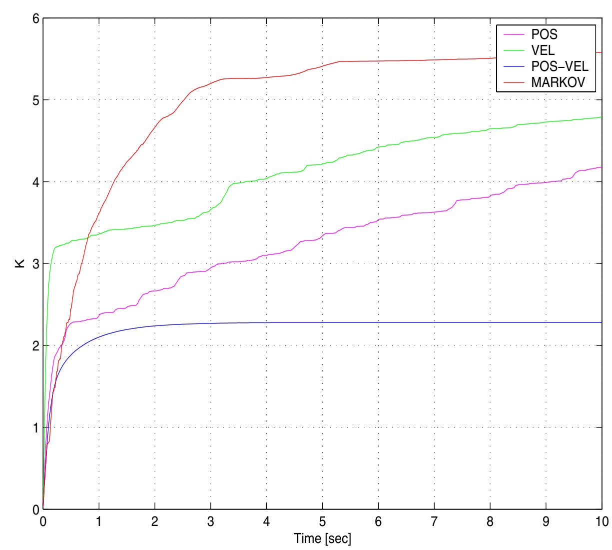

Fig. 4. Velocity Command