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# Model reduction for controller design for infinite-dimensional systems: theory and an example

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*Abstract*— **This paper presents a model reduction technique for infinite-dimensional systems called LQG-balancing. Theoretical results on the existence and uniqueness of LQG-balanced realizations are given as well as error bounds for truncated LQG-balanced realizations. We illustrate the theory by studying model reduction for an Euler-Bernoulli beam.**

## I. INTRODUCTION

Simple models are normally preferred over complex ones in control systems design. Sometimes it is obvious how to construct a simple model for a physical system, but sometimes it is not obvious what the characteristics essential to the controller design of a physical system are. One way of obtaining a simple model in this last case is to first obtain a sophisticated model that takes every aspect that could be of interest into account and then perform model reduction on this sophisticated model. A simple model reduction procedure was introduced by Moore [13] and is now a textbook subject (see e.g. Zhou and Doyle [23], Chapter 7). The method proposed by Moore consists of truncating a balanced realization. A balanced realization (also called Lyapunov- or internally balanced) is a realization for which the controllability and observability gramians are equal and diagonal. This procedure is only applicable to stable systems. Alternatively for unstable systems one can use truncations of a LQG-balanced realization, which for rational transfer functions always exists. A LQG-balanced realization is a realization for which the (linear quadratic regulator) optimal cost operator for a system and its dual system are equal and diagonal. This method was proposed by Verriest [20], [21] and further developed by Jonckheere and Silverman [10]. For an alternative treatment see Mustafa and Glover [14]. The discrete-time case was considered in Hoffmann et al. [9]. There is a relation between LQG-balanced truncation and Lyapunov-balanced truncation of a normalized coprime factor that we will comment on later.

In the case that the system is infinite-dimensional, the model approximation becomes essential. One would like to use the methods of balanced truncation and LQG-balanced truncation in this case too. The existence of Lyapunov-balanced and LQG-balanced realizations for irrational transfer functions is however nontrivial. Necessary and sufficient conditions for the existence of LQG-balanced realizations were proven in [16] for the discrete-time case and in [17] for the continuous-time case. Here we summarize the continuous-time result and apply the theory to an example: model reduction for an Euler-Bernoulli beam.

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In the next subsection we review the theory on finite-dimensional LQG-balancing. In Section II we introduce the class of systems we study. Section III contains the theoretical results on LQG-balancing for this class of systems. In Section IV the example is studied.

### A. Theoretical results in the finite-dimensional case

We consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where $A, B, C, D$ are matrices of compatible dimensions. We consider the cost functional

$$J(x_0, u) := \int_0^\infty \|u(t)\|^2 + \|y(t)\|^2 dt. \tag{3}$$

It is well-known that if the system is minimal, then, for every $x_0 \in X$, there exists a unique $u^{\mathrm{opt}}$ such that $J(x_0, u^{\mathrm{opt}}) < J(x_0, u)$ for all $u \neq u^{\mathrm{opt}}$. It is also well-known that there exists a nonnegative matrix $Q$ such that

$$\int_0^\infty \|u^{\mathrm{opt}}(t)\|^2 + \|y^{\mathrm{opt}}(t)\|^2 dt = \langle Qx_0, x_0\rangle.$$

This $Q$ is called *the optimal cost operator* of the system. The matrix $Q$ is the unique nonnegative solution of the following *control algebraic Riccati equation*

$$\begin{aligned} &A^*Q + QA + C^*C \\ &\quad = (QB + C^*D)(I + D^*D)^{-1}(D^*C + B^*Q). \end{aligned}$$

The dual Riccati equation, i.e.,

$$\begin{aligned} &AP + PA^* + BB^* \\ &\quad = (PC^* + BD^*)(I + DD^*)^{-1}(DB^* + CP), \end{aligned}$$

which is called the *filter algebraic Riccati equation* also has a unique nonnegative solution. It is easily seen that the eigenvalues of the product $PQ$ are similarity invariants and they thus only depend on the transfer function $G$ of the system and not on the particular realization. The square roots of these eigenvalues are called the *LQG-singular values* of the transfer function. It is proven in [10], [20], [21] that there exists a minimal realization of the transfer function $G$ such that the solution of the control algebraic Riccati equation and the solution of the filter algebraic Riccati equation are equal and diagonal. Thus, there exist matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and a diagonal matrix $X$ with nonnegative eigenvalues such that

$$C(sI-A)^{-1}B + D = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B} + \tilde{D},$$

$$\begin{aligned} &A^*X + XA + C^*C \\ &\quad = (XB + C^*D)(I + D^*D)^{-1}(D^*C + B^*X), \end{aligned}$$

and

$$\begin{aligned} &AX + XA^* + BB^* \\ &\quad = (XC^* + BD^*)(I + DD^*)^{-1}(DB^* + CX). \end{aligned}$$

Such a realization is called *LQG-balanced*. We obtain a $k$-dimensional LQG-balanced truncation of the LQG-balanced realization by truncating the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ in such a way that the truncations correspond to the largest $k$ LQG-singular values. In [10], [20], [21] it is proposed that this is a good approximation technique, which can be understood from the following error-bound (see [14]). We have

$$\delta_g(\Sigma, \Sigma_k) \le 2 \sum_{i=k+1}^{n} \frac{\mu_i}{\sqrt{1+\mu_i^2}}, \tag{4}$$

where the $\mu_i$'s are the LQG-singular values and $\delta_g(\Sigma, \Sigma_k)$ is the distance in the gap metric between the system $\Sigma$ and its $k$-dimensional LQG-balanced truncation $\Sigma_k$. See [23, Chapter 17] for a definition of the gap metric.

Define $\varepsilon_{\max} := (1+\mu_1^2)^{-1/2}$. In Chapter 9 of [5] it is shown that for a system $\Sigma$ and any $\varepsilon < \varepsilon_{\max}$ there exists a controller that stabilizes all systems $\Sigma'$ with $\delta(\Sigma, \Sigma') < \varepsilon$.

This shows that if

$$2 \sum_{i=k+1}^{n} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < (1+\mu_1^2)^{-1/2},$$

then there exists a $k$ dimensional controller that stabilizes the system $\Sigma$.

We note that there is a connection between LQG-balanced truncations and Lyapunov-balanced truncations. Let $[M; N]$ be a normalized coprime factor of $G$ and let $[M_k; N_k]$ be the transfer function of the $k$-dimensional truncation of a Lyapunov-balanced realization of $[M; N]$. Then $G_k := N_k M_k^{-1}$ is the transfer function of a $k$-dimensional truncation of a LQG-balanced realization of $G$. This implies that LQG-balanced approximation boils down to Lyapunov-balanced approximation of the normalized coprime factors.

## II. The class of systems

The class of systems we consider was introduced in [15]. It contains basically all systems described by linear partial differential equations whose coefficients do not depend on time and all delay equations.

We start with the intuition behind the definition. A finite-dimensional linear system is usually described by specifying four matrices $A, B, C, D$ and defining for a given initial state $x_0$ and an input function $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{u}})$ the state $x \in C(0,\infty;\mathbb{C}^{\mathrm{x}})$ and the output $y \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{y}})$ as the unique solutions of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{5}$$

As is well-known, these unique solutions are given explicitly by

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,ds, \tag{6}$$

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\,ds + Du(t).$$

If we Laplace transform the equations (5) and solve for $x$ and $y$ we obtain

$$\hat{x}(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}B\hat{u}(s) \tag{7}$$

$$\hat{y}(s) = C(sI-A)^{-1}x_0 + \left(C(sI-A)^{-1}B + D\right)\hat{u}(s).$$

Our approach to infinite-dimensional systems will be to generalize the situation (7) rather than the situation (5) or (6).

We first study the generalizations of the matrix-valued functions $(sI-A)^{-1}$, $(sI-A)^{-1}B$, $C(sI-A)^{-1}$ and $C(sI-A)^{-1}B + D$.

*Definition 2.1:* A *resolvent linear system* on a triple of Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ consists of a nonempty connected open subset $\Lambda$ of the complex plane and four operator valued functions $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ satisfying

$\mathfrak{a} : \Lambda \to \mathcal{L}(\mathcal{X})$ satisfies

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{8}$$

$\mathfrak{b} : \Lambda \to \mathcal{L}(\mathcal{U}, \mathcal{X})$ satisfies

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{9}$$

$\mathfrak{c} : \Lambda \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ satisfies

$$\mathfrak{c}(\beta) - \mathfrak{c}(\alpha) = (\alpha - \beta)\mathfrak{c}(\alpha)\mathfrak{a}(\beta) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{10}$$

$\mathfrak{d} : \Lambda \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ satisfies

$$\mathfrak{d}(\beta) - \mathfrak{d}(\alpha) = (\alpha - \beta)\mathfrak{c}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11}$$

The function $\mathfrak{a}$ is called the *pseudoresolvent*, $\mathfrak{b}$ the *incoming wave function*, $\mathfrak{c}$ the *outgoing wave function* and $\mathfrak{d}$ the *characteristic function* of the resolvent linear system.

It is not possible to make sense out of the equations (7) without some additional assumption. One possible assumption is the following.

*Definition 2.2:* An *integrated resolvent linear system* is a resolvent linear system with the additional property that there exist constants $\gamma, C > 0$ and $n \in \mathbb{N}$ such that

$$\Lambda_H := \{s \in \mathbb{C} : \mathrm{Re}\ s \ge \gamma\} \subset \Lambda \tag{12}$$

and

$$\|\mathfrak{a}(s)\| \le C(1+|s|)^n \quad \forall\ s \in \Lambda_H. \tag{13}$$

The above definition in words is: an integrated resolvent linear system is a resolvent linear system whose pseudoresolvent is polynomially bounded on some right half-plane. The term “integrated” stems from the connection with integrated semigroups. Note that due to the above functional equations the wavefunctions and characteristic function of an integrated resolvent linear system are also polynomially bounded on $\Lambda_H$. To define the state and output we will need the following well-known characterization of Laplace transformable Banach space valued distributions by Schwartz. The image of the Schwartz-Laplace transformable Banach space valued distributions is exactly the set of polynomially bounded analytic functions defined on some right half-plane. For details see [18].

*Definition 2.3:* The state $x$ and output $y$ of an integrated resolvent linear system corresponding to the initial state $x_0 \in \mathcal{X}$ and the input $u$ (a $\mathcal{U}$-valued Laplace transformable distribution) are defined through their Laplace transforms as

$$\hat{x}(s) := \mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s), \ \hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s). \quad (14)$$

We now consider the linear quadratic regulator problem for integrated resolvent linear systems. The *finite cost condition* is: for every $x_0 \in \mathcal{X}$ there exists a $u \in L^2(0,\infty;\mathcal{U})$ such that $y \in L^2(0,\infty;\mathcal{Y})$. It is shown in [15] that if the finite cost condition is satisfied, then for every $x_0 \in \mathcal{X}$ there exists a unique minimizing input $u^{\rm opt}$ for the quadratic cost functional (3) and a bounded operator $Q$ such that the optimal cost is given by $\langle Qx_0, x_0\rangle$. This operator $Q$ is called the *optimal cost operator*. The dual of a resolvent linear system is given by

$$\mathfrak{a}^d(s) := \mathfrak{a}(\bar{s})^*, \quad \mathfrak{b}^d(s) := \mathfrak{c}(\bar{s})^*,$$
$$\mathfrak{c}^d(s) := \mathfrak{b}(\bar{s})^*, \quad \mathfrak{d}^d(s) := \mathfrak{d}(\bar{s})^*.$$

If the finite cost condition for this dual system is satisfied, then its optimal cost operator $P$ exists which is called the *dual optimal cost operator* of the original integrated resolvent linear system.

## III. LQG-BALANCING: THEORETICAL RESULTS

In this section we state theoretical results on LQG-balancing for infinite-dimensional systems. Proofs can be found in [17].

As in the finite-dimensional case, it can be shown that the square roots $\mu_i$ of the eigenvalues of the product $PQ$ (with the possible exception of zero) do not depend on the particular realization, but only on the characteristic function.

*Definition 3.1:* An integrated resolvent linear system for which the finite cost condition and the dual finite cost condition are satisfied is called *LQG-balanced* if its optimal cost operator $Q$ and dual optimal cost operator $P$ are equal. It is called *daigonally LQG-balanced* if in addition $P = Q$ has a set of eigenvectors that form a basis for the state space. The following theorem gives necessary and sufficient conditions for the existence of a LQG-balanced realization.

*Theorem 3.2:* The characteristic function of any integrated resolvent linear system for which the finite cost condition and the dual finite cost condition are satisfied has a LQG-balanced realization.

This existence theorem is complemented by a uniqueness theorem: minimal LQG-balanced realizations are unique up to a unitary transformation in the state space.

It is interesting to note that under the conditions of Theorem 3.2 the characteristic function has a normalized coprime factorization. The proof of Theorem 3.2 is based on the relation between LQG-balanced realizations and Lyapunov-balanced realizations of the normalized coprime factor.

Let $\Sigma$ be a diagonally LQG-balanced integrated resolvent linear system. Define $\Sigma_k$ to be the projection of $\Sigma$ on the space generated by the eigenvectors corresponding to the $k$ largest LQG-singular values. $\Sigma_k$ is called the $k$-dimensional truncated LQG-balanced realization.

The following theorem gives sufficient conditions for the existence of a diagonally LQG-balanced realization and an error bound in the gap metric. We note that an operator $T$ is called *nuclear* if it is compact and its singular values (the eigenvalues of $T^*T$) form a $l^1$ sequence.

*Theorem 3.3:* If an integrated resolvent linear system and its dual both satisfy the finite cost condition, the product of the optimal cost operator $Q$ and the dual optimal cost operator $P$ is nuclear, and the input and output spaces are finite-dimensional, then the characteristic function has a diagonally LQG-balanced realization and

$$\delta_g(G, G^k) \le 2 \sum_{i=k+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}, \quad (15)$$

(the right-hand side being finite) where $G^k$ is the characteristic function of a $k$-dimensional truncated LQG-balanced realization of $G$.

Define $\varepsilon_{\max} := (1+\mu_1^2)^{-1/2}$. In [7] it is shown that for a system $\Sigma$ that satisfies the assumptions of Theorem 3.3 and any $\varepsilon < \varepsilon_{\max}$ there exists a controller that stabilizes all systems $\Sigma'$ with $\delta_g(\Sigma, \Sigma') < \varepsilon$. The result in [7] is an extension of the results in Chapter 9 of [5] and of [6].

The above shows that if

$$2 \sum_{i=k+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < (1+\mu_1^2)^{-1/2},$$

then there exists a $k$-dimensional controller that stabilizes the system $\Sigma$.

## IV. LQG-BALANCING: AN EXAMPLE

In this section we present a simple but nontrivial example.

### A. The model

The system we consider is a one-dimensional Euler-Bernoulli beam with Voight-damping and with free ends. The measurements are the displacement and the angle of rotation of the middle of the beam. As actuators we choose a force and a moment at the middle of the beam.

We obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + \beta \frac{\partial^5 w}{\partial x^4 \partial t} + \alpha \frac{\partial^4 w}{\partial x^4} = \frac{u_1 \delta - u_2 \delta'}{\rho a}$$
$$\frac{\partial^3 w}{\partial x^3}(-1,t) = 0, \quad \frac{\partial^3 w}{\partial x^3}(1,t) = 0$$
$$\frac{\partial^2 w}{\partial x^2}(-1,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(1,t) = 0$$

$$y(t) = \left[ \begin{array}{c} w(0,t) \\ \frac{\partial w}{\partial x}(0,t) \end{array} \right],$$

where $w(t,x)$ is the displacement of the beam at position $x \in (-1,1)$ at time $t$, $u_1(t)$ is the force applied and $u_2(t)$ the moment applied to the middle ($x = 0$) of the beam, $y(t)$ holds the measurements, $\rho, a, \alpha$ and $\beta$ are physical parameters and $\delta$ is the Dirac delta distribution and $\delta'$ is its distributional derivative. A derivation of this model from physical considerations is given in Bontsema [1].

### B. Theoretical results

The next proposition shows that our beam system has a diagonally LQG-balanced realization and the error bound (15) holds.

*Proposition 4.1:* The system considered satisfies all the assumptions of 3.3.

*Proof:* If follows from [1, Lemma 2.13] that the system under consideration is a well-posed linear system, which implies that it is an integrated resolvent linear system. The input and output space are both two-dimensional. It remains to show that the finite cost condition and the dual finite cost condition are satisfied, and that $PQ$ is nuclear.

A spectral decomposition of the main operator $A$ as performed in [1] shows that $A$ has $\alpha/\beta$ in its continuous spectrum, the other spectral points are eigenvalues and these are either located on a circle with center $\alpha/\beta$ or on the real line (see figure 1). All spectral points are in the open left

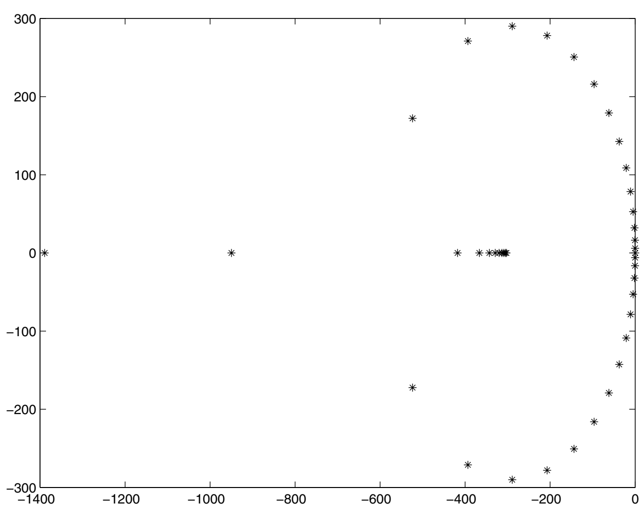


Fig. 1. Eigenvalues of the $A$ operator of the beam

half-plane, except for a quadruple eigenvalue at zero. From the above spectral decomposition one can conclude that the operator $A$ generates an analytic semigroup (this follows as in the appendix of [3]). It is shown in [1] that the control operator $B$ is unbounded, but not maximally unbounded and that the observation operator $C$ is bounded. Using the spectral decomposition of $A$ we can split the system into a stable part and an unstable part as in [5, Section 5.2]. Since the unstable part is controllable we conclude that the system is exponentially stabilizable, which implies that it satisfies the finite cost condition. That the system satisfies the dual finite cost condition follows similarly. From the fact that the semigroup is analytic and the control operator not maximally unbounded we conclude that the optimal state feedback is bounded (see [11]). From this it follows that the optimal closed-loop system as considered in [15] has an analytic semigroup, a control operator that is not maximally unbounded and a bounded observation operator. We invoke [4, Theorem 6] to show that the Hankel operator of this closed-loop system is nuclear. This shows that $PQ$ is nuclear. ■

Proposition 4.1 shows that the controller design method mentioned in Section III performed on a high enough LQG-balanced truncation results in a finite-dimensional stabilizing controller for the beam.

### C. Numerical results

For the numerical part of this article we choose the physical parameters in accordance with De Silva [19], see also Bontsema et. al. [2]. We analyze different approximation techniques using LQG-singular values and Bode diagrams. Due to lack of space we only show the Bode diagrams from the first input to the first output, the response from the second input to the second output is similar and the other two responses are zero. Also, we only show the Bode magnitude diagram.

*1) Modal approximation:* It is relatively easy to obtain a modal approximation of our model based on the eigenvectors of the fourth derivative operator with boundary conditions as above. For more complicated models of physical systems it will not be easy (or even possible) to obtain a modal approximation. In figure 2 the solid line is a Bode-diagram of the 30 dimensional modal approximation. Table I shows the largest ten LQG-characteristic values for modal approximations.

TABLE I
LARGEST 10 LQG-CHARACTERISTIC VALUES FOR MODAL APPROXIMATIONS

| 6 modes | 10 modes | 14 modes | 22 modes | 30 modes |
|---|---|---|---|---|
| 2.4142 | 2.4134 | 2.4134 | 2.4134 | 2.4134 |
| 2.4135 | 2.4123 | 2.4116 | 2.4111 | 2.4109 |
| 0.4143 | 0.4146 | 0.4147 | 0.4147 | 0.4148 |
| 0.4142 | 0.4144 | 0.4144 | 0.4144 | 0.4144 |
| 0.1071 | 0.1071 | 0.1071 | 0.1071 | 0.1071 |
| 0.1068 | 0.1068 | 0.1068 | 0.1068 | 0.1068 |
| - | 0.1010 | 0.1010 | 0.1010 | 0.1010 |
| - | 0.1004 | 0.1004 | 0.1004 | 0.1004 |
| - | 0.0009 | 0.0104 | 0.0104 | 0.0104 |
| - | 0.0009 | 0.0102 | 0.0102 | 0.0102 |

If we construct the controller mentioned in Section III based on a 4 mode approximation it stabilizes the 30 mode approximation, for a lower order approximation this is no longer the case. Since the unstable subspace is four-dimensional this is of course not very surprizing.

*2) Finite-difference approximation:* We have obtained finite-difference approximations of our model. In figure 2 the dashed line is a 30 dimensional finite-difference approximation and in figure 3 the dashed line is a 6 dimensional finite-difference approximation.

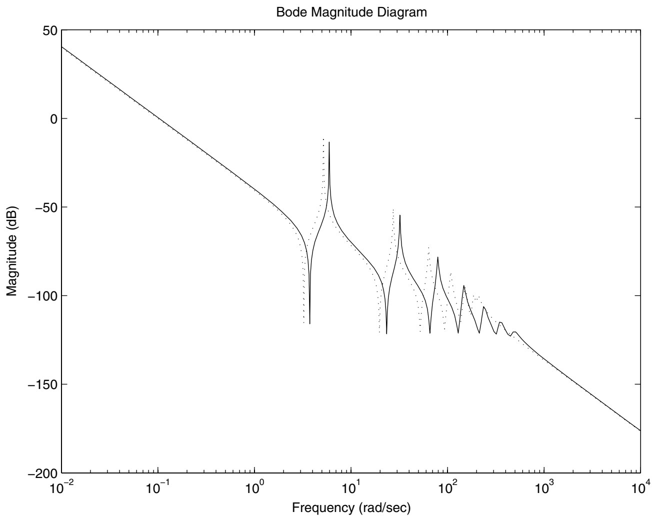


Fig. 2. 30 mode approximation (-) and 30 dimensional finite difference approximation (:)

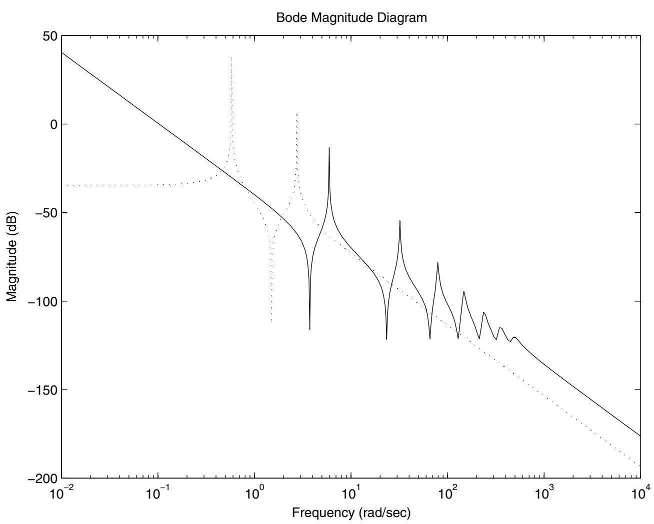


Fig. 3. 30 mode approximation (-) and 6 dimensional finite difference approximation (:)

From this and the 'intermediate' Bode diagrams not shown it can be seen that the resonance peaks are at too low a frequency and this error converges slowly to zero. The 6 dimensional finite-difference approximation also has an incorrect slope for low frequencies. In table II the LQG-characteristic values for finite difference approximations are given.

We can see here also that the 6 dimensional finite-difference approximation is not good and that convergence is slower then in the modal approximation. However, from the 10 dimensional finite-difference approximation on the first four LQG-characteristic values are fairly accurate and the other LQG-characteristic values seem to converge to their

TABLE II

LARGEST 10 LQG-CHARACTERISTIC VALUES FOR FINITE DIFFERENCE APPROXIMATIONS

| 6 dim f-d | 10 dim f-d | 14 dim f-d | 22 dim f-d | 30 dim f-d |
|---|---|---|---|---|
| 2.4142 | 2.4129 | 2.4129 | 2.4131 | 2.4132 |
| 0.9964 | 2.4125 | 2.4122 | 2.4116 | 2.4113 |
| 0.9799 | 0.4146 | 0.4146 | 0.4147 | 0.4147 |
| 0.6408 | 0.4144 | 0.4145 | 0.4144 | 0.4144 |
| 0.6394 | 0.3189 | 0.2286 | 0.1711 | 0.1503 |
| 0.4142 | 0.3183 | 0.2282 | 0.1708 | 0.1500 |
| - | 0.1133 | 0.1255 | 0.1225 | 0.1183 |
| - | 0.1129 | 0.1250 | 0.1219 | 0.1177 |
| - | 0.0089 | 0.0073 | 0.0109 | 0.0114 |
| - | 0.0088 | 0.0072 | 0.0108 | 0.0112 |

correct values. It turns out that the controller mentioned in Section III when based on a 6 dimensional finite difference approximation is not stabilizing and that the one based on a 10 dimensional finite difference approximation is. We conclude that controller-design using finite-difference approximations leads to a controller of more then 6 dimensions.

*3) LQG-balanced approximation:* We have shown that our model has a diagonally LQG-balanced realization. Computing this realization exactly is however impossible. The method of LQG-balancing can however be used to obtain good low-order approximations of good high-order approximations. We compute a LQG-balanced realization for the 30 dimensional finite-difference approximation of the beam (this is finite-dimensional LQG-balancing, so it can be done using an algorithm from finite-dimensional theory). The Bode diagram of a 14 dimensional LQG-balanced truncation of the 30 dimensional finite-difference approximation of the beam is shown in figure 4 and that of a 4 dimensional LQG-balanced truncation of the 30 dimensional finite-difference approximation of the beam is shown in figure 5.

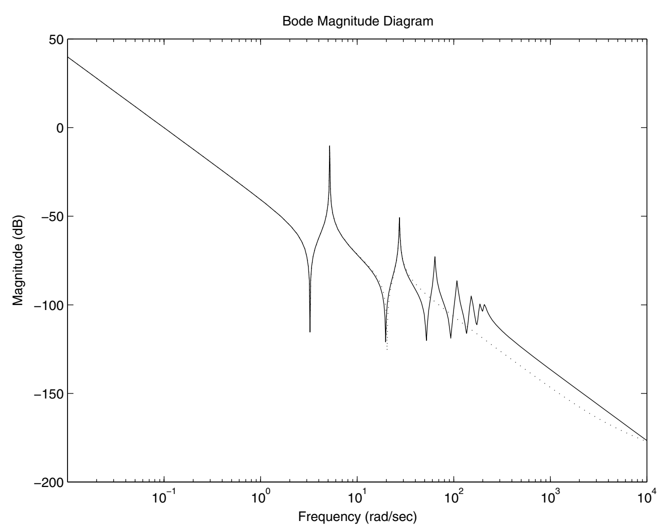


Fig. 4. 30 dimensional finite difference approximation (-) and its 14 dimensional LQG-balanced truncation

As can be seen the approximation is about as good as can be expected given the order of the approximation. The

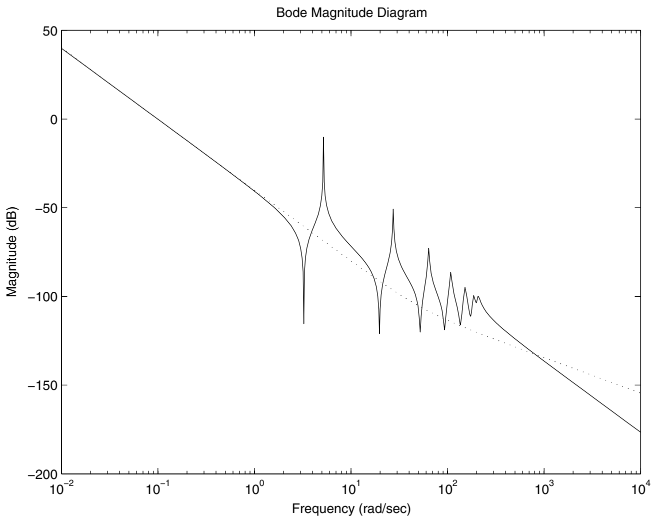


Fig. 5. 30 dimensional finite difference approximation (-) and its 4 dimensional LQG-balanced truncation

controller mentioned in Section III when based on a 4 dimensional LQG-balanced truncation of a 30 dimensional finite difference approximation stabilizes the 30 dimensional modal approximation. Thus it can be expected that it will stabilize the beam.

*4) Conclusions for the example:* We showed that the beam has a diagonally LQG-balanced realization and we obtained an error bound that showed that there is a finite-dimensional stabilizing controller. To explicitly compute such a controller we had to resort to numerical approximations (as is usual). It was shown that a finite difference approximation followed by a LQG-balanced truncation gives a stabilizing 4 dimensional controller. This is as good as can be obtained using a modal approximation. A stabilizing controller based only on a finite difference approximation must have more then 6 states. This shows that the combination of a finite difference approximation and LQG-balancing is better than a finite difference approximation alone.

## V. Concluding Remarks

We have shown that a very large class of infinite dimensional systems have a LQG-balanced realization. Systems in a large subclass of this class have a diagonally LQG-balanced realization. For these systems we do not only have an existence result, but also an error-bound. We studied the example of a beam which shows an application of these theoretical results.

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