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On Delayed Observers for Linear Systems with Unknown Inputs

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Abstract— **We present a method for constructing reduced-order state observers for linear systems with unknown inputs. Our design provides a characterization of observers with delay, which eases the established necessary conditions for existence of unknown input observers with zero-delay. In order to obtain the observer parameters, we develop a systematic design procedure that is quite general in that it encompasses the design of full-order observers via appropriate choices of design matrices.**

I. Introduction

In practice, it is often the case that a dynamic system can be modeled as having unknown inputs. For example, in decentralized control, it may not be possible to have knowledge of the control signals generated by different controllers [3]. Unknown inputs can also be used to represent uncertain system dynamics [1] and faults [11].

The problem of constructing an observer for such systems has received considerable attention over the past few decades [4], [5], [2], [13]. Various methods of realizing both full and reduced-order observers have been presented in the literature. It is known that in order to reconstruct the entire state vector in the presence of unknown inputs, a fairly strict condition must be met. In [7], it was shown that this necessary condition can be relaxed by allowing delays in the observer. While [7] established necessary and sufficient conditions for the existence of observers with delays, no design procedure was provided. In [10], the authors handled delayed observers by constructing a higher dimensional system which incorporated the delayed states into the new state vector. An observer was then constructed for this augmented system, and geometric conditions were given for the existence of such observers. In this paper, we provide a unified design procedure for both reduced and full-order observers with delays. In contrast to the work in [10], the dimension of our observer is no greater than the dimension of the original system, and we present algebraic existence conditions. Our approach generalizes recently published work on full-order zero-delay observers [13], and allows us to treat the full-order observer as a special case of a reduced-order observer where the dynamic portion reconstructs the entire state vector.

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II. Preliminaries

Consider a discrete-time linear system $\mathcal{S}$ in the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \ , \end{aligned} \tag{1}$$

with state vector $x \in \mathbb{R}^n$, unknown input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and system matrices (A, B, C, D) of appropriate dimensions. Note that we omit known inputs in the above equations for clarity of development. We also assume without loss of generality that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is of full column rank. This assumption can always be enforced by an appropriate transformation and renaming of the unknown input signals.

The response of system (1) over $\alpha + 1$ time units is given by

$$\begin{aligned} \underbrace{\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+\alpha} \end{bmatrix}}_{Y_{k:k+\alpha}} &= \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha} \end{bmatrix}}_{\Theta_\alpha} x_k \\ &+ \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1}B & CA^{\alpha-2}B & \cdots & D \end{bmatrix}}_{M_\alpha} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+\alpha} \end{bmatrix}}_{U_{k:k+\alpha}} . \end{aligned} \tag{2}$$

The matrices Θ_α and M_α in the above equation can be expressed in a variety of ways. We will be using the following identities in our derivations:

$$\Theta_\alpha = \begin{bmatrix} C \\ \Theta_{\alpha-1}A \end{bmatrix} = \begin{bmatrix} \Theta_{\alpha-1} \\ CA^{\alpha} \end{bmatrix} \ , \tag{3}$$

$$M_\alpha = \begin{bmatrix} D & 0 \\ \Theta_{\alpha-1}B & M_{\alpha-1} \end{bmatrix} = \begin{bmatrix} M_{\alpha-1} & 0 \\ C\zeta_{\alpha-1} & D \end{bmatrix} \ , \tag{4}$$

where

$$\zeta_{\alpha-1} \equiv \begin{bmatrix} A^{\alpha-1}B & A^{\alpha-2}B & \cdots & B \end{bmatrix} .$$

We are now ready to proceed with the construction of an observer to estimate the states in $\mathcal{S}$.

III. Unknown Input Observer

We start by determining the set of states which can be directly obtained from the output of the system over $\alpha + 1$ time-steps. The following theorem provides an answer to this problem.

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Theorem 1: For system (1) with response over $\alpha+1$ time-steps given by (2), let

$$t = \text{rank}\begin{bmatrix}\Theta_\alpha & M_\alpha\end{bmatrix} - \text{rank}\begin{bmatrix}M_\alpha\end{bmatrix} \ .$$

Then it is possible to perform a similarity transform on the system $\mathcal{S}$ to obtain a new system $\bar{\mathcal{S}}$ such that exactly t of the states in $\bar{\mathcal{S}}$ are directly obtainable from the output of the system.

Proof: Assume $\text{rank}\begin{bmatrix}\Theta_\alpha & M_\alpha\end{bmatrix} - \text{rank}\begin{bmatrix}M_\alpha\end{bmatrix} = t$. This implies that there are t linearly independent vectors in the matrix Θ_α that cannot be written as a linear combination of vectors in M_α. Thus there exists a matrix $\mathcal{P}$ of dimension $t\times(\alpha+1)p$ such that $\mathcal{P}\Theta_\alpha$ has full row-rank, and $\mathcal{P}M_\alpha = \mathbf{0}$. Define the similarity transformation matrix

$$\mathcal{T} \equiv \begin{bmatrix}\mathcal{P}\Theta_\alpha \\ \mathcal{H}\end{bmatrix} \tag{5}$$

where the matrix $\mathcal{H}$ is chosen so that $\mathcal{T}$ has full rank. If desired, $\mathcal{P}$ and $\mathcal{H}$ can be chosen so that $\mathcal{T}$ is orthogonal. Consider the system $\bar{\mathcal{S}}$ with state-vector $\bar{x}_k = \begin{bmatrix}\bar{x}_{1,k} \\ \bar{x}_{2,k}\end{bmatrix} = \mathcal{T}x_k$. The system matrices in $\bar{\mathcal{S}}$ are given by

$$\begin{aligned}
\bar{A} &\equiv \mathcal{T}A\mathcal{T}^{-1} = \begin{bmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \ , \\
\bar{B} &\equiv \mathcal{T}B = \begin{bmatrix}\mathcal{P}\Theta_\alpha B \\ \mathcal{H}B\end{bmatrix} \ , \\
\bar{C} &\equiv C\mathcal{T}^{-1}, \ \ \bar{D} \equiv D \ .
\end{aligned} \tag{6}$$

Now it is readily seen from (2) that

$$\begin{aligned}
\mathcal{P}Y_{k:k+\alpha} &= \mathcal{P}\Theta_\alpha\mathcal{T}^{-1}\bar{x}_k \\
&= \begin{bmatrix}I_t & 0\end{bmatrix}\bar{x}_k \ ,
\end{aligned}$$

and thus the first t states of $\bar{x}_k$ are immediately obtained. ■

Remark 1: The problem of determining a particular set of states from the (delayed) output was studied in [15], and the special case of perfect observability (i.e., $t = n$) was studied in [3], [9]. The result in Theorem 1 appears to be new in that it deals with reconstructing a maximal subset of states from the output.

To estimate the remaining $(n-t)$ states of $\bar{x}_k$ (i.e., $\bar{x}_{2,k}$), we construct a reduced-order observer of the form

$$\begin{aligned}
z_{k+1} &= Ez_k + FY_{k:k+\alpha} \\
\psi_k &= z_k + GY_{k:k+\alpha} \ ,
\end{aligned} \tag{7}$$

where matrices E, F and G are chosen such that $\psi_k \to \bar{x}_{2,k}$ as $k \to \infty$. Using (2), the observer error is given by

$$\begin{aligned}
e_{k+1} &\equiv \psi_{k+1} - \bar{x}_{2,k+1} \\
&= Ez_k + FY_{k:k+\alpha} + GY_{k+1:k+\alpha+1} \\
&\quad - \begin{bmatrix}A_{21} & A_{22}\end{bmatrix}\bar{x}_k - \mathcal{H}Bu_k \\
&= Ee_k + (F - EG)\,\Theta_\alpha x_k + G\Theta_\alpha Ax_k \\
&\quad + E\mathcal{H}x_k - \begin{bmatrix}A_{21} & A_{22}\end{bmatrix}\mathcal{T}x_k \\
&\quad + (F - EG)\,M_\alpha U_{k:k+\alpha} + G\Theta_\alpha Bu_k \\
&\quad + GM_\alpha U_{k+1:k+\alpha+1} - \mathcal{H}Bu_k \ .
\end{aligned}$$

Using the identities (3) and (4), the expression for the error can be written as

$$\begin{aligned}
e_{k+1} &= Ee_k + \begin{bmatrix}F - EG & 0\end{bmatrix}\Theta_{\alpha+1}x_k + \begin{bmatrix}0 & G\end{bmatrix}\Theta_{\alpha+1}x_k \\
&\quad + E\mathcal{H}x_k - \begin{bmatrix}A_{21} & A_{22}\end{bmatrix}\mathcal{T}x_k \\
&\quad + \begin{bmatrix}F - EG & 0\end{bmatrix}M_{\alpha+1}U_{k:k+\alpha+1} \\
&\quad + \begin{bmatrix}0 & G\end{bmatrix}M_{\alpha+1}U_{k:k+\alpha+1} - \mathcal{H}Bu_k \ .
\end{aligned}$$

Partition the matrices F and G as

$$\begin{aligned}
F &= \begin{bmatrix}F_0 & F_1 & \cdots & F_\alpha\end{bmatrix} \ , \\
G &= \begin{bmatrix}G_0 & G_1 & \cdots & G_\alpha\end{bmatrix}
\end{aligned}$$

where each F_i and G_i are of dimension $(n-t)\times p$, and define

$$\begin{aligned}
K \equiv \big[F_0 - EG_0 \quad & F_1 - EG_1 + G_0 \quad \cdots \\
& \cdots \quad F_\alpha - EG_\alpha + G_{\alpha-1} \quad G_\alpha\big] \ .
\end{aligned} \tag{8}$$

Note that since we are free to choose F and G, the matrix K can be chosen to have any value we require. The error can then be expressed as

$$\begin{aligned}
e_{k+1} &= Ee_k + \left(E\mathcal{H} - \begin{bmatrix}A_{21} & A_{22}\end{bmatrix}\mathcal{T} + K\Theta_{\alpha+1}\right)x_k \\
&\quad + KM_{\alpha+1}U_{k:k+\alpha+1} - \mathcal{H}Bu_k \ .
\end{aligned}$$

In order to force the error to go to zero, regardless of the values of x_k and the inputs, the following two conditions must hold:

1) E must be a stable matrix,
2) The matrix K must satisfy

$$KM_{\alpha+1} = \begin{bmatrix}\mathcal{H}B & 0 & \cdots & 0\end{bmatrix}, \tag{9}$$

$$E\mathcal{H} = \begin{bmatrix}A_{21} & A_{22}\end{bmatrix}\mathcal{T} - K\Theta_{\alpha+1} \ . \tag{10}$$

The solvability of condition (9) is given by the following theorem.

Theorem 2: There exists a matrix K such that

$$KM_{\alpha+1} = \begin{bmatrix}\mathcal{H}B & 0 & \cdots & 0\end{bmatrix}$$

if and only if

$$\text{rank}\,[M_{\alpha+1}] - \text{rank}\,[M_\alpha] = m \ . \tag{11}$$

Proof: There exists a K satisfying (9) if and only if the matrix

$$R \equiv \begin{bmatrix}\mathcal{H}B & 0 & \cdots & 0\end{bmatrix}$$

is in the space spanned by the rows of $M_{\alpha+1}$. This is equivalent to the condition

$$\text{rank}\begin{bmatrix}M_{\alpha+1} \\ R\end{bmatrix} = \text{rank}\,[M_{\alpha+1}] \ .$$

Using (4), we get

$$\text{rank}\begin{bmatrix} M_{\alpha+1} \\ R \end{bmatrix} = \text{rank}\begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \\ \mathcal{H}B & 0 \end{bmatrix}$$
$$= \text{rank}\left(\begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{P} & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \\ \mathcal{H}B & 0 \end{bmatrix}\right)$$
$$= \text{rank}\left(\begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{T} & 0 \\ 0 & \Theta_\alpha & I \end{bmatrix}\begin{bmatrix} D & 0 \\ B & 0 \\ 0 & M_\alpha \end{bmatrix}\right) \quad .$$

By our assumption that the matrix $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ has full column rank, we get

$$\text{rank}\begin{bmatrix} M_{\alpha+1} \\ R \end{bmatrix} = m + \text{rank}\left[M_\alpha\right] \quad ,$$

thereby completing the proof. ■

Note that (11) is the condition for inversion of the inputs with known initial state, as given in [12]. If we set $\alpha = 0$, condition (11) becomes

$$\text{rank}\begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} = m + \text{rank}\left[D\right] \quad ,$$

which is the well known necessary condition for unknown-input observers with zero delay [4]. This is a fairly strict condition, and demonstrates the utility of a delayed observer. When designing such an observer, one can start with $\alpha = 0$ and increase α until a value is found that satisfies (11). An upper bound on α is provided by the following theorem from [14], which considered the problem of system invertibility.

Theorem 3: Let q be the dimension of the nullspace of D. Then there exists an α satisfying (11) if and only if

$$\text{rank}\left[M_{n-q+1}\right] - \text{rank}\left[M_{n-q}\right] = m \quad .$$

Thus the largest delay that will be required by our observer is $n-q$ time-steps. If (11) is not satisfied even for $\alpha = n-q$, it is not possible to estimate all the states in the system.

Remark 2: Condition (11) was also obtained in [7] through a different method. The approach in that paper was to define a new output equation for the system, with Θ_α and M_α taking the place of the 'C' and 'D' matrices, respectively. These matrices were then substituted into the necessary conditions for zero-delay observers, and reduced to produce equation (11). While this approach is quite intuitive, it may result in unnecessarily large and redundant matrices when designing the observer parameters. Furthermore, the upper bound on the observer delay provided in [7] is $\alpha = n-1$, which can be tightened by applying the results of Theorem 3.

We now turn our attention to condition (10). Right-multiplying by $\mathcal{T}^{-1}$, we get the equivalent condition

$$\begin{bmatrix} 0 & E \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - K\Theta_{\alpha+1}\mathcal{T}^{-1} \quad . \tag{12}$$

From the above equation it is apparent that there is an additional constraint on K; namely, K times the first t columns of $\Theta_{\alpha+1}\mathcal{T}^{-1}$ must produce A_{21}. To satisfy this constraint, we define

$$\mathcal{T}_y \equiv \begin{bmatrix} \mathcal{P} \\ \mathcal{Q} \end{bmatrix}, \quad \mathcal{J} \equiv \begin{bmatrix} \mathcal{T}_y & 0 \\ 0 & I_p \end{bmatrix} \quad , \tag{13}$$

where the matrix $\mathcal{Q}$ is chosen so that $\mathcal{T}_y$ is square and invertible. Using (3) and (4), we get

$$\widehat{M} \equiv \mathcal{J}M_{\alpha+1} = \begin{bmatrix} 0 & 0 \\ \mathcal{Q}M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} \quad , \tag{14}$$

$$\widehat{\Theta} \equiv \mathcal{J}\Theta_{\alpha+1}\mathcal{T}^{-1} = \begin{bmatrix} I_t & 0 \\ L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \quad , \tag{15}$$

where

$$\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} = \begin{bmatrix} \mathcal{Q}\Theta_\alpha\mathcal{T}^{-1} \\ CA^{\alpha+1}\mathcal{T}^{-1} \end{bmatrix} \quad .$$

Since $\mathcal{J}$ is invertible, we can define a matrix $\bar{K}$ such that $K = \bar{K}\mathcal{J}$. Partitioning $\bar{K}$ as

$$\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \quad ,$$

where $\bar{K}_1$ has t columns, equations (9) and (12) become

$$\begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \mathcal{Q}M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 & \cdots & 0 \end{bmatrix} \quad , \tag{16}$$

$$\begin{bmatrix} 0 & E \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix}\begin{bmatrix} I_t & 0 \\ L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \quad . \tag{17}$$

We see from the above equations that $\bar{K}_1$ must be chosen such that

$$\bar{K}_1 = A_{21} - \bar{K}_2\begin{bmatrix} L_1 \\ L_3 \end{bmatrix} \quad ,$$

and so the problem is reduced to finding the matrix $\bar{K}_2$ satisfying equations (16) and (17).

Recall that the first m columns of $M_{\alpha+1}$ must be linearly independent of each other and of the remaining $(\alpha+1)m$ columns (by Theorem 2), and so the rank of

$$\begin{bmatrix} \mathcal{Q}M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} \tag{18}$$

is $m + \text{rank}[M_\alpha]$. Let $\mathcal{N}$ be a matrix whose rows form a basis for the left nullspace of the last $(\alpha+1)m$ columns of (18). In particular, we can assume without loss of generality that $\mathcal{N}$ satisfies

$$\mathcal{N}\begin{bmatrix} \mathcal{Q}M_\alpha & 0 \\ C\zeta_\alpha & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} \quad . \tag{19}$$

From (16), we see that $\bar{K}_2$ must be of the form

$$\bar{K}_2 = \widehat{K}\mathcal{N}$$

for some $\widehat{K} = \begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}$, where $\widehat{K}_2$ has m columns. Equation (16) then becomes

$$\begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} \quad , \tag{20}$$

from which it is obvious that $\widehat{K}_2 = \mathcal{H}B$ and $\widehat{K}_1$ is a free matrix.

Returning to equation (17), we have

$$E = A_{22} - \bar{K}_2 \begin{bmatrix} L_2 \\ L_4 \end{bmatrix}$$
$$= A_{22} - \begin{bmatrix} \widehat{K}_1 & \mathcal{H}B \end{bmatrix} \mathcal{N} \begin{bmatrix} L_2 \\ L_4 \end{bmatrix} \ .$$

Defining

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \equiv \mathcal{N} \begin{bmatrix} L_2 \\ L_4 \end{bmatrix} \ , \tag{21}$$

where ν_2 has m rows, we come to the final equation

$$E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}_1 \nu_1 \ . \tag{22}$$

Recall that we require E to be a stable matrix, and this is only possible if the pair $(A_{22} - \mathcal{H}B\nu_2, \nu_1)$ is detectable. This detectability condition can be stated in terms of the original system matrices as follows.

Theorem 4: The rank condition

$$\text{rank} \begin{bmatrix} zI - A_{22} + \mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} = n - t, \ \ \forall z \in \mathbb{C}, \ \ |z| \geq 1$$

is satisfied if and only if

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \ \ \forall z \in \mathbb{C}, \ \ |z| \geq 1 \ .$$

To prove the theorem, we make use of the following lemma, which is obtained by a simple modification of a lemma from [7].

Lemma 1: The rank condition

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} = n + m, \ \ \forall z \in \mathbb{C}, \ \ |z| \geq 1$$

is satisfied if and only if

$$\text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = n + m + \text{rank}[M_\alpha],$$
$$\forall z \in \mathbb{C}, \ \ |z| \geq 1 \ .$$

We are now in place to prove Theorem 4.

Proof: We start by noting from (14) and (19) that

$$\begin{bmatrix} I_t & 0 \\ 0 & \mathcal{N} \end{bmatrix} \mathcal{J} \begin{bmatrix} D & 0 \\ \Theta_\alpha B & M_\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_m & 0 \end{bmatrix} \ .$$

Let $\bar{\mathcal{N}}$ be a matrix whose rows form a basis for the left nullspace of M_α. We can then write

$$\begin{bmatrix} I_t & 0 \\ 0 & \mathcal{N} \end{bmatrix} \mathcal{J} = \mathcal{W} \begin{bmatrix} I_p & 0 \\ 0 & \bar{\mathcal{N}} \end{bmatrix} \tag{23}$$

for some invertible matrix $\mathcal{W}$. Through a series of nonsingular transformations, we obtain

$$\text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = \text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} zI - A & -B & 0 & 0 \\ C & D & 0 & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 & 0 \\ zC & 0 & D & 0 \\ z\Theta_{\alpha-1} A & 0 & \Theta_{\alpha-1} B & M_{\alpha-1} \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} zI - A & -B & 0 & 0 \\ C & D & 0 & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 & 0 \\ 0 & 0 & D & 0 \\ z\Theta_{\alpha-1} A & z\Theta_{\alpha-1} B & \Theta_{\alpha-1} B & M_{\alpha-1} \end{bmatrix} \ .$$

Continuing in the above manner, we get

$$\text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = \text{rank} \begin{bmatrix} zI - A & -B & 0 \\ C & D & 0 \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B & 0 \\ 0 & 0 & M_\alpha \end{bmatrix}$$

and the rank of the top left submatrix in the above expression is given by

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} = \text{rank} \begin{bmatrix} zI - \mathcal{T}A\mathcal{T}^{-1} & -\mathcal{T}B \\ C\mathcal{T}^{-1} & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1} & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} zI - A_{11} & -A_{12} & -\mathcal{P}\Theta_\alpha B \\ -A_{21} & zI - A_{22} & -\mathcal{H}B \\ C\mathcal{T}^{-1}(:,1) & C\mathcal{T}^{-1}(:,2) & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,1) & \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,2) & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} \ ,$$

where $\mathcal{T}^{-1}(:,1)$ represents the first t columns of $\mathcal{T}^{-1}$, and $\mathcal{T}^{-1}(:,2)$ represents the last $n-t$ columns. By the definition of $\mathcal{P}$, there exists a matrix $\mathcal{V}$ such that $\mathcal{P} = \mathcal{V}\bar{\mathcal{N}}$. Using the fact that $\mathcal{V}\bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$, we get

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} zI_t & 0 & 0 \\ -A_{21} & zI - A_{22} & -\mathcal{H}B \\ C\mathcal{T}^{-1}(:,1) & C\mathcal{T}^{-1}(:,2) & D \\ \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,1) & \bar{\mathcal{N}}\Theta_\alpha A\mathcal{T}^{-1}(:,2) & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} \ .$$

Using (14), (15), (21) and (23), we left-multiply the last two block rows in the above matrix by $\mathcal{W}$ to obtain

$$\text{rank} \begin{bmatrix} zI - A & -B \\ C & D \\ \bar{\mathcal{N}}\Theta_\alpha A & \bar{\mathcal{N}}\Theta_\alpha B \end{bmatrix} = \text{rank} \begin{bmatrix} zI_t & 0 & 0 \\ -A_{21} & zI - A_{22} & -\mathcal{H}B \\ I_t & 0 & 0 \\ * & \nu_1 & 0 \\ * & \nu_2 & I_m \end{bmatrix}$$
$$= t + \text{rank} \begin{bmatrix} zI - A_{22} & -\mathcal{H}B \\ \nu_1 & 0 \\ \nu_2 & I_m \end{bmatrix}$$
$$= t + m + \text{rank} \begin{bmatrix} zI - A_{22} + \mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} \ ,$$

where $*$ represents unimportant matrices. This gives

$$\text{rank}\begin{bmatrix} zI-A & -B & 0 \\ C & D & 0 \\ \Theta_\alpha A & \Theta_\alpha B & M_\alpha \end{bmatrix} = t+m+\text{rank}[M_\alpha] + \text{rank}\begin{bmatrix} zI-A_{22}+\mathcal{H}B\nu_2 \\ \nu_1 \end{bmatrix} .$$

Using Lemma 1, we get the desired result. ∎

We can now state the following theorem, whose proof is immediately given by the discussion so far.

Theorem 5: The system $\mathcal{S}$ in (1) has an observer with delay α if and only if

1) rank $[M_{\alpha+1}]$ − rank $[M_\alpha] = m$,
2) rank $\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix} = n+m, \ \forall z \in \mathbb{C}, \ |z| \geq 1$.

Recall that the first condition in the above theorem means that the system is invertible with delay $\alpha+1$. In fact, it has been shown in [8] that condition 2 is sufficient for the existence of a stable inverse for system $\mathcal{S}$. This fact leads to the following theorem.

Theorem 6: The system $\mathcal{S}$ in (1) has an observer (possibly with delay) if and only if

$$\text{rank}\begin{bmatrix} zI-A & -B \\ C & D \end{bmatrix} = n+m, \ \forall z \in \mathbb{C}, \ |z| \geq 1 .$$

Remark 3: The result in the above theorem has also been noted in [6], which studied the problem of reconstructing the unknown inputs. The difference between Theorem 5 and Theorem 6 is that the latter does not provide a characterization of the delay in the observer. Note that the conditions in Theorem 5 (and the equivalent condition in Theorem 4) are a generalization of those given in [13], [5], [4] for the existence of zero-delay observers, and verify the conditions in [7]. It is of interest to note that while we have pursued the development of a reduced-order observer, the above approach and conditions immediately apply to full-order observers as well. This is because a full-order observer can be viewed as a special case of a reduced-order observer, where the dynamic portion reconstructs the entire state. This can be accomplished by setting $\mathcal{P}$ to be an empty matrix (i.e., by choosing $t=0$, $\mathcal{T}=\mathcal{H}=I_n$, $\mathcal{T}_y = I_{(\alpha+1)p}$).

IV. Design Procedure

We now summarize the design steps that can be used in designing a delayed observer for the system given in (1).

1) Find the smallest α such that rank$[M_{\alpha+1}]$ − rank $[M_\alpha] = m$. If the condition is not satisfied for $\alpha = n - \text{nullity}\,[D]$, it is not possible to reconstruct the entire state of the system.
2) Find the matrix $\mathcal{P}$ such that $\mathcal{P}\Theta_\alpha$ is full row rank and $\mathcal{P}M_\alpha$ is zero. Use Theorem 1 for a reduced-order observer, or set $\mathcal{P}$ to be the empty matrix for a full-order observer. Choose $\mathcal{H}$ and form the matrix $\mathcal{T}$ in (5) to obtain the transformed system given by (6). Also choose $\mathcal{Q}$ and form the matrix $\mathcal{T}_y$ given in (13).
3) Find the matrix $\mathcal{N}$ satisfying (19).
4) Form the matrices $\widehat{\Theta}$ and $\left[\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right]$ from (15) and (21).
5) If the detectability condition in Theorem 4 is satisfied, choose the matrix $\widehat{K}_1$ such that the eigenvalues of $E = (A_{22} - \mathcal{H}B\nu_2) - \widehat{K}_1\nu_1$ are stable.
6) Set

$$\bar{K}_2 = \begin{bmatrix}\widehat{K}_1 & \mathcal{H}B\end{bmatrix}\mathcal{N}$$
$$\bar{K}_1 = A_{21} - \bar{K}_2\begin{bmatrix}L_1 \\ L_3\end{bmatrix}$$
$$K = \begin{bmatrix}\bar{K}_1 & \bar{K}_2\end{bmatrix}\begin{bmatrix}\mathcal{T}_y & 0 \\ 0 & I_p\end{bmatrix} . \tag{24}$$

7) Use (8) to map this K matrix to F and G. Note that this mapping is not unique. In particular, one can choose $G_0 = G_1 = \cdots = G_{\alpha-1} = 0$, thereby getting

$$K = \begin{bmatrix}F_0 & F_1 & \cdots & F_\alpha - EG_\alpha & G_\alpha\end{bmatrix} .$$

This choice corresponds to using only the most delayed measurement in the output of the observer. Similarly, one can choose $F_1 = F_2 = \cdots = F_\alpha = 0$, which corresponds to using only the earliest measurement in the dynamic portion of the observer. Other combinations are also possible. Note that this freedom does not exist when designing a zero-delay observer.
8) The final observer is given by equation (7). The estimate of the original system states is obtained as

$$\hat{x}_k = \mathcal{T}^{-1}\begin{bmatrix}\mathcal{P}Y_{k:k+\alpha} \\ \psi_k\end{bmatrix} . \tag{25}$$

V. Example

Consider the system given by the matrices

$$A = \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, \ B = \begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix},$$
$$C = \begin{bmatrix}0&1&-1\\1&0&0\end{bmatrix}, \ D = \begin{bmatrix}0&1\\0&1\end{bmatrix} .$$

It is found that condition (11) holds for $\alpha = 1$, so our observer must have a minimum delay of one time-step. Using Theorem 1, we find $t = 2$ and choose

$$\mathcal{P} = \begin{bmatrix}-1&1&0&0\\0&0&-1&1\end{bmatrix},$$
$$\mathcal{H} = \begin{bmatrix}0&1&1\end{bmatrix},$$
$$\mathcal{Q} = \begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix} .$$

Performing the similarity transformation, we get

$$\left[\begin{array}{c|c} A_{21} & A_{22}\end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 1\end{array}\right] .$$

The matrices $\widehat{M}$ and $\widehat{\Theta}$ from equations (14) and (15) are found to be

$$\widehat{M} = \begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\\0&1&0&1&0&0\\1&1&0&1&0&1\\0&1&0&1&0&1\end{bmatrix}, \ \widehat{\Theta} = \begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\\-1&2&1\\0&0&1\\-1&2&1\end{bmatrix} .$$

In this example, the last $(\alpha+1)m = 4$ columns of $\widehat{M}$ have a rank of two, and thus the matrix $\mathcal{N}$ in (19) will only have two rows:

$$\mathcal{N} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

Thus, equation (20) becomes

$$\begin{bmatrix} \widehat{K}_1 & \widehat{K}_2 \end{bmatrix} \begin{bmatrix} I_2 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{H}B & 0 \end{bmatrix} ,$$

and since $\widehat{K}_2$ has $m = 2$ columns, $\widehat{K}_1$ is the empty matrix. This implies that we will have no freedom in choosing the eigenvalues of our observer.

Next, we use equation (21) to obtain

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Again, since ν_2 has $m = 2$ rows, ν_1 is the empty matrix. We now check the detectability of our system by computing

$$E = A_{22} - \mathcal{H}B\nu_2 = 0 ,$$

which implies that we are able to design a stable observer. Using (24), we get

$$\begin{aligned} \bar{K}_2 &= \begin{bmatrix} 1 & 0 & 2 & -2 \end{bmatrix} , \\ \bar{K}_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} , \\ K &= \begin{bmatrix} 0 & 1 & -1 & 1 & 2 & -2 \end{bmatrix} . \end{aligned}$$

Finally, we obtain the F and G matrices by choosing $G_0 = 0$. Since $E = 0$, we have

$$\begin{aligned} F &= \begin{bmatrix} F_0 & F_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix} \\ G &= \begin{bmatrix} G_0 & G_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -2 \end{bmatrix} . \end{aligned}$$

The final observer is given by

$$\begin{aligned} z_{k+1} &= FY_{k:k+1} \\ \psi_k &= z_k + GY_{k:k+1} , \end{aligned}$$

and an estimate of the original system states can be obtained via (25).

To test this observer, the system is simulated with an initial non-zero state, and driven by a sinusoidal input. The observer is initialized with an initial state of zero, and as seen from the plots in Figure 1, catches up with the system state to produce a perfect estimate that is delayed by one time-step. Note that the observer starts operation at $k = 1$, to account for the one-step delay.

VI. Conclusions

We have provided a characterization of unknown input observers with delays, and have developed a streamlined design procedure to obtain the observer parameters. Our approach is quite general in that it treats both reduced and full-order observers by selecting the design matrices appropriately. While we have only considered discrete-time systems, it is worth noting that our development transfers readily to continuous-time systems simply by replacing delays with integrators (or equivalently, by replacing advances with differentiators).

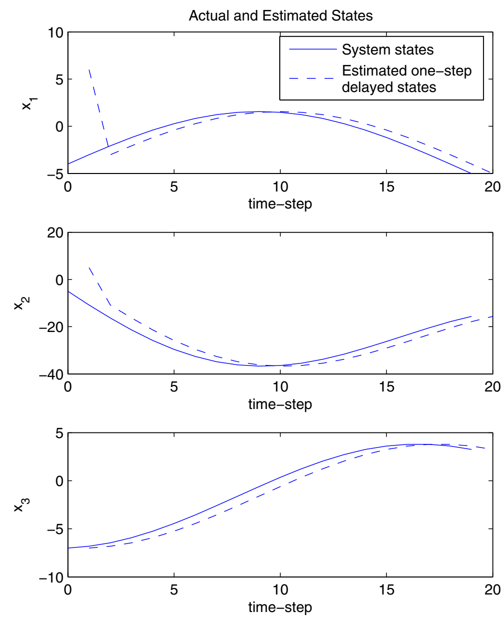

Fig. 1. Simulation of system and one-step delayed observer.

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