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Wave-based Analysis and Wave Control of Ladder Networks

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Abstract— **This paper investigates the wave-based analysis and the wave control for ladder electric networks. The results are a generalized version of our previous work for the cascade connected damped mass-spring systems. We first clarify the class of the ladder networks which satisfies the three condition for the propagation constants to be analyzed by the wave-based analysis. Secondly, for the class of the networks, we investigate the analyticity of the secondary constants and positive real property of the characteristic impedances. Properties of the impedance matching are also investigated. A numerical example for a mechanical system shows effectiveness of the impedance matching for vibration control.**

I. INTRODUCTION

In recent years, as an alternative approach to the modal control [1]–[3], the wave control [4]–[6] has received increasing attention in active vibration control of flexible structures. The wave control is a similar concept to the impedance matching in the electric circuit theory, and it is expected to be applicable for vibration control of lightly damping or large scale structures [5], [6]. The wave control of flexible structures was originally developed for simple uniform beam [4], and extended to structural networks consists of uniform slender structural members connected each other [5]. Aiming to treat more general structures, there are some researches which utilize the dereverberated mobility as a control design model [6]. The dereverberated mobility can be calculated numerically from input/output behavior through the cepstrum [6] for uniform members, however, validity of the method to apply for non-uniform structures is not clear from theoretical point of view. In this sense, the wave control so far has been basically developed based on the wave propagation of the uniform members. In addition, for practical use, the wave control for lumped parameter systems will be also needed.

In [7]–[10], aiming to clarify the possible structure of the systems whose dynamical response can be described by the traveling waves, we were concerned with the wave control of (non-uniform) cascade connected damped mass-spring systems. We considered three conditions for the propagation constants which play central roles in the wave-based analysis for the uniform case, and derived a necessary and sufficient condition for the physical parameters of the system to satisfy the three conditions [7], [8]. The condition for the physical parameters are that the masses, damping coefficients and spring constants vary with some constant ratio. The systems which satisfy the condition are called the *uniformly varying* damped mass-spring systems. For the system, we investigated analytical properties of the secondary constants such as the propagation constants and the characteristic impedances [9], [10]. Properties of the impedance matching were also investigated. From the analogy between mechanical systems and electric circuits, the cascade connected damped mass-spring system can be regarded as the RL-C ladder network, i.e., a special class of the ladder network. So, in this paper, we extend our previous results for the damped mass-spring systems to *general* ladder electric networks. The circuit elements of the network is only assumed to be linear, time invariant, finite and passive. We first clarify a class of the networks which satisfies the three conditions for the propagation constants. Similar to the damped mass-spring case, the uniformly varying condition is obtained. For this class of the networks, we next investigate analyticity of the secondary constants and positive real property of the characteristic impedances. Properties of the impedance matching are also investigated. A numerical example for a mechanical system shows effectiveness of the impedance matching for vibration control.

II. SYSTEM CHARACTERIZATION FROM PROPAGATION CONSTANTS

In this paper, $\mathcal{R}_+$ denotes the set of positive real number, and $\mathcal{C}_+$ denotes the set of complex number with positive real part. The Laplace operator is represented by s.

This paper considers the ladder network shown in Fig. 1. In the figure, $Z_\ell(s)$ and $Y_\ell(s)$ represent the impedance and the admittance of electric circuits whose circuit elements are linear, time invariant, finite and passive. The subscript ℓ ($\ell = 1, 2, \cdots, n$) represents the position of the circuit elements from the left end. The voltage and the current of the right port of the ℓ-th elements are represented by $v_\ell(t)$[V] and $i_\ell(t)$[A], respectively. From the assumption, notice that $Z_\ell(s)$ and $Y_\ell(s)$ are rational positive real functions.

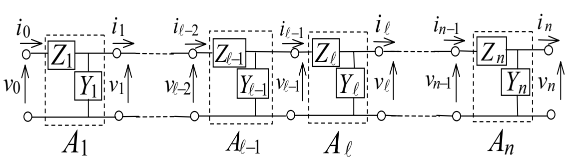

Fig. 1. Ladder network

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Let $v_\ell(s)$ and $i_\ell(s)$ be the Laplace transform of $v_\ell(t)$ and $i_\ell(t)$, respectively. The relation of $v_\ell(s)$ and $i_\ell(s)$ between ℓ and $\ell-1$ are given by the recurrent formula

$$\begin{bmatrix} v_\ell(s) \\ i_\ell(s) \end{bmatrix} = A_\ell(s) \begin{bmatrix} v_{\ell-1}(s) \\ i_{\ell-1}(s) \end{bmatrix}, \tag{1}$$

where

$$A_\ell(s) := \begin{bmatrix} 1 & -Z_\ell(s) \\ -Y_\ell(s) & 1+Z_\ell(s)Y_\ell(s) \end{bmatrix}. \tag{2}$$

Remark 1: In the case of the cascade connected damped mass-spring systems [7]–[10], the structure of the coefficient matrix of the recurrent formula is the same as (2), while the elements $Z_\ell(s)$ and $Y_\ell(s)$ are specifically $Z_\ell(s) = s/(d_\ell s + k_\ell)$ and $Y_\ell(s) = m_\ell s$ where m_ℓ, d_ℓ, $k_\ell \in \mathcal{R}_+$. Therefore, the ladder network shown in Fig. 1 is a generalized version of the damped mass-spring systems considered in [7]–[10].

Suppose there exists a transformation

$$\begin{bmatrix} v_\ell(s) \\ i_\ell(s) \end{bmatrix} = T_\ell(s) \begin{bmatrix} i_\ell^-(s) \\ i_\ell^+(s) \end{bmatrix}, \tag{3}$$

which transforms equation (1) into

$$\begin{bmatrix} i_\ell^-(s) \\ i_\ell^+(s) \end{bmatrix} = T_\ell(s)^{-1} A_\ell(s) T_{\ell-1}(s) \begin{bmatrix} i_{\ell-1}^-(s) \\ i_{\ell-1}^+(s) \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} \lambda_{1_\ell}(s) & 0 \\ 0 & \lambda_{2_\ell}(s) \end{bmatrix} \begin{bmatrix} i_{\ell-1}^-(s) \\ i_{\ell-1}^+(s) \end{bmatrix}. \tag{5}$$

As described in [7], [8], $i_\ell^-(s)$ and $i_\ell^+(s)$ can be regarded as the traveling waves, if $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$ has some preferable properties. In that case, this diagonalization ensures no internal reflection in the structure. In [7], [8], for the cascade connected damped mass-spring systems, from the analogy of the wave-based analysis of the uniform case, we considered the three conditions for $\lambda_{1_\ell}(s)$ and $\lambda_{2_\ell}(s)$ of the system to be analyzed by the traveling waves:

(a) $\lambda_1(s) \neq \lambda_2(s)$.
(b) $\lambda_1(s)\lambda_2(s) = a \in \mathcal{R}_+$.
(c) $\lambda_1(s), \lambda_2(s)$ are independent of ℓ.

In the above conditions, the subscript ℓ is omitted owing to (c). The condition (a) requires that the wave properties of $i_\ell^+(s)$ and $i_\ell^-(s)$ are different, and (b) requires that $i_\ell^+(s)$ and $i_\ell^-(s)$ travel in the opposite direction at the same speed. The condition (c) requires that the wave properties of $i_\ell^+(s)$ and $i_\ell^-(s)$ are independent of the position ℓ. Notice that if (c) is not satisfied, the wave forms of the traveling waves may be distorted, and the waves possibly change the direction at some locations. Although there are some possibilities of extension of the wave control to the systems with some distorted waves (possibly at different speed), the topic is beyond the scope of this paper.

The next theorem characterizes a condition for the circuit elements to satisfy the above three conditions.

Theorem 1: Consider the recurrent formula (1). There exists a transformation (3) which transforms (1) into (5) and the conditions (a), (b) and (c) are satisfied iff the impedance and the admittance of the ladder network satisfy

$$Z_\ell(s) = \frac{1}{a} Z_{\ell-1}(s),\ Y_\ell(s) = aY_{\ell-1}(s),\ \ell = 2, 3, \cdots, n. \tag{6}$$

Proof: The proof can be constructed by tracing the proof of Theorem 1 in [7] (or Theorem 2.1 in [8]) by replacing $Z_\ell(s) = s/(d_\ell s + k_\ell)$ and $Y_\ell(s) = m_\ell s$ with $Z_\ell(s)$ and $Y_\ell(s)$. ■

For the system satisfying (6), $\lambda_1(s)$ and $\lambda_2(s)$ are given by the roots of a polynomial

$$\lambda^2 - (a + 1 + Z_\ell(s)Y_\ell(s))\lambda + a, \tag{7}$$

and the transformation matrix is given by

$$T_\ell(s) := \begin{bmatrix} Z_\ell^-(s) & Z_\ell^+(s) \\ -1 & 1 \end{bmatrix}, \tag{8}$$

where

$$Z_\ell^-(s) := \frac{Z_\ell(s)}{\lambda_1(s) - a},\ Z_\ell^+(s) := -\frac{Z_\ell(s)}{\lambda_2(s) - a}. \tag{9}$$

The variables $\lambda_1(s)$, $\lambda_2(s)$ and $Z_\ell^-(s)$, $Z_\ell^+(s)$ are called the propagation constants and the characteristic impedances, respectively, and referred to the secondary constants. See [7], [8], for the precise derivation of these secondary constants.

From the definitions (7)–(9), some algebraic relationships hold for the secondary constants. From the relations between the roots and the coefficients for (7), we get

$$\lambda_1(s)\lambda_2(s) = a, \tag{10}$$

$$\lambda_1(s) + \lambda_2(s) = a + 1 + Z_\ell(s)Y_\ell(s). \tag{11}$$

In addition, from (6) and (9), since

$$Z_\ell^-(s) = \frac{1}{a} Z_{\ell-1}^-(s),\ Z_\ell^+(s) = \frac{1}{a} Z_{\ell-1}^+(s), \tag{12}$$

we get

$$T_{\ell-1}(s)T_\ell^{-1}(s) = Q, \quad Q := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

Finally, from (4) and (5), since $T_\ell^{-1}(s) = T_{\ell-1}^{-1}(s)Q$ from (13), we get

$$T_{\ell-1}^{-1} Q A_\ell T_{\ell-1} = \mathrm{diag}(\lambda_1, \lambda_2). \tag{14}$$

On the other hand, from (13), since $T_{\ell-1}(s) = QT_\ell(s)$ also holds, we get

$$T_\ell^{-1} A_\ell Q T_\ell = \mathrm{diag}(\lambda_1, \lambda_2). \tag{15}$$

Wave-based interpretation can be constructed as follows. Suppose the system is subjected to a harmonic excitation and is in the steady state. Notice that $\arg[\lambda_1(j\omega)] = -\arg[\lambda_2(j\omega)]$ from (b). From (5) and (c), $i_\ell^+(t)$ and $i_\ell^-(t)$ can be represented by $i_\ell^+(t) = \alpha(j\omega)\lambda_2(j\omega)^\ell e^{j\omega t}$ and $i_\ell^-(t) = \beta(j\omega)\lambda_1(j\omega)^\ell e^{j\omega t}$, where $\alpha(j\omega)$ and $\beta(j\omega)$ are some complex-valued functions of $s = j\omega$. From the above equations, by tracing the position where the argument of

$i_\ell^+(t)$ and $i_\ell^-(t)$ are constant, $i_\ell^+(t)$ and $i_\ell^-(t)$ can be regarded as the traveling waves towards the opposite direction of ℓ. From (3) and (8), since $i_\ell(t)$ and $v_\ell(t)$ are represented by the sum of these traveling waves, $i_\ell(t)$ and $v_\ell(t)$ can be evaluated by these traveling waves.

To make the above wave-based analysis valid, however, we need some investigation of the secondary constants as analytic functions of s.

III. Analytical Properties of Secondary Constants

From the knowledge of the complex analysis, roots of a polynomial can be regarded as the branches of an algebraic function [11]. In this section, we investigate analytical properties of the secondary constantans as the branches of the algebraic function. In the following, the variable s is omitted for notational simplicity.

A. Propagation constants

From (7), λ_1 and λ_2 can be regarded as the branches of the algebraic function $\boldsymbol{\lambda}$ defined by

$$\boldsymbol{\lambda}^2 - (a + 1 + Z_\ell Y_\ell)\boldsymbol{\lambda} + a = 0. \qquad (16)$$

As the branches of the algebraic function, we can show that λ_1 and λ_2 are analytic in $\mathcal{C}_+$.

Theorem 2: Consider λ_1 and λ_2, the branches of the algebraic function defined by (16). λ_1 and λ_2 are analytic in $\mathcal{C}_+$.

Proof: From the results of the complex analysis, in the polynomial coefficient form, the algebraic functions have at most the algebraic singularities at the zeros of the first coefficient, the zeros of the discriminant and infinity [11].

First, in the polynomial coefficient form, since the zeros of the first coefficient are at most the poles of the positive real functions Z_ℓ and Y_ℓ, we immediately know that the zeros are not in $\mathcal{C}_+$. Next, notice that the discriminant of (16) are given by

$$\begin{aligned} D_{\boldsymbol{\lambda}} &:= (a + 1 + Z_\ell Y_\ell)^2 - 4a \\ &= \left\{(1-\sqrt{a})^2 + Z_\ell Y_\ell\right\}\left\{(1+\sqrt{a})^2 + Z_\ell Y_\ell\right\}. \end{aligned} \qquad (17)$$

From the maximum modulus theorem [11], recall that the analytic function in a region where the function is analytic has its minimum of the real part only on its boundary. Therefore, since $\mathrm{Re}[Z_\ell] >$ and $\mathrm{Re}[Y_\ell] > 0$ in $\mathcal{C}_+$, we get $D_{\boldsymbol{\lambda}} \neq 0$ in $\mathcal{C}_+$. ■

The next lemma is concerned with the loci of λ_1 and λ_2 in $\mathcal{C}_+$.

Lemma 1: Consider λ_1 and λ_2, the branches of the algebraic function defined by (16). Suppose $|\lambda_1(s_0)| > \sqrt{a}$ at some $s_0 \in \mathcal{C}_+$. Then the following inequality holds.

$$0 < |\lambda_2| < \sqrt{a} < |\lambda_1| < \infty, \quad s \in \mathcal{C}_+ \qquad (18)$$

Proof: From (10) and analyticity, the lower and the upper bounds of (18) are obvious. In addition, since $\lambda_1\lambda_2 = a$ from (10), for the proof of the lemma, we only need to show that λ_1 can not across the circle with radius $\sqrt{a}$.

Suppose $|\lambda_1| = \sqrt{a}$ at some $s \in \mathcal{C}_+$. Then, from (10), λ_1 and λ_2 are conjugate roots of (16). From (11), this requires that the coefficient $a+1+Z_\ell Y_\ell$ is real, and thus $Z_\ell Y_\ell$ is real. Noticing that $\mathrm{Re}[Z_\ell] >$ and $\mathrm{Re}[Y_\ell] > 0$ in $\mathcal{C}_+$, we conclude that $Z_\ell Y_\ell \in R_+$. However, from (17), this requires that the discriminant of (16) to be $D_{\boldsymbol{\lambda}} > 0$. This is a contradiction. ■

B. Characteristic Impedances

From (9) and (7), notice that

$$\begin{aligned} aY_\ell {Z_\ell^+}^2 - (-a + 1 + Z_\ell Y_\ell)Z_\ell^+ - Z_\ell &= 0, \\ aY_\ell(-Z_\ell^-)^2 - (-a + 1 + Z_\ell Y_\ell)(-Z_\ell^-) - Z_\ell &= 0. \end{aligned}$$

Observing the same structure of the above equations, we see that Z_ℓ^+ and $-Z_\ell^-$ are also regarded as the branches the algebraic function $\mathbf{Z}_\ell$ defined by

$$aY_\ell {\mathbf{Z}_\ell}^2 - (-a + 1 + Z_\ell Y_\ell)\mathbf{Z}_\ell - Z_\ell = 0. \qquad (19)$$

As the branches of the algebraic function, we can show that Z_ℓ^+ and Z_ℓ^- are analytic in $\mathcal{C}_+$.

Theorem 3: Consider Z_ℓ^+ and $-Z_\ell^-$, the branches of the algebraic function defined by (19). Z_ℓ^+ and Z_ℓ^- are analytic in $\mathcal{C}_+$.

Proof: The proof is similar to the proof of Theorem 2. In the polynomial coefficient form, it is clear that the zeros of the first coefficient of (19) are not in $\mathcal{C}_+$. In addition, the discriminant $D_{\mathbf{Z}}$ is given by

$$\begin{aligned} D_{\mathbf{Z}} &:= (-a + 1 + Z_\ell Y_\ell)^2 + 4aZ_\ell Y_\ell \\ &= (a + 1 + Z_\ell Y_\ell)^2 - 4a = D_{\boldsymbol{\lambda}} \end{aligned} \qquad (20)$$

Therefore, from similar argument in the proof of Theorem 2, the zeros of the discriminant are not in $\mathcal{C}_+$. ■

Next, the loci of Z_ℓ^+ and Z_ℓ^- are investigated. For the purpose, from (6), notice that (19) is equivalent to

$$Y_{\ell+1}{\mathbf{Z}_\ell}^2 - (-a + 1 + Z_{\ell+1}Y_{\ell+1})\mathbf{Z}_\ell - aZ_{\ell+1} = 0. \qquad (21)$$

In addition, if we define

$$\mathbf{Y}_\ell := -\frac{1}{\mathbf{Z}_\ell}, \qquad (22)$$

from (19), $-1/Z_\ell^+$ and $1/Z_\ell^-$ are represented by the branches of the the algebraic function $\mathbf{Y}_\ell$ defined by

$$Z_\ell {\mathbf{Y}_\ell}^2 - (-a + 1 + Z_\ell Y_\ell)\mathbf{Y}_\ell - aY_\ell = 0. \qquad (23)$$

From (21) and (23), we get

$$\mathbf{Z}_\ell = Z_{\ell+1} + \cfrac{1}{Y_{\ell+1} + \cfrac{a}{\mathbf{Z}_\ell}} \qquad (24)$$

$$= \frac{(1 + Z_{\ell+1}Y_{\ell+1})\mathbf{Z}_\ell/a + Z_{\ell+1}}{Y_{\ell+1}\mathbf{Z}_\ell/a + 1}, \qquad (25)$$

$$\mathbf{Y}_\ell = Y_\ell + \cfrac{1}{Z_\ell + \cfrac{a}{\mathbf{Y}_\ell}} \qquad (26)$$

$$= -\frac{-aY_\ell + (1 + Z_\ell Y_\ell)(-\mathbf{Y}_\ell)}{a - Z_\ell(-\mathbf{Y}_\ell)}. \qquad (27)$$

Solutions of the equations (25) and (27) can be evaluated by the fixed-pint iteration [12]. Corresponding to (25) and (27), consider the iterative sequences $\mathbf{Z}_\ell^{(k)}$ and $\mathbf{Y}_\ell^{(k)}$ (with respect to k) defined by

$$\mathbf{Z}_\ell^{(k)}=\frac{(1+Z_{\ell+1}Y_{\ell+1})\mathbf{Z}_\ell^{(k-1)}/a+Z_{\ell+1}}{Y_{\ell+1}\mathbf{Z}_\ell^{(k-1)}/a+1}, \tag{28}$$

$$\mathbf{Y}_\ell^{(k)}=-\frac{-aY_\ell+(1+Z_\ell Y_\ell)(-\mathbf{Y}_\ell^{(k-1)})}{a-Z_\ell(-\mathbf{Y}_\ell^{(k-1)})}. \tag{29}$$

The next lemma relates the limits of these continued fractions with Z_ℓ^+ and Z_ℓ^-.

Lemma 2: Consider $\mathbf{Z}_\ell^{(k)}$ and $\mathbf{Y}_\ell^{(k)}$ defined by (28) and (29), and Z_ℓ^+ and Z_ℓ^- defined by (9). Suppose $|Z_\ell^+|<\infty$ and $|Z_\ell^-|<\infty$. The following equations hold:
(i) $|\lambda_2/\lambda_1|<1: \lim_{k\to\infty}\mathbf{Z}_\ell^{(k)}=Z_\ell^+,\ \lim_{k\to\infty}\mathbf{Y}_\ell^{(k)}=1/Z_\ell^-$.
(ii) $|\lambda_2/\lambda_1|>1: \lim_{k\to\infty}\mathbf{Z}_\ell^{(k)}=-Z_\ell^-,\ \lim_{k\to\infty}\mathbf{Y}_\ell^{(k)}=-1/Z_\ell^+$.

Proof: See Appendix. ■

Owing to this lemma, we see that loci of Z_ℓ^- and Z_ℓ^+ have positive real part in $\mathcal{C}_+$.

Lemma 3: Consider Z_ℓ^+ and $-Z_\ell^-$, the branches of the algebraic function defined by (19). Let λ_1 be the branch of the algebraic function defined by (16) satisfying $|\lambda_1(s_0)|>\sqrt{a}$ at some $s_0\in\mathcal{C}_+$, and Z_ℓ^+ be the corresponding branch satisfying (9). Then, $\mathrm{Re}[Z_\ell^+]\geq 0$ and $\mathrm{Re}[Z_\ell^-]\geq 0$ in $\mathcal{C}_+$.

Proof: From Lemma 1, Lemma 2 and $|Z_\ell^+|<\infty$, $|Z_\ell^-|<\infty$ from the analyticity, notice that $\lim_{k\to\infty}\mathbf{Z}_\ell^{(k)}=Z_\ell^+$, $\lim_{k\to\infty}\mathbf{Y}_\ell^{(k)}=1/Z_\ell^-$. On the other hand, observing (24) and (26), we see that $\mathbf{Z}_\ell^{(k)}$ and $\mathbf{Y}_\ell^{(k)}$ can be recurrently calculated with the calculations: inverse, multiplication with positive real number, and sum with complex number with positive real part. Let $\mathbf{Z}_\ell^{(0)}$ and $\mathbf{Y}_\ell^{(0)}$ be some complex number with positive real part. From the property of the calculations in (24) and (26) described the above, we see that $\mathrm{Re}[\mathbf{Z}_\ell^{(k)}]>0$ and $\mathrm{Re}[\mathbf{Y}_\ell^{(k)}]>0$ in $\mathcal{C}_+$ for all positive integer k. Taking the limit of both sides, we get $\lim_{k\to\infty}\mathrm{Re}[\mathbf{Z}_\ell^{(k)}]=\mathrm{Re}[Z_\ell^+]\geq 0$ and $\lim_{k\to\infty}\mathrm{Re}[\mathbf{Y}_\ell^{(k)}]=\mathrm{Re}[1/Z_\ell^-]\geq 0$. ■

Combining Theorem 3 and Lemma 3, we see that Z_ℓ^- and Z_ℓ^+ are positive real functions.

Theorem 4: Under the assumptions of Lemma 3, Z_ℓ^- and Z_ℓ^+ are positive real functions.

Proof: From the definition of the positive real function [13], we need to show:
(i) Z_ℓ^+ and Z_ℓ^- are analytic in $\mathcal{C}_+$.
(ii) Z_ℓ^+ and Z_ℓ^+ are real for $s\in\mathcal{R}_+$.
(iii) $\mathrm{Re}[Z_\ell^+]\geq 0$ and $\mathrm{Re}[Z_\ell^-]\geq 0$ for $s\in\mathcal{C}_+$.
Owing to Theorem 3 and Lemma 3, the remainder of the proof is to show the condition (ii). Notice that the discriminant of (19) is given by (20). For $s\in\mathcal{R}_+$, we get

$$D_{\mathbf{Z}}=(a+1+Z_\ell Y_\ell)^2-4a>(a-1)^2\geq 0.$$

This proves (ii). ■

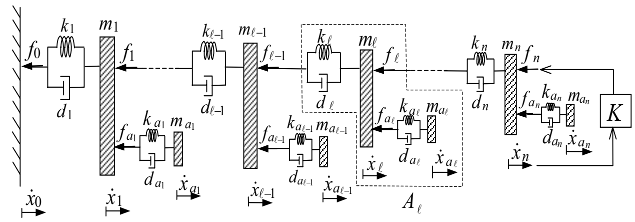

Fig. 2. Damped mass-spring system with auxiliary masses

IV. Impedance Matching

In this section, we investigate properties of the network terminated by the characteristic impedances. This termination is referred to the impedance matching. For the purpose, define $Z_0^-:=\frac{Z_0}{\lambda_1-a}=\frac{aZ_1}{\lambda_1-a}$ and $Z_0^+:=-\frac{Z_0}{\lambda_2-a}=-\frac{aZ_1}{\lambda_2-a}$ by extending the definition (9). The next theorem show that these termination can eliminate one of the traveling waves.

Theorem 5: Consider the ladder network shown in Fig. 1. Under the assumptions of Lemma 3, the following statements hold.
(i) Consider terminating the right end of the network by Z_n^+, i.e., $v_n=Z_n^+i_n$. Then the *right* reflection coefficient $\rho_\ell^R:=i_\ell^-/i_\ell^+$ renders $\rho_\ell^R=0$ in $\mathcal{C}_+$ for all $\ell=0,1,\cdots,n$.
(ii) Consider terminating the left end of the network by Z_0^-, i.e., $v_0=-Z_0^-i_0$. Then the *left* reflection coefficient $\rho_\ell^L:=i_\ell^+/i_\ell^-$ renders $\rho_\ell^L=0$ in $\mathcal{C}_+$ for all $\ell=0,1,\cdots,n$.

Proof: Notice that $i_\ell^-=(v_\ell-Z_\ell^+i_\ell)/(Z_\ell^++Z_\ell^-)$ and $i_\ell^+=(v_\ell+Z_\ell^-i_\ell)/(Z_\ell^++Z_\ell^-)$ from (3) and (8), and that $i_\ell^-=\lambda_1 i_{\ell-1}^-$ and $i_\ell^+=\lambda_2 i_{\ell-1}^+$ from (5) and (c). Notice also that $0<|\lambda_1|<\infty$, $0<|\lambda_2|<\infty$ in $\mathcal{C}_+$ from Lemma 1, and that $Z_\ell^+\neq -Z_\ell^-$ in $\mathcal{C}_+$ since $\mathbf{Z}_\ell$ has no branch point in $\mathcal{C}_+$ from Theorem 3.

For (i), from $v_n=Z_n^+i_n$, we get $i_n^-=0$ and $i_n^+\neq 0$. Moreover, since $i_{\ell-1}^-=1/\lambda_1 i_\ell^-$ and $i_{\ell-1}^+=1/\lambda_2 i_\ell^+$, we get $i_\ell^-=0$ and $i_\ell^+\neq 0$ for all $\ell=0,1,\cdots,n$. Therefore, $\rho_\ell^R=0$ in $\mathcal{C}_+$ for all $\ell=1,2,\cdots,n$. On the other hand, for (ii), from $v_0=-Z_0^-i_0$, we get $i_0^+=0$ and $i_0^-\neq 0$. Since $i_\ell^+=\lambda_2 i_{\ell-1}^+$ and $i_\ell^-=\lambda_1 i_{\ell-1}^-$, we get $i_\ell^+=0$ and $i_\ell^-\neq 0$ for all $\ell=0,1,\cdots,n$. Therefore, $\rho_\ell^L=0$ in $\mathcal{C}_+$ for $\ell=0,1,\cdots,n$. ■

Remark 2: Notice that the termination considered in Theorem 5 is realized by the positive real functions Z_ℓ^+ and Z_ℓ^-. As described in [9], [10] for the cascade connected damped mass-spring systems case, by defining the inputs and the outputs of the network and regarding the terminations as a controller, we may construct the condition for the internal stability based on the Nyquist criterion. On the other hand, although the termination of the right end by $-Z_n^-$ or the termination of the left end by $-Z_0^+$ realizes $\rho_\ell^L=0$ or $\rho_\ell^R=0$, these terminations may violate the stability of the network.

V. Numerical Example

In this section, we consider vibration control of a mechanical system shown in Fig. 2 by the impedance match-

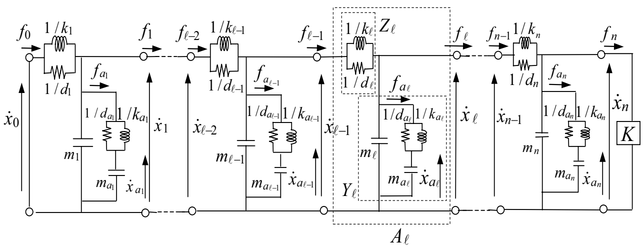

Fig. 3. Electric circuit analogy

ing. The system is composed of cascade connected damped mass-spring systems with auxiliary damped mass-spring systems installed on each mass. Mass, damping coefficient and spring constant of the ℓ-th stage are represented by m_ℓ[Kg], d_ℓ[Ns/m] and k_ℓ[N/m], respectively, and the auxiliary mass, damping coefficient and spring coefficient are represented by m_{a_ℓ}[Kg], d_{a_ℓ}[Ns/m] and k_{a_ℓ}[N/m]. $\dot{x}_\ell$[m/s] and $\dot{x}_{a_\ell}$[m/s] are the velocity of the masses m_ℓ and m_{a_ℓ}, and f_ℓ[N] and f_{a_ℓ} [N] are the reaction force produced by the corresponding springs and dampers. The left end is fixed on the ground ($\dot{x}_n = 0$) and a control force f_n is acting on the right end. With the measurement $\dot{x}_n$, the control force is given by $f_n = K(s)\dot{x}_n$.

An electric circuit analogy for the mechanical system by using mobility analogy (velocity-voltage and force-current correspondence) is shown in Fig. 3. Circuit elements of the ℓ-th network is composed of the inductor $1/k_\ell$[H], resistor $1/d_\ell[\Omega]$ and capacitance m_ℓ[F] with parallel connection of the auxiliary inductor $1/k_{a_\ell}$[H], resistance $1/d_{a_\ell}[\Omega]$ and capacitance m_{a_ℓ}[F]. $\dot{x}_\ell$[V] and $\dot{x}_{a_\ell}$ [V] are the voltage of the capacitances m_ℓ and m_{a_ℓ}, and f_ℓ[A] and f_{a_ℓ}[A] are the current through the corresponding inductors and resisters.

The circuit can be regarded as the ladder network shown in Fig. 1, and the corresponding impedance and admittance are given by

$$Z_\ell = \frac{s}{d_\ell s + k_\ell}, \quad Y_\ell = m_\ell s + \frac{1}{\dfrac{s}{d_{a_\ell} s + k_{a_\ell}} + \dfrac{1}{m_{a_\ell} s}}$$

(See Fig. 3). In this case, the uniformly varying condition (6) is reduced to $m_\ell/m_{\ell-1} = d_\ell/d_{\ell-1} = k_\ell/k_{\ell-1} = a$ and $m_{a_\ell}/m_{a_{\ell-1}} = d_{a_\ell}/d_{a_{\ell-1}} = k_{a_\ell}/k_{a_{\ell-1}} = a$. From Theorem 5, notice that the impedance matching at the right end is achieved by $f_n = 1/Z_n^+ \dot{x}_n$.

In the numerical example, we consider the case, $n = 20$, $m_1 = 1$, $d_1 = 2 \times 0.001$, $k_1 = 1$, $m_{a_1} = 0.1$, $d_{a_1} = 2 \times 0.005$, $k_{a_1} = 0.1$ with $a = 0.9$. Frequency response of the characteristic impedance Z_n^+ is evaluated by the solution of (19) with positive real part, and can be approximated by a rational positive real function through the complex curve fitting. Fig. 4 shows the Bode plots of the impedance matching controller $K_n^+ := 1/Z_n^+$ (dashed line) and an approximation $\widehat{K}_n^+(s)$ (solid line) with degree 12. K_n^+ and $\widehat{K}_n^+$ coincide in this figure.

Fig. 5 shows the gain plots of the transfer functions from

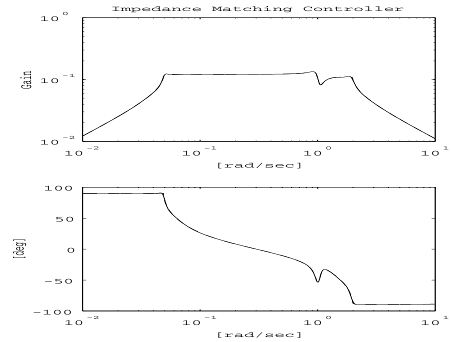

Fig. 4. K_n^+ (dashed) and $\widehat{K}_n^+$ (solid)

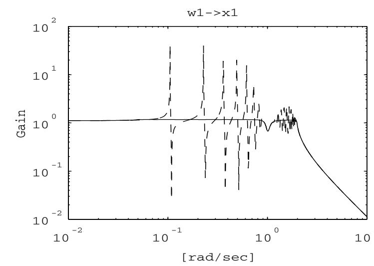

Fig. 5. Open (dashed) and closed loop (solid)

the disturbance at m_1 to x_1 (the displacement of m_1). The dashed line represents the gain without control and the solid line represents the gain with control by $\widehat{K}_n^+$. From the figure, the impedance matching controller well suppresses the peak of the vibration modes.

Fig. 6 shows the impulse response disturbed at m_1. Solid lines represent the response with control and dashed lines represent the response without control. x_1, x_{10}, x_{20} and $x_{a_{20}}$ are the displacement of the corresponding masses, and f_n is the control force. From the response with control, reflection is suppressed at position 20 where controller exists and vibration vanishes rapidly compare to the response without control. Vibration of the auxiliary mass is also suppressed. To evaluate the robustness of the controller against parameter variations, Fig. 7 shows the impulse response for the system whose physical parameters vary at random within 25[%] from its nominal values. From the figure, the impedance matching controller is still effective even for the system with relatively large perturbations. Although the class of the system considered in this paper is still somewhat restricted, this motivates to apply our impedance matching controller to some perturbed systems.

VI. Conclusion

In this paper, we extended our previous results for the damped mass-spring systems to the ladder electric networks, whose circuit elements are linear, time invariant, finite and passive. We first clarified a class of the network that satisfies the three conditions for the propagation constants. For the class of the ladder networks, we next investigated analyticity of the secondary constants and positive real property of

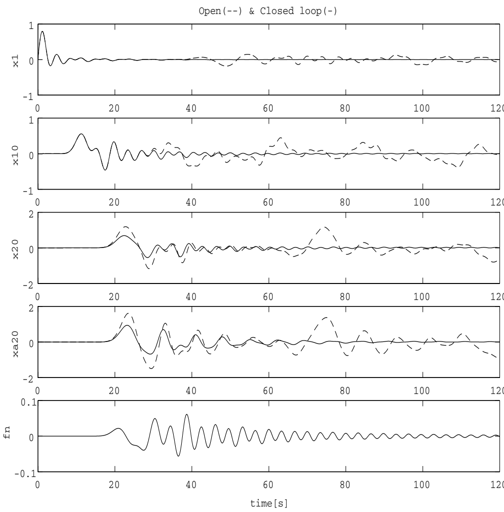

Fig. 6. Impulse response with control (solid) and without control (dashed)

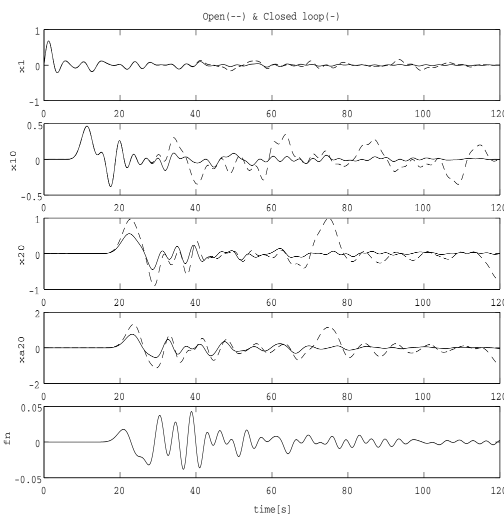

Fig. 7. Impulse response with $\pm 25[\%]$ parameter variation

the characteristic impedances. Properties of the impedance matching were also investigated. A numerical example for a mechanical system showed effectiveness of the impedance matching for vibration control.

One of our future researches is a straightforward extension of the results to other more general networks. Investigation of the possibility to relax the conditions (a)-(c), and the robust analysis and design of the wave control are also interesting.

REFERENCES

[1] M. J. Balas, "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications*, vol. 25, no. 3, pp. 415–436, 1978.

[2] M. J. Balas, "Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on Automatic Control*, vol. AC-27, no. 3, pp. 522–535, 1982.

[3] L. Meirovitch, H. Baruh and H. Öz, "A Comparison of Control Techniques for Large Flexible Systems," *Journal of Guidance*, vol. 6, no. 4, pp. 302–310, 1983.

[4] D. R. Vaughan, "Application of Distributed Parameter Concepts to Dynamic Analysis and Control of Bending Vibrations," *Journal of Basic Engineering*, June 1968, 157–166, 1968.

[5] A. H. von Flotow, "Traveling Wave Control for Large Spacecraft Structures," *Journal of Guidance, Control, and Dynamics*, vol. 9, no. 4, pp. 462–468, 1986.

[6] D. G. MacMartin and S. R. Hall, "Control of Uncertain Structures Using an H_∞ Power Flow Approach," *Journal of Guidance*, vol. 14, no. 3, pp. 521–530, 1991.

[7] H. Ojima, K. Nagase and Y. Hayakawa, "Wave-based Analysis and Wave Control of Damped Mass-Spring Systems —Characterization from the propagation constants—," *Transactions of the Society of Instrument and Control Engineers*, vol. 38, no. 3, pp. 239–246, 2002.

[8] H. Ojima, K. Nagase and Y. Hayakawa, "Wave-based Analysis and Wave Control of Damped Mass-Spring Systems," *Proc. of the 40th IEEE Conf. on Decision and Control*, pp. 2574–2579, 2001.

[9] H. Ojima, K. Nagase and Y. Hayakawa, "On the Analysis of the Impedance Matching Controller for Uniformly Varying Damped Mass-Spring Systems," *Transactions of the Society of Instrument and Control Engineers*, vol. 40, no.3, pp. 315–321, 2002.

[10] H. Ojima, K. Nagase and Y. Hayakawa, "A Study on the Impedance Matching Controller for the Uniformly Varying Damped Mass-Spring Systems," *Proc. of 2003 American Control Conference*, pp. 2821–2826, 2003.

[11] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill Book Company, 1979.

[12] E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, Inc., 925–936, 1993.

[13] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*, Prentice Hall, 1973.

APPENDIX

Proof of Lemma 2: From (2) and (13), notice that (28) and (29) are equivalent to

$$\mathbf{Z}_\ell^{(k)} = \frac{\xi_1^{(k)}}{\xi_2^{(k)}} \quad \text{where} \quad \begin{bmatrix} \xi_1^{(k)} \\ \xi_2^{(k)} \end{bmatrix} = A_{\ell+1}^{-1} Q^{-1} \begin{bmatrix} \xi_1^{(k-1)} \\ \xi_2^{(k-1)} \end{bmatrix}, \quad (30)$$

and

$$\mathbf{Y}_\ell^{(k)} = -\frac{\zeta_2^{(k)}}{\zeta_1^{(k)}} \quad \text{where} \quad \begin{bmatrix} \zeta_1^{(k)} \\ \zeta_2^{(k)} \end{bmatrix} = A_\ell Q \begin{bmatrix} \zeta_1^{(k-1)} \\ \zeta_2^{(k-1)} \end{bmatrix}. \quad (31)$$

From (14) and (15), since (30) and (31) are transformed to

$$\begin{aligned}
\begin{bmatrix} \xi_1^{(k)} \\ \xi_2^{(k)} \end{bmatrix} &= (QA_{\ell+1})^{-k} \begin{bmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{bmatrix} = \left(T_\ell \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T_\ell^{-1} \right)^{-k} \begin{bmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{bmatrix} \\
&= T_\ell \begin{bmatrix} \frac{1}{\lambda_1^k} & 0 \\ 0 & \frac{1}{\lambda_2^k} \end{bmatrix} T_\ell^{-1} \begin{bmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{bmatrix}, \\
\begin{bmatrix} \zeta_1^{(k)} \\ \zeta_2^{(k)} \end{bmatrix} &= (A_\ell Q)^k \begin{bmatrix} \zeta_1^{(0)} \\ \zeta_2^{(0)} \end{bmatrix} = T_\ell \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} T_\ell^{-1} \begin{bmatrix} \zeta_1^{(0)} \\ \zeta_2^{(0)} \end{bmatrix},
\end{aligned}$$

we get

$$\begin{aligned}
\mathbf{Z}_\ell^{(k)} &= \frac{(Z_\ell^+ \lambda_1^k + Z_\ell^- \lambda_2^k)\xi_1^{(0)} + Z_\ell^- Z_\ell^+ (\lambda_1^k - \lambda_2^k)\xi_2^{(0)}}{(\lambda_1^k - \lambda_2^k)\xi_1^{(0)} + (Z_\ell^- \lambda_1^k + Z_\ell^+ \lambda_2^k)\xi_2^{(0)}}, \\
\mathbf{Y}_\ell^{(k)} &= -\frac{(\lambda_2^k - \lambda_1^k)\zeta_1^{(0)} + (Z_\ell^- \lambda_2^k + Z_\ell^+ \lambda_1^k)\zeta_2^{(0)}}{(Z_\ell^+ \lambda_2^k + Z_\ell^- \lambda_1^k)\zeta_1^{(0)} + Z_\ell^- Z_\ell^+ (\lambda_2^k - \lambda_1^k)\zeta_2^{(0)}}.
\end{aligned}$$

From the above equations, with the assumptions $|Z_\ell^+| < \infty$, $|Z_\ell^-| < \infty$ and $\lim_{k\to\infty} |\lambda_2/\lambda_1|^k \to 0$ for (i) or $\lim_{k\to\infty} |\lambda_1/\lambda_2|^k \to 0$ for (ii), we get (i) and (ii). ■