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# On a proof of the general version of the spectral theorem in max-plus algebra

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*Abstract*— **The so-called spectral theorem for square irreducible matrices is well-known in the max-plus community. The theorem is a fundamental result concerning matrix powers and eigenvalues in the context of max-plus algebra and forms the basis for many results.**

**This paper aims at giving a complete proof of the above spectral property in its full generality. In particular, the distinction will be highlighted between the graph cyclicity of a matrix (the cyclicity of the graph of the matrix) and the cyclicity of the matrix itself.**

## I. INTRODUCTION

The so-called spectral theorem for square irreducible matrices is well-known in the max-plus community. The theorem is a fundamental result concerning matrix powers and eigenvalues in the context of max-plus algebra and forms the basis for many results. And although the property is generally accepted to be true, actual proofs of the property in its full generality are difficult to find. Most of the proofs available in literature, c.f. [1], focus on special cases and state that the general case can obtained by a similar reasoning, or refer to publications that are difficult to track. Also the difference between the cyclicity of a matrix and the cyclicity of its graph is not always treated in a transparent way.

The present paper is based on Chapter 3 in [6]. The paper aims at giving a complete proof of the above spectral property in its full generality. In particular, the distinction will be highlighted between the graph cyclicity of matrix $A$ (the cyclicity of the graph of matrix $A$) and the cyclicity of matrix $A$ itself.

In addition to the above references, more information on max-plus algebra, the spectral theorem, its (partial) proof(s) and the two notions of cyclicity of a max-plus matrix can be found in [3], [4] and [5].

## II. MAX-PLUS ALGEBRA

### A. Basic definitions

To introduce the max-plus algebra, define $\mathbb{R}_{\max} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ denotes the set of real numbers. For $a,b \in \mathbb{R}_{\max}$, define $a \oplus b \stackrel{\text{def}}{=} \max\{a,b\}$ and $a \otimes b \stackrel{\text{def}}{=} a+b$. Introduce $\varepsilon \stackrel{\text{def}}{=} -\infty$ and $e \stackrel{\text{def}}{=} 0$, then $\varepsilon$ is the neutral element with respect to $\oplus$ and $e$ is the unit element with respect $\otimes$, i.e., $a \oplus \varepsilon = \varepsilon \oplus a = \varepsilon$ and $a \otimes e = e \otimes a = a$, for all $a \in \mathbb{R}_{\max}$. The quintuple $(\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e)$ forms the max-plus algebra.

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### B. Vectors and matrices

The natural numbers are denoted by $\mathbb{N} = \{0,1,2,\dots\}$, and for any integer $m \geq 1$, define $\underline{m} \stackrel{\text{def}}{=} \{1,\dots,m\}$. Let $\mathbb{R}_{\max}^{k\times l}$ be the set of $k \times l$ matrices with entries in $\mathbb{R}_{\max}$. If $A \in \mathbb{R}_{\max}^{k\times l}$, then $a_{ij}$ denotes the element of $A$ in row $i \in \underline{k}$ and column $j \in \underline{l}$. The element $a_{ij}$ will also be referred to as the $(i,j)$-th entry of $A$. Likewise for matrix $B$ with entries $b_{ij}$, matrix $C$ with entries $c_{ij}$, etc. Given matrices $A,B \in \mathbb{R}_{\max}^{k\times l}$, then their sum $A \oplus B$, when denoted by $C$, is defined by $c_{ij} \stackrel{\text{def}}{=} a_{ij} \oplus b_{ij}$, for all $j \in \underline{k}$ and $j \in \underline{l}$. Given matrices $A \in \mathbb{R}_{\max}^{k\times m}$ and $B \in \mathbb{R}_{\max}^{m\times l}$, then their product $A \otimes B$, when denoted by $D$, is defined by $d_{ij} \stackrel{\text{def}}{=} \bigoplus_{s\in\underline{m}} a_{is} \otimes b_{sj}$, for all $i \in \underline{k}$ and $j \in \underline{l}$. If $A \in \mathbb{R}_{\max}^{n\times n}$ and $k$ is a non-negative integer, then $A^{\otimes k} \stackrel{\text{def}}{=} A \otimes A \otimes \cdots \otimes A$ ($k$ times). Finally, if $A \in \mathbb{R}_{\max}^{n\times n}$, then $A^{+} \stackrel{\text{def}}{=} \bigoplus_{s\geq 1} A^{\otimes s}$.

The real number $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}_{\max}^{n\times n}$ if there is a vector $v \in \mathbb{R}^n$, with real components $v_i$, $i \in \underline{n}$, such that $A \otimes v = \lambda \otimes v$, where the $i$-th component of $A \otimes v$ equals $\bigoplus_{s\in\underline{n}} a_{is} \otimes v_s$ and the $i$-th component of $\lambda \otimes v$ is given by $\lambda \otimes v_i$.

The matrix $A_\lambda$ denotes the matrix obtained from $A$ by subtracting $\lambda$ from all (finite) entries of $A$. Clearly, $\lambda$ is an eigenvalue of $A \in \mathbb{R}_{\max}^{n\times n}$ if and only if $e$ is an eigenvalue of $A_\lambda \in \mathbb{R}_{\max}^{n\times n}$, where for both the same eigenvector can be used!

## III. CIRCUITS AND GRAPH CYCLICITY

### A. Max-plus and graphs

Let $\mathscr{G}(A)$ represent the graph of matrix $A \in \mathbb{R}_{\max}^{n\times n}$. This graph consists of a node set $\mathscr{N}(A) = \{1,\dots,n\}$ and a directed arc set $\mathscr{D}(A) = \{(j,i) \in \underline{n} \times \underline{n} | a_{ij} \neq \varepsilon\}$. In the following $\mathscr{N}(A)$ and $\underline{n}$ will be used interchangeably to denote the set $\{1,\dots,n\}$. If $(j,i)$ is an arc of $\mathscr{G}(A)$, then it starts in node $j$, ends in node $i$ and its weight is $a_{ij}$ (no typo!). In $\mathscr{G}(A)$ the notions of arcs, paths, circuits and their concatenations are defined as in the usual way. The same holds for the length and weight of paths and circuits. A path is called elementary if every node of the path is traversed only once. A circuit is said to be *elementary* if in the succession of nodes each of its nodes occurs only once. Then the circuit has no self intersections. It can be shown constructively that every circuit can be seen as the concatenation of elementary circuits.

If $A,B \in \mathbb{R}_{\max}^{n\times n}$, then matrices $A$ and $B$ are said to be similar, denoted as $A \sim B$, if $\mathscr{G}(A)$ can be obtained from $\mathscr{G}(B)$ by a mere relabelling of its nodes. Hence, matrices $A,B \in \mathbb{R}_{\max}^{n\times n}$ are similar, if $B$ can be obtained from $A$ by a permutation applied to the rows and the columns of $A$ simultaneously.

In this paper the $(i,j)$-th entry of $A^{\otimes k}$ and $A^{+}$ will be denoted by $[A^{\otimes k}]_{ij}$ and $[A^{+}]_{ij}$, respectively, for $i,j \in \underline{n}$. It

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is easy to prove, see [1] or [6], that $[A^{\otimes k}]_{ij}$ is equal to the maximal weight of a path in $\mathscr{G}(A)$ of length $k$ from node $j$ to node $i$. Similarly, $[A^{+}]_{ij}$ is equal to the maximal weight of a path in $\mathscr{G}(A)$ of any positive length from node $j$ to node $i$. From these interpretations it follows that

$$[A^{\otimes(k+l)}]_{ij} \geq [A^{\otimes k}]_{is} \otimes [A^{\otimes l}]_{sj},$$

for all integers $k,l \geq 1$ and all $i,j,s \in \underline{n}$. Further, it follows that $[A^{\otimes k}]_{ij} \leq [A^{+}]_{ij}$, for all $i,j \in \underline{n}$, implying that

$$[A^{\otimes k}]_{ij} \leq \bigoplus_{s\in\underline{n}} [A^{+}]_{is} \otimes [A^{+}]_{sj}, \qquad (1)$$

for all integers $k \geq 1$, and $i,j \in \underline{n}$.

### B. Connectedness and irreducibility

The graph $\mathscr{G}(A)$ is called strongly connected if every two nodes in $\mathscr{N}(A)$ are connected to each other by means of a path. If $\mathscr{G}(A)$ is not strongly connected, there exist 'largest' subsets of nodes that are connected to each other by means of a path. The associated subgraphs are referred to as maximally strongly connected subgraphs (m.s.c.s.).

The matrix $A$ is called *irreducible* if the graph $\mathscr{G}(A)$ is strongly connected. The matrix $A$ is called reducible if it is not irreducible, in which case the graph $\mathscr{G}(A)$ is not strongly connected.

If the graph $\mathscr{G}(A)$ contains no circuits at all, then the matrix $A$ is called *acyclic*. Clearly, if $A$ is not acyclic, then $\mathscr{G}(A)$ contains at least one circuit. Obviously, any irreducible matrix is not acyclic.

### C. Graph cyclicity

If $A \in \mathbb{R}_{\max}^{n\times n}$ is *irreducible*, and consequently not acyclic, the *graph cyclicity* of $A$ is defined to be equal to the g.c.d. of the lengths of all circuits in $\mathscr{G}(A)$, where g.c.d. stands for 'greatest common divisor'. Since every circuit is the concatenation of elementary circuits, it follows that the graph cyclicity of an irreducible $A$ is also given by the g.c.d. of the lengths of all *elementary* circuits in $\mathscr{G}(A)$. (To justify this alternative definition, the next observation is useful: If $a_i$, $i \in \underline{q}$, are $q$ positive integers, and $\beta_i \in \mathbb{N}$, $i \in \underline{q}$, are natural numbers of which at least one is positive, then the g.c.d. of $\{a_1,\dots,a_q\}$ is equal to the g.c.d. of $\{a_1,\dots,a_q,\sum_{i\in\underline{q}}\beta_i a_i\}$. In the present context, the $a_i$, $i \in \underline{q}$, can be seen as lengths of elementary circuits, whereas $\sum_{i\in\underline{q}}\beta_i a_i$ represents the length of a non-elementary circuit.)

If $A$ is *not irreducible*, but at least not acyclic, the *graph cyclicity* of $A$ is defined to be equal to the l.c.m. of the graph cyclicities of all m.s.c.s.'s in $\mathscr{G}(A)$, where l.c.m. stands for 'least common multiple'.

The graph cyclicity of a matrix $A$ that is not acyclic is denoted by $\sigma_{\mathscr{G}}(A)$. For acyclic matrices the graph cyclicity has no meaning and therefore is not defined!

## IV. CRITICAL CIRCUITS AND MATRIX CYCLICITY

### A. Critical circuits and critical graphs

If $A \in \mathbb{R}_{\max}^{n\times n}$ is irreducible then it has a unique eigenvalue, see [1], [6]. In terms of the associated graph $\mathscr{G}(A)$, the eigenvalue of an irreducible matrix $A$ is equal to the maximal circuit mean in $\mathscr{G}(A)$, where circuit mean is defined as circuit weight divided by circuit length.

Assume that $A$ is not acyclic. Circuits in $\mathscr{G}(A)$ of which the mean is maximal are called critical circuits. The critical graph $\mathscr{G}(A)$ of such a matrix $A$, denoted by $\mathscr{G}^{c}(A)$, is given by the node set $\underline{n}$ (also denoted by $\mathscr{N}(A)$) and the set of arcs that are contained in any of the critical circuits. Note that the node set of the graph and the critical graph is the same. This is merely done to make that matrix $A$ and the matrix that can be associated with the critical graph have the same dimensions. The latter matrix is called critical matrix. More specifically, if $A$ is not acyclic, the critical matrix of $A$, denoted by $A^{c}$, is the restriction of $A$ to those entries that correspond to arcs contained in any of the critical circuits, while all other entries of $A^{c}$ have value $\varepsilon$. Clearly, $A^{c} \in \mathbb{R}_{\max}^{n\times n}$ and the graph of $A^{c}$ is equal to the critical graph of $A$, i.e., $\mathscr{G}^{c}(A) = \mathscr{G}(A^{c})$. It can be shown easily, see [1] or [6], that any circuit in $\mathscr{G}(A^{c})$ is critical, so that $\mathscr{G}^{c}(A^{c}) = \mathscr{G}(A^{c})$. Hence, $\mathscr{G}^{c}(A) = \mathscr{G}(A^{c}) = \mathscr{G}^{c}(A^{c})$ for any matrix $A$ which is not acyclic.

### B. Matrix cyclicity

The matrix cyclicity of a matrix $A$ that is not acyclic is defined to be equal to the graph cyclicity of the associated critical matrix $A^{c}$. The matrix cyclicity of a 'non-acyclic' matrix $A$ is denoted by $\sigma_{\mathscr{M}}(A)$. Hence, $\sigma_{\mathscr{M}}(A) = \sigma_{\mathscr{G}}(A^{c})$.

If $A = A^{c}$, then obviously $\mathscr{G}(A) = \mathscr{G}(A^{c}) = \mathscr{G}^{c}(A) = \mathscr{G}^{c}(A^{c})$, implying $\sigma_{\mathscr{G}}(A) = \sigma_{\mathscr{G}}(A^{c}) = \sigma_{\mathscr{M}}(A) = \sigma_{\mathscr{M}}(A^{c})$. Hence, for matrices that themselves are critical, the graph and matrix cyclicity coincide!

If $A \neq A^{c}$, the graph and matrix cyclicity may be different:

**Example 4.1:** *Consider*

$$A = \begin{pmatrix} e & 1 \\ 1 & \varepsilon \end{pmatrix} \text{ with } A^{c} = \begin{pmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{pmatrix}.$$

*The graph $\mathscr{G}(A)$ of matrix $A$ consists of two circuits. One circuit, indicated by $1 \to 1$, has length one and mean $e$. The other circuit, indicated by $1 \to 2 \to 1$, has length two and mean one. The critical graph $\mathscr{G}^{c}(A)$ of matrix $A$ consists of one circuit, indicated by $1 \to 2 \to 1$, with length two. It follows that the graph cyclicity of $A$ is one, whereas the matrix cyclicity of $A$ is two.*

Above it is stated that $\mathscr{G}^{c}(A) = \mathscr{G}(A^{c}) = \mathscr{G}^{c}(A^{c})$ for any matrix $A$ which is not acyclic. This implies that $\sigma_{\mathscr{M}}(A) = \sigma_{\mathscr{G}}(A^{c}) = \sigma_{\mathscr{M}}(A^{c})$ for any matrix $A$ which is not acyclic. Hence, the matrix cyclicity of a matrix and its critical matrix coincide, whereas the graph cyclicity of a matrix and its critical matrix are generally different.

### C. Some properties of circuits and critical circuits

1) Let a circuit of length $l$ be given. Choose a node on the circuit and traverse the circuit with steps of length $k$, possibly going around the circuit more than once. Then after a certain number of these steps of length $k$ the chosen node is reached again. If the number of

steps of length $k$ to reach the chosen node is denoted by $\gamma$, then the product $k\gamma$ equals the l.c.m. of $\{k,l\}$.

2) Let $\xi$ be a path contained in a critical circuit $\beta$ of $\mathscr{G}(A)$. Assume that $\xi$ is a path of length $k$ from node $j$ to node $i$. Then the weight of $\xi$ must be equal to the maximal weight of any path of length $k$ from node $j$ to node $i$., i.e., the weight of $\xi$ must be equal to $[A^{\otimes k}]_{ij}$. Indeed, if the previous is not the case, then $\xi$ can be replaced by another path of length $k$ from node $j$ to node $i$, with a weight larger than the weight of $\xi$. The new circuit obtained in this way has a circuit mean larger than the circuit mean of $\beta$. However, this is contradicted by the fact that $\beta$ is critical and, consequently, has maximal circuit mean.

### D. Critical graphs and powers of matrices

Recall that $\sigma_{\mathscr{M}}(A) = \sigma_{\mathscr{G}}(A^{c}) = \sigma_{\mathscr{M}}(A^{c})$ for any matrix $A$ that is not acyclic. In this subsection this observation will be generalized. For that purpose, denote the maximal circuit mean in the graph of a matrix $A$ that is not acyclic by $\mu(A)$. Clearly, if $A$ is irreducible, then $\mu(A)$ equals the eigenvalue of $A$. Now the following generalization can be stated.

**Lemma 4.2:** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be not acyclic with critical matrix $A^{c}$. Then for all integers $k \geq 1$:*

(1) $\mathscr{G}^{c}((A^{c})^{\otimes k}) = \mathscr{G}^{c}(A^{\otimes k})$,

(2) $\mathscr{G}^{c}((A^{c})^{\otimes k}) = \mathscr{G}((A^{c})^{\otimes k})$,

*so that $\sigma_{\mathscr{M}}(A^{\otimes k}) = \sigma_{\mathscr{G}}((A^{c})^{\otimes k}) = \sigma_{\mathscr{M}}((A^{c})^{\otimes k})$, for all integers $k \geq 1$.*

*Proof:* Note that if $A$ is not acyclic, then $A^{c}$ is not acyclic, and also $A^{\otimes k}$ and $(A^{c})^{\otimes k}$ are not acyclic, for any integer $k \geq 1$. Indeed, if $\mathscr{G}(A)$ contains at least one circuit, it also contains a critical circuit, which must be contained in $\mathscr{G}(A^{c})$. Hence, $A^{c}$ is not acyclic. Next consider a critical circuit in $\mathscr{G}(A)$, choose an arbitrary node on the circuit and traverse the circuit with steps of length $k$ until the chosen node is reached again. If $l$ such steps of length $k$ are needed, then there exists a circuit in $\mathscr{G}(A^{\otimes k})$ of length $l$. Hence, if $A$ is not acyclic, then $A^{\otimes k}$ is not acyclic. In a similar way it follows that then also $(A^{c})^{\otimes k}$ is not acyclic.

Note that any arc in $\mathscr{G}(A^{\otimes k})$ from node $j$ to node $i$ corresponds to at least one path in $\mathscr{G}(A)$ of length $k$ from node $j$ to node $i$ with maximal weight $[A^{\otimes k}]_{ij}$. Then, if a number of arcs in $\mathscr{G}(A^{\otimes k})$ form a critical circuit, say of length $l$, the corresponding paths in $\mathscr{G}(A)$ form a circuit of length $kl$. Since the weight of the path is not altered by the change of interpretation, it follows that $\mu(A^{\otimes k}) \leq \mu(A)$. Conversely, consider a critical circuit in $\mathscr{G}(A)$. Next, choose an arbitrary node on the circuit and traverse the circuit with steps of length $k$ until the chosen node is reached again. Because the circuit is critical the weight each step of length $k$, say from node $j$ to node $i$, must be equal to the maximal weight of any path of length $k$ from node $j$ to node $i$ (see also the previous subsection). If $l$ steps of length $k$ are needed, then there exists a circuit in $\mathscr{G}(A^{\otimes k})$ of length $l$. It now follows that $\mu(A) \leq \mu(A^{\otimes k})$, so that $\mu(A^{\otimes k}) = \mu(A)$.

The above implies that any critical circuit in $\mathscr{G}(A^{\otimes k})$ of length $l$ can be 'expanded' to at least one critical circuit in $\mathscr{G}(A)$ of length $kl$ and, conversely, any critical circuit in $\mathscr{G}(A)$ of length $kl$ can be 'contracted' to a critical circuit in $\mathscr{G}(A^{\otimes k})$ of length $l$. More explicitly, if $\alpha$ is a critical circuit in $\mathscr{G}(A^{\otimes k})$ of length $l$, there is at least one critical circuit $\alpha'$ in $\mathscr{G}(A)$ of length $kl$, that can be contracted to $\alpha$. Because any critical circuit in $\mathscr{G}(A)$ is also a critical circuit in $\mathscr{G}^{c}(A)$, it follows that $\alpha'$ is a critical circuit in $\mathscr{G}(A^{c})$ of length $kl$. The latter implies that $\alpha$ is a critical circuit in $\mathscr{G}((A^{c})^{\otimes k})$ of length $l$. Hence, any critical circuit in $\mathscr{G}(A^{\otimes k})$ is a critical circuit of $\mathscr{G}((A^{c})^{\otimes k})$. By the same reasoning it follows that any critical circuit in $\mathscr{G}((A^{c})^{\otimes k})$ is also a critical circuit of $\mathscr{G}(A^{\otimes k})$. Since critical graphs are completely determined by their circuits, this concludes part (1) of the lemma.

Part (2) can be proved in a similar way as part (1).

Hence, $\mathscr{G}^{c}(A^{\otimes k}) = \mathscr{G}((A^{c})^{\otimes k}) = \mathscr{G}^{c}((A^{c})^{\otimes k})$, implying that $\sigma_{\mathscr{M}}(A^{\otimes k}) = \sigma_{\mathscr{G}}((A^{c})^{\otimes k}) = \sigma_{\mathscr{M}}((A^{c})^{\otimes k})$, for all $k \geq 1$. ∎

## V. FUNDAMENTAL SPECTRAL THEOREM

With the concepts introduced above the celebrated spectral theorem for irreducible max-plus matrices can be stated now.

### A. General formulation of spectral theorem

**Theorem 5.1:** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix with eigenvalue $\lambda$ and matrix cyclicity $\sigma$, i.e., $\sigma_{\mathscr{M}}(A) = \sigma$. Then there is an integer $N \in \mathbb{N}$ such that*

$$A^{\otimes(k+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes k},$$

*for all integers $k \geq N$.*

The purpose of this paper is to present a complete proof of this fundamental result and to emphasize the distinction between the graph and matrix cyclicity of matrix $A$. To that end, in this section, first a special case of Theorem 5.1 will be treated. The special case deals with the situation that the eigenvalue of $A$ equals $e$ and the matrix cyclicity of $A$ is one.

In Section VI the *graph* cyclicity is studied and it is shown that if the graph cyclicity of an irreducible matrix $A$ is $\sigma$, then the graph cyclicity of $A^{\otimes\sigma}$ is one. In the same spirit, in Section VII the *matrix* cyclicity is studied and it is shown that if the matrix cyclicity of an irreducible matrix $A$ is $\sigma$, then the matrix cyclicity of $A^{\otimes\sigma}$ is one. It is furthermore shown that in that case the matrix cyclicity corresponding to each associated m.c.s.c. is one too. Then for each m.s.c.s. the special case formulated below can be applied and the proof of Theorem 5.1 can be completed. This is done in Section VIII.

The proof of the special case is inspired by [5]. In the proof the next lemma, see [2], plays an important role.

**Lemma 5.2:** *Let $a_1, a_2, \ldots, a_q$ be positive integers such that their g.c.d. is one i.e.,* g.c.d.$\{a_1, a_2, \ldots, a_q\} = 1$. *Then there exists an integer $N \in \mathbb{N}$ such that for all integers $k \geq N$ there are $n_1, n_2 \ldots, n_q \in \mathbb{N}$ such that*

$$k = n_1 a_1 + n_2 a_2 + \cdots + n_q a_q.$$

### B. Special case of spectral theorem

As already indicated the following special case of Theorem 5.1 will be treated first.

**Special case 5.3:** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix with eigenvalue e and matrix cyclicity one, i.e., $\sigma_{\mathscr{M}}(A) = 1$. Then there exists an integer $N \in \mathbb{N}$ such that*

$$A^{\otimes(k+1)} = A^{\otimes k},$$

*for all integers $k \geq N$.*

*Proof:* Let $\mathscr{G}^c(A)$ be the critical graph of $A$ and let $\mathscr{N}^c \subseteq \underline{n}$ be the set of nodes contained in any of the critical circuits. Note that $\mathscr{N}^c$ and $\underline{n}$ are not necessarily the same!

It will subsequently be shown below that there exists an integer $N \in \mathbb{N}$ such that for all integers $k \geq N$:

(1) $[A^{\otimes k}]_{ii} = [A^+]_{ii} = e$ for all $i \in \mathscr{N}^c$,
(2) $[A^{\otimes k}]_{ij} = [A^+]_{ij}$ for all $i \in \mathscr{N}^c$ and $j \in \underline{n}$ ,
(3) $[A^{\otimes k}]_{ij} = \bigoplus_{l\in\mathscr{N}^c} [A^+]_{il} \otimes [A^+]_{lj}$ for all $i, j \in \underline{n}$.

From the above three statements it follows that $A^{\otimes(k+1)} = A^{\otimes k}$ for all integers $k \geq N$. Clearly, only Case (3) is required to complete the proof. However, Case (2) will be used in proving Case (3), while Case (1) plays a role in the proof of Case (2). The reason for considering the three statements is that they give the proof a nice structure. Therefore, to complete the proof, the above three statements will be proved successively.

**Case (1).** Consider node $i \in \mathscr{N}^c$, then there is a strongly connected component of $\mathscr{G}^c(A)$, say $\mathscr{G}_1$, with node set $\mathscr{N}_1$, such that $i \in \mathscr{N}_1$. Note that $\mathscr{G}_1$ is a critical (sub)graph, implying that all its circuits are critical. Since the matrix cyclicity of $A$ is one, it follows that the graph (and matrix) cyclicity of the matrix corresponding to $\mathscr{G}_1$ is one too. Hence, there exist (elementary) circuits in $\mathscr{G}_1$, say $\beta_1, \ldots, \beta_q$, whose lengths have a g.c.d. equal to one, i.e., g.c.d. $\{|\beta_1|_1, \ldots, |\beta_q|_1\} = 1$, where $|\beta_i|_l$ stands for the length of circuit $\beta_j$, for $j \in \underline{q}$. Since $\mathscr{G}_1$ is strongly connected, there is a circuit $\alpha$ in $\mathscr{G}_1$ such that $i$ is a node in $\alpha$ and $\alpha \cap \beta_j \neq \emptyset$, for $j \in \underline{q}$, i.e., $\alpha$ passes through $i$ and through all circuits $\beta_1, \ldots, \beta_q$. By Lemma 5.2, it follows that there is an integer $N \in \mathbb{N}$ such that for each integer $k \geq N$, there exist $n_1, \ldots, n_q \in \mathbb{N}$, such that

$$k - |\alpha|_1 = n_1\,|\beta_1|_1 + \cdots + n_q\,|\beta_q|_1.$$

For these $n_1, \ldots, n_q$, construct a circuit passing through $i$, built from circuit $\alpha$, $n_1$ copies of circuit $\beta_1$, $n_2$ copies of circuit $\beta_2$, etc., up to $n_q$ copies of circuit $\beta_q$. It is clear that the circuit is a circuit in $\mathscr{G}_1$. Therefore, it is itself also a critical circuit with weight $e$. Recall that $A$ is irreducible with eigenvalue $e$. Hence, the maximal circuit mean in $\mathscr{G}(A)$ is $e$, implying that $[A^{\otimes k}]_{ii} = e$, for all integers $k \geq N$, by the definition of $[A^+]_{ii}$, also implying that $[A^+]_{ii} = e$.

**Case (2).** By the definition of $[A^+]_{ij}$ there exists an $l$ such that $[A^{\otimes l}]_{ij} = [A^+]_{ij}$. In fact, since the eigenvalue of $A$ is $e$, it can be shown by contradiction that even $l \leq n$. (To that end, the interpretation of the eigenvalue as the maximal circuit mean is useful.) Then it follows for $k$ large enough, for $i \in \mathscr{N}^c$ and $j \in \underline{n}$, that

$$[A^{\otimes(k+l)}]_{ij} \geq [A^{\otimes k}]_{ii} \otimes [A^{\otimes l}]_{ij} = [A^{\otimes l}]_{ij} = [A^+]_{ij}\,,$$

see Case (1). Clearly,

$$[A^+]_{ij} = \bigoplus_{s\geq 1} [A^{\otimes s}]_{ij} \geq [A^{\otimes(k+l)}]_{ij} \geq [A^+]_{ij}.$$

If $k+l$ is replaced by $k$, it therefore follows that $[A^{\otimes k}]_{ij} = [A^+]_{ij}$, for all $i \in \mathscr{N}^c, j \in \underline{n}$, with $k$ large enough. Dually, it follows, of course, that $[A^{\otimes m}]_{ij} = [A^+]_{ij}$, for all $i \in \underline{n}, j \in \mathscr{N}^c$ and $m$ large enough.

**Case (3).** Take $k$ and $m$ large enough such that $[A^{\otimes k}]_{il} = [A^+]_{il}$ and $[A^{\otimes m}]_{lj} = [A^+]_{lj}$, for $l \in \mathscr{N}^c$, see Case (2). Then

$$[A^{\otimes(k+m)}]_{ij} \geq [A^{\otimes k}]_{il} \otimes [A^{\otimes m}]_{lj} = [A^+]_{il} \otimes [A^+]_{lj},$$

for all $l \in \mathscr{N}^c$. If $k+m$ is replaced by $k$, it therefore follows for $k$ large enough that

$$[A^{\otimes k}]_{ij} \geq \bigoplus_{l\in\mathscr{N}^c} [A^+]_{il} \otimes [A^+]_{lj}. \tag{2}$$

Now consider a path from $j$ to $i$ not passing through $\mathscr{N}^c$. Such a path consists of an elementary path, and a number of circuits all having negative weight, i.e., a weight less than $e$. Let the circuit mean of a non-critical circuit be maximally $\delta$, then the weight of any path from $j$ to $i$ of length $k$ not passing through a node in $\mathscr{N}^c$ can be bounded from above by $[A^+]_{ij} + k\,\delta = [A^+]_{ij} \otimes \delta^{\otimes k}$, where $[A^+]_{ij}$ is a fixed upper bound for the weight of the elementary path and $k\,\delta$ is an upper bound for the total weight of the circuits. Since $\delta < e$, i.e., $\delta < 0$ in conventional notation, it follows that for $k$ large enough

$$[A^+]_{ij} \otimes \delta^{\otimes k} \leq \bigoplus_{l\in\mathscr{N}^c} [A^+]_{il} \otimes [A^+]_{lj}\,. \tag{3}$$

Indeed, the right-hand side of the inequality is fixed, while the left-hand side tends to $-\infty$ for $k$ going to infinity. Hence, for $k$ large enough it follows that

$$\bigoplus_{l\in\underline{n}} [A^+]_{il} \otimes [A^+]_{lj} = \bigoplus_{l\in\mathscr{N}^c} [A^+]_{il} \otimes [A^+]_{lj}\,,$$

for all $i, j \in \underline{n}$, because for $k$ large the weights of the paths that do not pass through $\mathscr{N}^c$ are dominated by the weights of the paths that do pass through $\mathscr{N}^c$. Combining the obtained results in (1), (2) and (3) it follows that

$$[A^{\otimes k}]_{ij} = \bigoplus_{l\in\mathscr{N}^c} [A^+]_{il} \otimes [A^+]_{lj}\,,$$

for all $i, j \in \underline{n}$, and $k$ large enough. ■

## VI. GRAPH CYCLICITY FOR MATRIX POWERS

To complete the proof of Theorem 5.1 the graph and matrix cyclicity of powers of $A$ will be investigated. For the graph cyclicity the next lemma follows.

**Lemma 6.1:** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be irreducible with eigenvalue e and graph cyclicity $\sigma$, i.e., $\sigma_{\mathscr{G}}(A) = \sigma$. Then $A^{\otimes\sigma}$ is similar to a block diagonal matrix with $\sigma$ square diagonal blocks that are irreducible with eigenvalue e and graph cyclicity* *one, i.e.,* $A^{\otimes\sigma} \sim \mathrm{diag}(B_1, B_2, \ldots, B_\sigma)$*, where* $B_i$ *is irreducible with eigenvalue e and* $\sigma_{\mathscr{G}}(B_i) = 1$*, for all* $i \in \underline{\sigma}$*.*

*Proof:* The proof is inspired by [5]. In $\mathscr{G}(A)$, consider the following relation between nodes $i, j \in \mathscr{N}(A)$:

$$i\mathscr{R}j \iff \begin{cases} \text{the length of every path from node } i \\ \text{to node } j \text{ is a multiple of } \sigma. \end{cases} \tag{4}$$

It can easily be shown that this relation is an equivalence relation on $\mathscr{N}(A)$. Further, let $m \in \mathscr{N}(A)$ be an arbitrarily chosen, but fixed, node. Then, given this node, equivalence classes $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{\sigma-1}$ associated with the equivalence relation (4) can be introduced as follows:

$$i \in \mathscr{C}_l \iff \begin{cases} \text{every path from node } m \text{ to node } i \\ \text{has length (mod } \sigma) \text{ equal to } l, \end{cases} \tag{5}$$

for $l = 0, 1, \ldots, \sigma - 1$. It is not difficult to show for any $i, j \in \mathscr{N}(A)$ that $i\mathscr{R}j \iff i, j \in \mathscr{C}_l$ for some $l = 0, 1, \ldots, \sigma - 1$.

Assume that there is a path from $i$ to $j$ of length $\sigma$. Then it follows that every path from $i$ to $j$ has a length that is a multiple of $\sigma$. Indeed, concatenation of the previously mentioned paths with one and the same path from $j$ to $i$ yields circuits whose lengths must be multiples of $\sigma$. It then follows that every path from $i$ to $j$ must have a length that is a multiple of $\sigma$. Hence, every path of length $\sigma$ must end in the same class as the class from which it starts. Because $A^{\otimes\sigma}$ can be computed by considering all paths of length $\sigma$, it follows that $A^{\otimes\sigma}$ is block diagonal, possibly after an appropriate relabelling of the nodes according to the classes $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{\sigma-1}$. For instance, by first labelling all nodes in $\mathscr{C}_0$, next all nodes in $\mathscr{C}_1$, and so on.

Further, since for all $i, j \in \mathscr{C}_l$ there is a path from $i$ to $j$ whose length is a multiple of $\sigma$, it follows that the block in $A^{\otimes\sigma}$ corresponding to class $\mathscr{C}_l$ is irreducible. Indeed, the previous path from $i$ to $j$ can be seen as a concatenation of a number of subpaths, all of length $\sigma$, and each going from one node in $\mathscr{C}_l$ to another node in $\mathscr{C}_l$. Now considering all such subpaths of maximal weight, it follows that the graph of the block in $A^{\otimes\sigma}$ corresponding to class $\mathscr{C}_l$ is strongly connected, and that the block itself is irreducible.

Because $e$ is the eigenvalue of $A$, $e$ is also the eigenvalue of $A^{\otimes\sigma}$ and also the eigenvalue of the block associated with $\mathscr{C}_l$, for all $l \in \underline{\sigma}$. Indeed, if $A^{\otimes\sigma} \sim \mathrm{diag}(B_1, B_2, \ldots, B_\sigma)$ and $v$ is a real vector such that $A \otimes v = e \otimes v = v$, then $A^{\otimes\sigma} \otimes v = e \otimes v = v$ and $B_l \otimes v_l = e \otimes v_l = v_l$, for some appropriate real sub-vector $v_l$ of $v$, for $l \in \underline{\sigma}$.

Finally, every circuit in $\mathscr{G}(A)$ must go through each of the equivalence classes $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{\sigma-1}$. Indeed, suppose there is a circuit going through just $\tau$ of the classes, where $\tau < \sigma$. Then there must be a class $\mathscr{C}_l$ and nodes $i, j \in \mathscr{C}_l$ such that there is a path from $i$ to $j$ of length less than or equal to $\tau$. However, this is in contradiction with the fact that any path between nodes of the same class must be a multiple of $\sigma$. Hence, it follows that all circuits in $\mathscr{G}(A)$ must go through class $\mathscr{C}_l$, for any $l \in \underline{\sigma}$. Observe that circuits in $\mathscr{G}(A)$ of length $\kappa\sigma$ can be associated with circuits in $\mathscr{G}(A^{\otimes\sigma})$ of length $\kappa$. Since the greatest common divisor of all circuit lengths in $\mathscr{G}(A)$ is $\sigma$, it follows that the graph of the block in $A^{\otimes\sigma}$ corresponding to class $\mathscr{C}_l$, for any $l \in \underline{\sigma}$, has cyclicity one. ■

The following result is relevant in relation to the extension of Lemma 6.1 to powers of $A$ that are multiples of its graph cyclicity.

**Proposition 6.2:** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *be irreducible with with eigenvalue e and graph cyclicity one, i.e.,* $\sigma_{\mathscr{G}}(A) = 1$*. Then* $A^{\otimes k}$ *is irreducible with eigenvalue e and graph cyclicity one, for all integers* $k \geq 1$*.*

*Proof:* Take an integer $k \geq 1$ and $i, j \in \underline{n}$. Because $\mathscr{G}(A)$ is strongly connected there exists a path from $j$ to $i$, say $\alpha$, that passes through all nodes of $\mathscr{G}(A)$. Further, because $\sigma_{\mathscr{G}}(A) = 1$, there are (elementary) circuits in $\mathscr{G}(A)$, say $\beta_1, \ldots, \beta_q$, whose lengths have a g.c.d. equal to one. By Lemma 5.2 it follows that for $k\gamma$ large enough, with $\gamma$ a positive integer, there exist $n_1, \ldots, n_q \in \mathbb{N}$, such that

$$k\gamma - |\alpha|_l = n_1|\beta|_l + \cdots + n_q|\beta|_q.$$

For these $n_1, \ldots, n_q$, construct a path from node $j$ to node $i$ by combining, i.e., concatenating, the path $\alpha$ with $n_1$ copies of circuit $\beta_1$, $n_2$ copies of circuit $\beta_2$, etc., up to $n_q$ copies of circuit $\beta_q$. Then a path in $\mathscr{G}(A)$ is obtained from node $j$ to node $i$ of length $k\gamma$. The path implies the existence of a path of length $\gamma$ in $\mathscr{G}(A^{\otimes k})$ from node $j$ to node $i$. Since $i, j \in \underline{n}$ were chosen arbitrarily, it follows that $\mathscr{G}(A^{\otimes k})$ is strongly connected and that $A^{\otimes k}$ is irreducible.

Since $e$ is eigenvalue of $A$, $e$ is also eigenvalue of $A^{\otimes k}$. Indeed, if $v \in \mathbb{R}^n$ is such that $A \otimes v = v$, then also $A^{\otimes k} \otimes v = v$.

Let $\mathscr{S}_1$ be the set of all circuits in $\mathscr{G}(A)$ whose lengths are a multiple of $k$. Then $\mathscr{S}_1$ contains $\mathscr{S}_2$, with $\mathscr{S}_2$ being the set of circuits obtained by going around $k$ times in each of the elementary circuits $\beta_1, \ldots, \beta_q$. Clearly, the lengths of the circuits in $\mathscr{S}_2$ have a g.c.d. equal to $k$. From this it follows that the lengths of circuits in $\mathscr{S}_1$ have a g.c.d. equal to $k$ too. Hence, the g.c.d. of the lengths of all circuits in $\mathscr{G}(A)$ whose length is a multiple of $k$ is actually $k$ itself, implying that the g.c.d. of the lengths of all circuits in $\mathscr{G}(A^{\otimes k})$ is equal to one. So, it follows that $A^{\otimes k}$ has graph cyclicity one. ■

The following corollary now follows immediately by applying Proposition 6.2 to each of the blocks in Lemma 6.1.

**Corollary 6.3:** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *be irreducible with eigenvalue e and graph cyclicity* $\sigma$*, i.e.,* $\sigma_{\mathscr{G}}(A) = \sigma$*. Then, for all integers* $k \geq 1$*,* $A^{\otimes(k\sigma)} \sim \mathrm{diag}(C_1, C_2, \ldots, C_\sigma)$*, where* $C_i$ *is irreducible with eigenvalue e and* $\sigma_{\mathscr{G}}(C_i) = 1$*, for* $i \in \underline{\sigma}$*, implying that the graph cyclicity of* $A^{\otimes(k\sigma)}$ *is also one, for all integers* $k \geq 1$*.*

## VII. MATRIX CYCLICITY FOR MATRIX POWERS

The previous result dealt with the graph cyclicity in relation to powers of a matrix. In the following a similar result will be established with respect the matrix cyclicity. But first a relation between the two types of cyclcity is presented.

**Proposition 7.1:** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *be irreducible. Then the matrix cyclicity of A is a multiple of the graph cyclicity of A, i.e.,* $\sigma_{\mathscr{M}}(A)$ *is a multiple of* $\sigma_{\mathscr{G}}(A)$*.*

*Proof:* Let $\mathscr{G}_1$ be a m.s.c.s. of $\mathscr{G}^c(A)$ with graph (and matrix) cyclicity $\sigma_1$. Because $\mathscr{G}_1$ and $\mathscr{G}(A)$ are both strongly connected and every circuit in $\mathscr{G}_1$ is also a circuit in $\mathscr{G}(A)$, it follows that $\sigma_1$ is a multiple of $\sigma_{\mathscr{G}}(A)$. Then the l.c.m. of the cyclicities of all m.s.c.s.'s of $\mathscr{G}^c(A)$ is a multiple of $\sigma_{\mathscr{G}}(A)$. Hence, $\sigma_{\mathscr{M}}(A)$ is a multiple of $\sigma_{\mathscr{G}}(A)$. ■

The following corollary is an immediate consequence of Lemma 4.2, Lemma 6.1 and Proposition 6.2.

**Lemma 7.2:** *Let $A \in \mathbb{R}^{n\times n}_{\max}$ be irreducible with with eigenvalue e, graph cyclicity one and matrix cyclicity $\sigma$, i.e., $\sigma_{\mathscr{G}}(A) = 1$ and $\sigma_{\mathscr{M}}(A) = \sigma$. Then $A^{\otimes\sigma}$ is irreducible with eigenvalue e and graph and matrix cyclicity one.*

*Proof:* According to Proposition 6.2, $A^{\otimes\sigma}$ is irreducible with eigenvalue $e$ and graph cyclicity one.

The matrix cyclicity of $A$ is equal to the graph cyclicity of $A^c$. Hence, $\sigma_{\mathscr{G}}(A^c) = \sigma$.

If $A^c$ is irreducible, it follows, according to Lemma 6.1, that $(A^c)^{\otimes\sigma}$ is similar to a block diagonal matrix with square diagonal blocks that are irreducible with eigenvalue $e$ and graph cyclicity one, i.e., $(A^c)^{\otimes\sigma} \sim \mathrm{diag}\big(B_1, B_2, \ldots, B_\sigma\big)$, where $B_i$ is irreducible with eigenvalue $e$ and $\sigma_{\mathscr{G}}(B_i) = 1$, $i \in \underline{\sigma}$. Hence, the graph cyclicity of $(A^c)^{\otimes\sigma}$ is one too.

If $A^c$ is not irreducible, the graph $\mathscr{G}(A^c)$ contains a number of m.s.c.s.'s, say $q$, that are completely isolated from each other (the graph $\mathscr{G}(A^c)$ contains only (critical) circuits!). By relabelling the nodes in $\mathscr{G}(A^c)$ matrix $A^c$ can be made block diagonal. Hence, $A^c$ is similar to a block diagonal matrix $\mathrm{diag}(D_1, \ldots, D_q, D_{q+1})$, where matrix $D_i$ is irreducible with eigenvalue $e$, for $i \in \underline{q}$, and where $D_{q+1}$ is a square matrix consisting of $\varepsilon$ only. Matrix $D_{q+1}$ appears when there are nodes that are not contained in any m.s.c.s.c. If the graph cyclicity of matrix $D_i$ is denoted by $\sigma_i$, for $i \in \underline{q}$, then $\sigma = \mathrm{l.c.d.}\{\sigma_1, \ldots, \sigma_q\}$. Like in the above part of this proof, it follows from Lemma 6.1 and Proposition 6.2, applied to matrix $D_i$, that $(D_i)^{\otimes\sigma}$ is irreducible with eigenvalue $e$ and graph cyclicity one, for $i \in \underline{q}$. Since $(A^c)^{\otimes\sigma} \sim \mathrm{diag}\big((D_1)^{\otimes\sigma}, \ldots, (D_q)^{\otimes\sigma}, (D_{q+1})^{\otimes\sigma}\big)$, with $D_{q+1}$ containing $\varepsilon$ only, this implies that the graph cyclicity of $(A^c)^{\otimes\sigma}$ is one too.

Hence, $\sigma_{\mathscr{G}}((A^c)^{\otimes\sigma}) = 1$, for $A^c$ irreducible and not irreducible. By Lemma 4.2, it follows that the matrix cyclicity of $A^{\otimes\sigma}$ is also one, i.e., $\sigma_{\mathscr{M}}(A^{\otimes\sigma}) = 1$. ■

Now the next lemma can be proved.

**Lemma 7.3:** *Let $A \in \mathbb{R}^{n\times n}_{\max}$ be irreducible with with eigenvalue e, graph cyclicity $\rho$ and matrix cyclicity $\sigma$, i.e., $\sigma_{\mathscr{G}}(A) = \rho$ and $\sigma_{\mathscr{M}}(A) = \sigma$. Then $A^{\otimes\sigma}$ is similar to a block diagonal matrix with $\rho$ square diagonal block matrices that are irreducible and have eigenvalue e and graph and matrix cyclicity one.*

*Proof:* According to Lemma 6.1, it follows that $A^{\otimes\rho}$ is similar to a block diagonal matrix with $\rho$ square diagonal block matrices that are irreducible and have eigenvalue $e$ and graph cyclicity one, i.e., $A^{\otimes\sigma} \sim \mathrm{diag}(B_1, B_2, \ldots, B_\rho)$, where $B_i$ is irreducible with eigenvalue $e$ and $\sigma_{\mathscr{G}}(B_i) = 1$, $i \in \underline{\rho}$. If the matrix cyclicity of $B_i$ is denoted by $\tau_i$, $i \in \underline{\rho}$, then, as in the proof of Proposition 7.1, the l.c.m. of $\{\tau_1, \ldots, \tau_\rho\}$, when denoted as $\tau$, is such that $\sigma = \tau\rho$. According to the proof of Lemma 7.2, $(B_i^c)^{\otimes\tau_i}$ has a graph cyclicity equal to one, for $i \in \underline{\rho}$, where $B_i^c$ denotes the critical matrix of $B_i$. By Proposition 6.2, the same applies to $(B_i^c)^{\otimes\tau}$, for $i \in \underline{\rho}$. According to Lemma 4.2 it follows that the matrix cyclicity of $(B_i)^{\otimes\tau}$ is one, for $i \in \underline{\rho}$. Hence, the $\tau$th power of $A^{\otimes\rho}$, being equal to $A^{\otimes(\tau\rho)} = A^{\otimes\sigma}$, has matrix cyclicity one. ■

## VIII. PROOF OF THEOREM 5.1

*Proof:* Observe that $A_\lambda$ is irreducible with eigenvalue $e$ and matrix cyclicity $\sigma$. Then, according to Lemma 7.3, the matrix $B \overset{\mathrm{def}}{=} (A_\lambda)^{\otimes\sigma}$ is similar to a block diagonal matrix with square diagonal block matrices that are irreducible and have eigenvalue $e$ and matrix cyclicity one. Hence, by applying Special case 5.3 to each diagonal block, it ultimately follows that an $M$ exists such that $B^{\otimes(l+1)} = B^{\otimes l}$, for all integers $l \geq M$. The latter implies that

$$\big((A_\lambda)^{\otimes\sigma}\big)^{\otimes(l+1)} = \big((A_\lambda)^{\otimes\sigma}\big)^{\otimes l},$$

which can be further written as $(A_\lambda)^{\otimes(l\sigma+\sigma)} = (A_\lambda)^{\otimes(l\sigma)}$, or

$$A^{\otimes(l\sigma+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes(l\sigma)},$$

for all integers $l \geq M$. Finally, note that $A^{\otimes(l\sigma+j+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes(l\sigma+j)}$, for any integer $j$, $0 \leq j \leq \sigma - 1$, implying that for all integers $k \geq N \overset{\mathrm{def}}{=} M\sigma$ it follows that

$$A^{\otimes(k+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes k}.$$

This completes the proof of Theorem 5.1. ■

## IX. EXTENSIONS AND CONCLUSIONS

In this paper a proof is given for the well-known spectral theorem in max-plus algebra, see Theorem 5.1. In the proof the distinction between the graph and matrix cyclicity of an irreducible matrix is highlighted.

For reasons of space limitation the observation that the matrix cyclicity of an irreducible matrix $A$ with eigenvalue $\lambda$ coincides with the minimal integer $\tau \geq 1$ such that $A^{\otimes(k+\tau)} = \lambda^{\otimes\tau} \otimes A^{\otimes k}$ for all integers $k \geq N$ for some large $N \in \mathbb{N}$, is not treated.


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