Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Stabilization of swelling porous elastic soils with fluid saturation by one internal damping

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Abstract— **This article considers the stabilization of a system of one-dimensional swelling porous elastic soils with fluid saturation. This system is strongly coupled by vibrations of both fluid and solid elastic materials. Using Riesz basis approach, it is shown that the system can be exponentially stabilized by one internal damping with variable feedback gain imposed in the fluid equation. The result improved greatly at the first time the previous results in literature where two dampings are needed to get the same result.**

I. INTRODUCTION

It is generally recognized that the swelling of soils, plants, drying of fibres, wood, paper, etc belong to the porous media theory. A complete formulation of a mixture theory for porous elastic solids filled with fluid and gas was developed in [2]. This formulation has many applications in various practical problems such as field of swelling, oil explanation, slurred and consolidation problems. Several literatures are available recently to cope with the stability of one-dimensional problems, for instance [4],[8]. One of problems in this theory is the interactions between two various components ([8]). In [4] the exponential stability was obtained for one-dimensional problem by imposing three internal damping with constant feedback gains in both solid and liquid equations. This article, on the other hand, shows that only one internal damping imposed in fluid equation is sufficient to stabilize exponentially the system. The advantage of this article is that the feedback gain is not constant but a function in spatial variable. Moreover, it requires only that this function is positive in any measurable subset with positive measure in spatial space, which can not be treated by the method in [4]. The Riesz basis approach is adopted in investigation. We obtained more elaborate results. The generalized eigenfunctions of the system form a Riesz basis for the energy state space. The asymptotic of eigenvalues is explicitly obtained. The spectrum-determined growth condition that is a difficult problem in partial differential equation system controls is established. The exponential stability is a consequence of these results. To our knowledge, this is a first attempt to exponentially stabilize the two coupled wave equations by one internal damping.

The plan of this paper is as follows: in Section 2, we formulate the problem in the energy state space as an abstract evolution equation. Section 3 is devoted to the spectral analysis of the system, which is the main body of the article. The Riesz basis generation as well as exponential stability are presented in Section 4.

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II. FORMULATION OF THE PROBLEM

We consider a linear field equation of swelling porous elastic soils in fluid saturation of the following ([2], [4]):

$$\rho_z \frac{\partial^2 z}{\partial t^2} = a_1 \frac{\partial^2 z}{\partial x^2} + a_2 \frac{\partial^2 u}{\partial x^2} - \rho_z k_1(x) \frac{\partial z}{\partial t}, \qquad (1)$$

$$\rho_u \frac{\partial^2 u}{\partial t^2} = a_2 \frac{\partial^2 z}{\partial x^2} + a_3 \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

where z and u represent the displacements of fluid and solid elastic materials at space position $x \in (0, \ell)$ and time $t > 0$, respectively. The constants $\rho_z, \rho_u > 0$ are the densities of each constituent. The parameters $a_1, a_3 > 0$ and a_2 are the constitutive constants and $k_1(x)$ is a viscous damping function.

The initial and boundary conditions of the system (1)-(2) are

$$z(x,0) = z_0(x),\ u(x,0) = u_0(x), \quad x \in [0,\ell], \qquad (3)$$

$$\frac{d}{dt} z(x,0) = z_1(x),\ \frac{d}{dt} u(x,0) = u_1(x), \quad x \in [0,\ell], \qquad (4)$$

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and

$$z(0,t) = \frac{d}{dx}z(\ell,t) = u(0,t) = \frac{d}{dx}u(\ell,t) = 0. \quad (5)$$

The total energy function for the system (1)-(5) is given by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^\ell \Bigg[\rho_z|z_t(x,t)|^2 + \rho_u|u_t(x,t)|^2 + \left\langle \Upsilon \begin{pmatrix} z_x(x,t) \\ u_x(x,t) \end{pmatrix}, \begin{pmatrix} z_x(x,t) \\ u_x(x,t) \end{pmatrix} \right\rangle_{\mathbb{C}^2} \Bigg] dx \quad (6)$$

where

$$\Upsilon := \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \quad (7)$$

is positive definite, i.e., $a_1a_3 > a_2^2$.

For simplicity, we assume $\ell = 1$ throughout the paper. We begin by formulating the system (1)-(5) into an abstract evolution equation on the state Hilbert space $\mathcal{H}$:

$$\begin{cases} \mathcal{H} := \Big(H_E^1(0,1) \times L^2(0,1)\Big)^2, \\ H_E^1(0,1) := \big\{f \in H^1(0,1) \mid f(0) = 0\big\} \end{cases} \quad (8)$$

where $H^1(0,1)$ denotes the usual Sobolev space. Due to the energy function (6), it is natural to introduce the following inner product on $\mathcal{H}$:

$$\langle Y_1,\ Y_2\rangle_{\mathcal{H}} := \int_0^1 \Bigg[\rho_z w_1\overline{w}_2 + \rho_u v_1\overline{v}_2 + \left\langle \Upsilon \begin{pmatrix} z_1' \\ u_1' \end{pmatrix}, \begin{pmatrix} z_2' \\ u_2' \end{pmatrix} \right\rangle_{\mathbb{C}^2} \Bigg] dx \quad (9)$$

where $Y_i := [z_i, w_i, u_i, v_i]$, $i = 1, 2$, and the prime "$\prime$" denotes the differentiation with respect to x. Define the operators $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^\top := \begin{bmatrix} w \\ \frac{1}{\rho_z}\Big(a_1z'' + a_2u''\Big) \\ v \\ \frac{1}{\rho_u}\Big(a_2z'' + a_3u''\Big) \end{bmatrix}^\top \quad (10)$$

for all $[z, w, u, v] \in \mathcal{D}(\mathcal{A})$, where

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{array}{c} [z, w, u, v] \in \mathcal{H} \mid z, u \in H^2(0,1), \\ w, v \in H_E^1(0,1), \\ z'(1) = u'(1) = 0 \end{array} \right\}, \quad (11)$$

and for all $[z, w, u, v] \in \mathcal{D}(\mathcal{B}) = \mathcal{H}$,

$$\mathcal{B}\begin{bmatrix} z \\ w \\ u \\ v \end{bmatrix}^\top := \begin{bmatrix} 0 \\ -k_1(x)w \\ 0 \\ 0 \end{bmatrix}^\top. \quad (12)$$

Let $Y(t) := [z(\cdot,t), z_t(\cdot,t), u(\cdot,t), u_t(\cdot,t)]$. Then the system (1)-(5) can be formulated into an abstract evolution equation on $\mathcal{H}$:

$$\begin{cases} \frac{d}{dt}Y(t) = (\mathcal{A} + \mathcal{B})Y(t), \quad t > 0, \\ \mathcal{D}(\mathcal{A} + \mathcal{B}) = \mathcal{D}(\mathcal{A}), \\ Y(0) := [z_0, z_1, u_0, u_1]. \end{cases} \quad (13)$$

Lemma 2.1: The operator $\mathcal{A}$ defined by (10) and (11) is skew-adjoint in $\mathcal{H}$.

Theorem 2.1: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (10)-(12). Then $\mathcal{A}$ and $\mathcal{A} + \mathcal{B}$ are of compact resolvents and $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{A} + \mathcal{B})$. Therefore, the spectrums of $\mathcal{A}$ and $\mathcal{A} + \mathcal{B}$ consist of isolated eigenvalues only.

Theorem 2.2: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (10)-(12). Then $\mathcal{A}$ generates a C_0-group on $\mathcal{H}$ and so is $\mathcal{A} + \mathcal{B}$ due to the boundedness of $\mathcal{B}$.

III. Spectral analysis

In this section, we are devoted to the spectral analysis for the system (1)-(5). Let $\lambda \in \sigma(\mathcal{A} + \mathcal{B})$ and $Y_\lambda := [z, w, u, v]$ be an eigenfunction of $\mathcal{A}+\mathcal{B}$ corresponding to λ. Then $(\mathcal{A}+\mathcal{B})Y_\lambda = \lambda Y_\lambda$ implies $w = \lambda z$, $v = \lambda u$ and that z, u satisfy the following system of characteristic equations $(0 < x < 1)$

$$\begin{cases} \rho_z\lambda^2 z(x) - a_1z''(x) - a_2u''(x) + \rho_z k_1(x)\lambda z(x) = 0, \\ \rho_u\lambda^2 u(x) - a_2z''(x) - a_3u''(x) = 0, \end{cases} \quad (14)$$

and the boundary condition,

$$z(0) = u(0) = z'(1) = u'(1) = 0. \quad (15)$$

For brevity in notation, we set

$$\begin{cases} r_1 := \sqrt{\frac{\rho_z}{a_1}}, \quad r_2 := \sqrt{\frac{\rho_u}{a_3}}, \quad a_4 := \frac{a_2}{a_1}, \\ a_5 := \frac{a_2}{a_3}, \quad \delta := \frac{1}{1 - a_4a_5} > 0. \end{cases} \quad (16)$$

Note that if $a_2 = 0$, there is no coupling for the system (1)-(2). So we always assume that $a_2 \neq 0$ in the sequel. Now (14) becomes $(0 < x < 1)$

$$\begin{cases} r_1^2\lambda^2 z(x) - z''(x) - a_4u''(x) + r_1^2 k_1(x)\lambda z(x) = 0, \\ r_2^2\lambda^2 u(x) - a_5z''(x) - u''(x) = 0. \end{cases} \quad (17)$$

In order to solve the above equation, we solve the following equation that is equivalent to (17):

$$\begin{cases} r_1^2\lambda^2 z(x) - a_4 r_2^2 \lambda^2 u(x) - (1/\delta) z''(x) \\ \qquad\qquad + r_1^2 k_1(x)\lambda z(x) = 0, \\ r_2^2\lambda^2 u(x) - a_5 r_1^2\lambda^2 z(x) - (1/\delta) u''(x) \\ \qquad\qquad - a_5 r_1^2 k_1(x)\lambda z(x) = 0. \end{cases} \tag{18}$$

Set

$$z_1 := z,\ z_2 := z',\ u_1 := u,\ u_2 := u', \tag{19}$$

$$\Phi(x) := [z_1, z_2, u_1, u_2]^\top. \tag{20}$$

Then (18) becomes

$$T^D(x,\lambda)\Phi(x) = 0, \tag{21}$$

where

$$T^D(x,\lambda)\Phi(x) := \Phi'(x) + A(x,\lambda)\Phi(x), \tag{22}$$

$$A(x,\lambda) := A_0 - \lambda A_1(x) - \lambda^2 A_2 \tag{23}$$

and A_0, A_1 and A_2 are three matrix functions defined by

$$A_0 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

$$A_1(x) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \delta r_1^2 k_1(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_5\delta r_1^2 k_1(x) & 0 & 0 & 0 \end{bmatrix}, \tag{25}$$

$$A_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \delta r_1^2 & 0 & -a_4\delta r_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ -a_5\delta r_1^2 & 0 & \delta r_2^2 & 0 \end{bmatrix}. \tag{26}$$

Under the same formulation, the boundary condition (15) becomes

$$T^R\Phi(x) := W^0\Phi(0) + W^1\Phi(1) = 0, \tag{27}$$

with

$$W^0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

$$W^1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

As a conclusion, we have the following Theorem 3.1.

Theorem 3.1: The characteristic equation (14) together with boundary condition (15) is equivalent to the first order linear system (21) with boundary condition (27). Moreover, $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$ iff (21) and (27) have a nonzero solution.

Next we utilize a standard technique due to Birkhoff-Langer ([1]) and Tretter ([5], [6]) to expand the characteristic determinant of (21) and (27). To begin with, we diagonalize the leading term $\lambda^2 A_2$ in (23). Let

$$r_3 := \sqrt{\frac{\delta\left(r_1^2+r_2^2\right) + \sqrt{\delta^2(r_1^2+r_2^2)^2 - 4\delta r_1^2 r_2^2}}{2}}, \tag{30}$$

$$r_4 := \sqrt{\frac{\delta\left(r_1^2+r_2^2\right) - \sqrt{\delta^2(r_1^2+r_2^2)^2 - 4\delta r_1^2 r_2^2}}{2}}, \tag{31}$$

$$s := -\frac{\delta a_4 r_2^2}{r_3^2 - \delta r_1^2} = -\frac{r_3^2 - \delta r_2^2}{\delta a_5 r_1^2}, \tag{32}$$

$$t := -\frac{\delta a_4 r_2^2}{r_4^2 - \delta r_1^2} = -\frac{r_4^2 - \delta r_2^2}{\delta a_5 r_1^2}, \tag{33}$$

$$\delta_1 := \frac{1}{s-t},\ s\delta_1 > 0,\ t\delta_1 < 0 \tag{34}$$

with $r_3^2 + r_4^2 = \delta r_1^2 + \delta r_2^2$ and

$$r_3 \neq r_4,\ 1 + t a_5 = \frac{r_3^2}{\delta r_1^2} > 0,\ 1 + s a_5 = \frac{r_4^2}{\delta r_1^2} > 0. \tag{35}$$

Define an invertible matrix $P(\lambda)$ by

$$P(\lambda) := S \begin{bmatrix} P_1(\lambda) & \\ & P_2(\lambda) \end{bmatrix}, \quad \forall\, \lambda \in \mathbb{C}, \lambda \neq 0, \tag{36}$$

where

$$S := \begin{bmatrix} sI_2 & tI_2 \\ I_2 & I_2 \end{bmatrix},\ S^{-1} := \delta_1 \begin{bmatrix} I_2 & -tI_2 \\ -I_2 & sI_2 \end{bmatrix}, \tag{37}$$

I_2 is a 2×2 identity matrix and

$$P_1(\lambda) := \begin{bmatrix} r_3\lambda & r_3\lambda \\ r_3^2\lambda^2 & -r_3^2\lambda^2 \end{bmatrix}, P_2(\lambda) := \begin{bmatrix} r_4\lambda & r_4\lambda \\ r_4^2\lambda^2 & -r_4^2\lambda^2 \end{bmatrix},$$

$$P_1^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_3\lambda} & \frac{1}{2r_3^2\lambda^2} \\ \frac{1}{2r_3\lambda} & \frac{-1}{2r_3^2\lambda^2} \end{bmatrix},$$

$$P_2^{-1}(\lambda) := \begin{bmatrix} \frac{1}{2r_4\lambda} & \frac{1}{2r_4^2\lambda^2} \\ \frac{1}{2r_4\lambda} & \frac{-1}{2r_4^2\lambda^2} \end{bmatrix}.$$

It is easy to see that $P(\lambda)^{-1}$ exists whenever $\lambda \neq 0$ and

$$P(\lambda)^{-1} := \begin{bmatrix} P_1^{-1}(\lambda) & \\ & P_2^{-1}(\lambda) \end{bmatrix} S^{-1}. \tag{38}$$

So the matrix $P(\lambda)$ is a polynomial of degree 2 in λ. Defining

$$\Psi(x) := P^{-1}(\lambda)\Phi(x) \text{ (that is } \Phi(x) = P(\lambda)\Psi(x)) \tag{39}$$

and $\widehat{T}^D(x,\lambda) := P(\lambda)^{-1}T^D(x,\lambda)P(\lambda)$, we have

$$\widehat{T}^D(x,\lambda)\Psi(x) = \Psi'(x) + \widehat{A}(x,\lambda)\Psi(x) = 0 \tag{40}$$

where

$$\widehat{A}(x,\lambda) := P(\lambda)^{-1}A(x,\lambda)P(\lambda). \tag{41}$$

Since

$$S^{-1}A(x,\lambda)S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\lambda^2 r_3^2 + \lambda k_3(x) & 0 & \lambda k_4(x) & 0 \\ 0 & 0 & 0 & -1 \\ -\lambda k_5(x) & 0 & -\lambda^2 r_4^2 - \lambda k_6(x) & 0 \end{bmatrix}$$

where

$$\begin{cases} k_3(x) := -s\delta_1\delta r_1^2(1+ta_5)k_1(x), \\ k_4(x) := -t\delta_1\delta r_1^2(1+ta_5)k_1(x), \\ k_5(x) := -s\delta_1\delta r_1^2(1+sa_5)k_1(x), \\ k_6(x) := -t\delta_1\delta r_1^2(1+sa_5)k_1(x), \end{cases} \tag{42}$$

we obtain

$$\widehat{A}(\cdot,\lambda) = \begin{bmatrix} k_7 - r_3\lambda & k_7 & k_8 & k_8 \\ -k_7 & r_3\lambda - k_7 & -k_8 & -k_8 \\ -k_9 & -k_9 & -r_4\lambda - k_{10} & -k_{10} \\ k_9 & k_9 & k_{10} & r_4\lambda + k_{10} \end{bmatrix}$$

where k_i, $i = 7, 8, 9, 10$ are functions in $x \in [0,1]$ given by

$$\begin{cases} k_7(x) := \dfrac{k_3(x)}{2r_3},\ k_8(x) := \dfrac{r_4 k_4(x)}{2r_3^2}, \\ k_9(x) := \dfrac{r_3 k_5(x)}{2r_4^2},\ k_{10}(x) := \dfrac{k_6(x)}{2r_4}. \end{cases} \tag{43}$$

It is seen from the above that $\widehat{A}(x,\lambda)$ can decompose into

$$\widehat{A}(x,\lambda) := -\lambda\widehat{A}_1 - \widehat{A}_0(x) \tag{44}$$

with

$$\widehat{A}_1 := \begin{bmatrix} r_3 & & & \\ & -r_3 & & \\ & & r_4 & \\ & & & -r_4 \end{bmatrix}, \tag{45}$$

$$\widehat{A}_0(x) := \begin{bmatrix} -k_7 & -k_7 & -k_8 & -k_8 \\ k_7 & k_7 & k_8 & k_8 \\ k_9 & k_9 & k_{10} & k_{10} \\ -k_9 & -k_9 & -k_{10} & -k_{10} \end{bmatrix}. \tag{46}$$

Theorem 3.2: Let $0 \neq \lambda \in \mathbb{C}$, and let $\widehat{A}(x,\lambda)$ be defined by (44)-(46). For $x \in [0,1]$, set

$$E(x,\lambda) := \operatorname{diag}\left[e^{r_3\lambda x}, e^{-r_3\lambda x}, e^{r_4\lambda x}, e^{-r_4\lambda x}\right]. \tag{47}$$

Then there exists a fundamental matrix solution $\widehat{\Psi}(x,\lambda)$ to the system (40),

$$\Psi'(x,\lambda) = -\widehat{A}(x,\lambda)\Psi(x,\lambda) \tag{48}$$

such that for large enough $|\lambda|$,

$$\widehat{\Psi}(x,\lambda) := \left(\widehat{\Psi}_0(x) + \mathcal{O}(\lambda^{-1})\right) E(x,\lambda), \tag{49}$$

where

$$\begin{aligned} &\widehat{\Psi}_0(x) := \operatorname{diag}\left(e^{-k_{11}(x)}, e^{k_{11}(x)}, e^{k_{12}(x)}, e^{-k_{12}(x)}\right). \\ &k_{11}(x) := \textstyle\int_0^x k_7(\xi)d\xi,\ \ k_{12}(x) := \int_0^x k_{10}(\xi)d\xi. \end{aligned} \tag{50}$$

Corollary 3.1: Let $\widehat{\Psi}(x,\lambda)$ given by (49) be a fundamental matrix solution to the system (48). Then

$$\widehat{\Phi}(x,\lambda) := P(\lambda)\widehat{\Psi}(x,\lambda) \tag{51}$$

is a fundamental matrix solution to the first order linear system (22).

We are now in a position to estimate the asymptotic of the eigenvalues. Note that the eigenvalues of the first order linear system (22), (27) are given by the zeros of the characteristic determinant

$$\Delta(\lambda) := \det\left(T^R\widehat{\Phi}(x,\lambda)\right), \quad \lambda \in \mathbb{C}, \tag{52}$$

where the operator T^R is given by (27) and $\widehat{\Phi}(x,\lambda)$ is any fundamental matrix to the equation $T^D(x,\lambda)\Phi(x) = 0$ (see [5]). Note that

$$T^R\widehat{\Phi}(x,\lambda) = W^0P(\lambda)\widehat{\Psi}(0,\lambda) + W^1P(\lambda)\widehat{\Psi}(1,\lambda). \tag{53}$$

A simple computation by using (28), (29) and (36) gives

$$T^R\widehat{\Phi}(\cdot,\lambda) =$$

$$\begin{bmatrix} sr_3\lambda[1]_1 & sr_3\lambda[1]_1 & tr_4\lambda[1]_1 & tr_4\lambda[1]_1 \\ r_3\lambda[1]_1 & r_3\lambda[1]_1 & r_4\lambda[1]_1 & r_4\lambda[1]_1 \\ sr_3^2\lambda^2[1]_1 e^{r_3\lambda - k_{11}(1)} & -sr_3^2\lambda^2[1]_1 e^{-r_3\lambda + k_{11}(1)} & tr_4^2\lambda^2[1]_1 e^{r_4\lambda + k_{12}(1)} & -tr_4^2\lambda^2[1]_1 e^{-r_4\lambda - k_{12}(1)} \\ r_3^2\lambda^2[1]_1 e^{r_3\lambda - k_{11}(1)} & -r_3^2\lambda^2[1]_1 e^{-r_3\lambda + k_{11}(1)} & r_4^2\lambda^2[1]_1 e^{r_4\lambda + k_{12}(1)} & -r_4^2\lambda^2[1]_1 e^{-r_3\lambda - k_{12}(1)} \end{bmatrix}$$

where

$$[a]_1 := a + \mathcal{O}(\lambda^{-1}).$$

Thus, further computation gives

$$\Delta(\lambda) = \det\left(T^R\widehat{\Phi}(\cdot,\lambda)\right) = -r_3^3 r_4^3 (t-s)^2\lambda^6$$
$$\times \left\{ \left(e^{r_3\lambda - \int_0^1 k_7(\xi)d\xi} + e^{-r_3\lambda + \int_0^1 k_7(\xi)d\xi} \right) \right.$$
$$\left. \times \left((e^{r_4\lambda + \int_0^1 k_{10}(\xi)d\xi} + e^{-r_4\lambda - \int_0^1 k_{10}(\xi)d\xi} \right) + \mathcal{O}(\lambda^{-1}) \right\}.$$

Theorem 3.3: Let $\Delta(\lambda)$ be the characteristic determinant of the first order linear system (21) and the boundary condition (27). Then an asymptotic expression of $\Delta(\lambda)$ is given by

$$\Delta(\lambda) = \mu\lambda^6 \left\{ \Delta_1 \times \Delta_2 + \mathcal{O}(\lambda^{-1}) \right\} \tag{54}$$

where $\mu := -r_3^3 r_4^3 (t-s)^2$,

$$\Delta_1 := e^{r_3\lambda - \int_0^1 k_7(\xi)d\xi} + e^{-r_3\lambda + \int_0^1 k_7(\xi)d\xi}, \tag{55}$$

$$\Delta_2 := e^{r_4\lambda + \int_0^1 k_{10}(\xi)d\xi} + e^{-r_4\lambda - \int_0^1 k_{10}(\xi)d\xi}. \tag{56}$$

Now the characteristic determinant $\Delta(\lambda) = 0$ is

$$\Delta_1 \times \Delta_2 + \mathcal{O}(\lambda^{-1}) = 0$$

which is equivalent to

$$e^{r_3\lambda - \int_0^1 k_7(\xi)d\xi} + e^{-r_3\lambda + \int_0^1 k_7(\xi)d\xi} + \mathcal{O}(\lambda^{-1}) = 0 \tag{57}$$

or

$$e^{r_4\lambda + \int_0^1 k_{10}(\xi)d\xi} + e^{-r_4\lambda - \int_0^1 k_{10}(\xi)d\xi} + \mathcal{O}(\lambda^{-1}) = 0. \tag{58}$$

By the Rouché's Theorem, the roots of (57) can be estimated by those of

$$e^{r_3\lambda - \int_0^1 k_7(\xi)d\xi} + e^{-r_3\lambda + \int_0^1 k_7(\xi)d\xi} = 0$$

which can be found explicitly as

$$\tilde{\lambda}_{1k} = \frac{1}{r_3}\left(\int_0^1 k_7(\xi)d\xi + (k+\frac{1}{2})\pi i \right), \quad k \in \mathbb{Z}, \tag{59}$$

where $k_7(x)$ is defined by (43). Thus, the roots of (57) satisfy (for $|k| \geq N_1, k \in \mathbb{Z}$)

$$\lambda_{1k} = \frac{1}{r_3}\left(\int_0^1 k_7(\xi)d\xi + (k+\frac{1}{2})\pi i \right) + \mathcal{O}(k^{-1}), \tag{60}$$

where N_1 is a sufficiently large positive integer. Repeating the same discussion for equation (58), we can get the asymptotics of its eigenvalues (for $|k| \geq N_2, k \in \mathbb{Z}$):

$$\lambda_{2k} = \frac{-1}{r_4}\left(\int_0^1 k_{10}(\xi)d\xi - (k+\frac{1}{2})\pi i \right) + \mathcal{O}(k^{-1}), \tag{61}$$

where N_2 is a sufficiently large positive integer.

Eventually, we have obtained the following result for the spectrum of $\mathcal{A}+\mathcal{B}$.

Theorem 3.4: Let $\mathcal{A}+\mathcal{B}$ be defined by (10)-(12). Then each $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$ is algebraically simple when $|\lambda|$ is large enough, and the following asymptotic expressions hold (for $|k| \geq \max\{N_1, N_2\}, k \in \mathbb{Z}$)

$$\lambda_{1k} = \frac{1}{r_3}\left(\int_0^1 k_7(\xi)d\xi + (k+\frac{1}{2})\pi i \right) + \mathcal{O}(k^{-1}), \tag{62}$$

$$\lambda_{2k} = \frac{-1}{r_4}\left(\int_0^1 k_{10}(\xi)d\xi - (k+\frac{1}{2})\pi i \right) + \mathcal{O}(k^{-1}), \tag{63}$$

where N_1, N_2 are large enough positive integers. Furthermore, from (16), (32)-(35), (42) and (43), it follows that

$$\int_0^1 k_7(\xi)d\xi = \frac{-s\delta_1 r_3}{2}\int_0^1 k_1(\xi)d\xi \tag{64}$$

and

$$\int_0^1 k_{10}(\xi)d\xi = \frac{-t\delta_1 r_4}{2}\int_0^1 k_1(\xi)d\xi. \tag{65}$$

Therefore, as $k \to \infty$,

$$\mathrm{Re}\lambda_{1k} \to \frac{-s\delta_1}{2}\int_0^1 k_1(\xi)d\xi, \tag{66}$$

$$\mathrm{Re}\lambda_{2k} \to \frac{t\delta_1}{2}\int_0^1 k_1(\xi)d\xi. \tag{67}$$

IV. EXPONENTIAL STABILITY

In this section, we investigate the stability of (13).

Theorem 4.1: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (10)-(12). Then the generalized eigenfunctions of $\mathcal{A}+\mathcal{B}$ are complete in $\mathcal{H}$.

Proof. Since $\mathcal{A}$ is a skew-adjoint operator with compact resolvents and $0, \infty \in \rho(\mathcal{A})$, $(i\mathcal{A})^{-1}$ is compact, self-adjoint and $\mathrm{Ker}(i\mathcal{A})^{-1} = \{0\}$. Moreover, by (62) and (63) (where we can take $k_1(x) \equiv 0$), we have $\{\lambda_k((i\mathcal{A})^{-1})\}_{k=1}^{\infty} \in l^2$. Since

$$(i(\mathcal{A}+\mathcal{B}))^{-1} = (i\mathcal{A})^{-1}(I - \mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1})$$

$\mathcal{B}\mathcal{A}^{-1}$ and $\mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1}$ are compact and $I-\mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1}$ is invertible, so the required result follows from the Keldysh's theorem (see [3, pp.170, Theorem 4.1]). □

Finally the verification of the Riesz basis property for the system (13) will be done by Theorem 4.2 of [7].

Theorem 4.2: System (13) is a Riesz spectral system (in the sense that its generalized eigenfunctions form a Riesz basis in $\mathcal{H}$) and so it satisfies the spectrum determined growth condition.

Proof. In view of Theorem 3.4, we may take $\sigma_2(\mathcal{A}+\mathcal{B}) := \sigma(\mathcal{A}+\mathcal{B})$ and $\sigma_1(\mathcal{A}+\mathcal{B}) := \{\infty\}$. Then conditions 1), 2) and 3) in Theorem 4.2 of [7] are satisfied. Moreover, Theorem 4.1 implies that $X_1=\{0\}$. Thus, the first assertion of Theorem 4.2 of [7] says that there is a sequence of the generalized eigenfunctions of $\mathcal{A}+\mathcal{B}$, which forms a Riesz basis for $\mathcal{H}$. Finally, the spectrum determined growth condition is a direct consequence of the Riesz basis generation and the algebraic simplicity of the high eigenvalues. □

We are now ready to discuss the stability of the system (13). From (66) and (67), in order to achieve the exponential stability, we need the following necessary condition

$$\int_0^1 k_1(\xi)d\xi > 0. \tag{68}$$

Condition (68) together with (34) guarantees that

$$\frac{-s\delta_1}{2}\int_0^1 k_1(\xi)d\xi < 0,\ \frac{t\delta_1}{2}\int_0^1 k_1(\xi)d\xi < 0.$$

These imply that the high eigenvalues of the system (13) are located on the left half plane.

The last step is to consider the low eigenvalues of the system (13):

Suppose $k_1(x)\ge 0$ and $k_1(x)\Big|_I > 0$, where I is some measurable subset of $[0,1]$ with positive measure. Then with the assumption (68), for each $Y:=[z,w,u,v]\in\mathcal{D}(\mathcal{A}+\mathcal{B})$, one has

$$\mathrm{Re}\big\langle(\mathcal{A}+\mathcal{B})Y,\ Y\big\rangle_{\mathcal{H}} = -\int_0^1 k_1(\xi)|w|^2 d\xi \le 0.$$

So the operator $\mathcal{A}+\mathcal{B}$ defined by (10)-(12) is dissipative and hence the real part of all the spectrum is located on the left half plane, i.e., $\mathrm{Re}\big(\lambda(\mathcal{A}+\mathcal{B})\big)\le 0$. Furthermore, let $\lambda := i\tau$, $0\ne\tau\in\mathbb{R}$ be an eigenvalue of $\mathcal{A}+\mathcal{B}$ and let Y be its corresponding eigenfunction. Then it follows from above that

$$0\equiv \mathrm{Re}\big\langle(\mathcal{A}+\mathcal{B})Y,\ Y\big\rangle_{\mathcal{H}} = -\int_0^1 k_1(\xi)|w|^2 d\xi,$$

and hence $w\equiv 0$ in I and so is $w\equiv 0$ in $[0,1]$. By $(\mathcal{A}+\mathcal{B})Y = i\tau Y$, we further have that $z\equiv 0$ and

$$\begin{cases} u''(x)=0, & 0<x<1,\\ \rho_u\tau^2 u(x)+a_3u''(x)=0, & 0<x<1,\\ u(0)=u'(1)=0. \end{cases}$$

By a direct computation, we obtain $v=u\equiv 0$. Thus $Y\equiv 0$ that contradicts $\lambda=i\tau$ being an eigenvalue of $\mathcal{A}+\mathcal{B}$. Therefore, there is no eigenvalue on the imaginary axis and hence the system (13) is exponentially stable.

Theorem 4.3: Let $\mathcal{A}+\mathcal{B}$ be defined by (10)-(12). Suppose condition (68) is satisfied and $k_1(x)\ge 0$ with $k_1(x)\Big|_I > 0$. Then the system (13) is exponentially stable.

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