Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Monte-Carlo Option-Pricing Algorithm for Log-Uniform Jump-Diffusion Model†

Zongwu Zhu* and Floyd B. Hanson*

Abstract— **A reduced European call option pricing formula by risk-neutral valuation is given. It is shown that the European call and put options for jump-diffusion models are worth more than that for the Black-Scholes (diffusion) model with the common parameters. Due to the complexity of the jump-diffusion models, obtaining a closed option pricing formula like that of Black-Scholes is not viable. Instead, a Monte Carlo algorithm is used to compute European option prices. Monte Carlo variance reduction techniques such as both antithetic and control variates are used. The numerical results show that this is a practical, efficient and easily implementable algorithm.**

I. Background

Despite the great success of Black-Scholes options model [2], it suffers from many defects, one defect is quite obvious during market crashes or massive buying frenzies which contradict the continuity properties of the underlying geometric diffusion process. For statistical evidence of jumps in various financial markets see Ball and Torous [1], Jarrow and Rosenfeld [12] or Jorion [13]. Hence, some jump-diffusion models was proposed including Merton's log-normal [16], Kou and Wang's log-double-exponential [14], Hanson and Westman's log-uniform [6] and Zhu and Hanson's log-double-uniform [17] jump-diffusion mdoels. The model to govern the dynamics of the asset price $S(t)$ is the stochastic differential equation (SDE) :

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + J(Q)dN(t)\right), \tag{1}$$

where $S_0 = S(0) > 0$, μ is the drift coefficient, σ is the diffusive volatility, $W(t)$ is a Wiener process, $J(Q)$ is the jump-amplitude, Q is an underlying amplitude mark process such that $Q = \ln(J(Q)+1)$, $N(t)$ is the standard Poisson jump counting process with joint mean and variance $\mathrm{E}[N(t)] = \lambda t = \mathrm{Var}[N(t)]$. The jump term in (1) is a symbol for $S(t)J(Q)dN(t) = \sum_{k=1}^{dN(t)} S(T_k^-)J(Q_k)$, where T_k is the kth jump time, Q_k is the kth mark and $S(T_k^-) = \lim_{t\uparrow T_k} S(t)$.

Let the jump-amplitude mark density be uniform:

$$\phi_Q(q) = \frac{1}{b-a}\left\{\begin{array}{ll} 1, & a \le q \le b \\ 0, & \text{else} \end{array}\right\}, \tag{2}$$

where $a < 0 < b$. The mark Q has mean $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$ and variance $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12$. The jump-amplitude J has mean

$$\bar{J} \equiv \mathrm{E}[J(Q)] = (\exp(b) - \exp(a))/(b-a) - 1. \tag{3}$$

Note that in absence of any special explanation, $\overline{X}$ will denote the mean of random variable X, that is, $\overline{X} = E[X]$. For more details, see [7] and [9].

By the Itô chain rule [8] for jump-diffusions, the log-return process $\ln(S(t))$ satisfies the constant coefficient SDE

$$d\ln(S(t)) = (\mu - \sigma^2/2)dt + \sigma dW(t) + QdN(t)\,,$$

which can be immediately integrated and the logarithm inverted to yield the stock price solution

$$S(t) = S_0 \exp((\mu - \sigma^2/2)t + \sigma W(t) + QN(t)), \tag{4}$$

where $QN(t) = \sum_{k=1}^{N(t)} Q_k$, but is zero if $N(t) = 0$, and the Q_k here are independent identically uniformly distributed jump-amplitude marks See the jump-diffusion book [8, Chapter 5].

Our objective is to derive a reduced formula and practical algorithm for the discounted, expected European call option price $\mathcal{C}(S_0, T)$, which is a function of the current stock price S_0 and the option expiration time T. There are also suppressed arguments like the strike price K, the stock volatility σ and the risk-free interest rate r, but for jump-diffusions also depends on the jump rate λ and the mean jump amplitude $\bar{J}$. In contrast to the Black-Scholes [2] hedge for constructing a portfolio to eliminate the diffusion in the case of a pure diffusion process, Merton [16] argued that such hedging was not possible in the case of the jump-diffusion model, but the risk-neutral part of the Black-Scholes strategy could preserve the no arbitrage strategy to ensure that the discounted, expected return would be at the risk-free rate r. Consequently, the European call option price can be formulated as the discounted expectation of the terminal claim $\max[S(T) - K, 0]$,

$$\mathcal{C}(S_0, T) \equiv e^{-rT}\hat{\mathrm{E}}[\max[S(T) - K, 0]]\,, \tag{5}$$

where, $\hat{\mathrm{E}}$ denotes the expected value in a risk-neutral world [11].

II. Risk-Neutral Constant-Coefficient SDE

By the equation (4), the expected stock price at expiration time T is found in the following theorem:

Theorem 2.1: The **Expected Stock Price** is

$$\mathrm{E}[S(t)] = S_0 e^{(\mu + \lambda\bar{J})t}. \tag{6}$$

†The work is supported by the National Science Foundation under Grant DMS-02-07081. The content of this material is that of the authors and does not necessarily reflect the views of the National Science Foundation.

*Department of Mathematics, Statistics, and Computer Science, M/C 249, University of Illinois, Chicago, IL 60607-7045 hanson@math.uic.edu and zzhu@math.uic.edu

0-7803-9568-9/05/$20.00 ©2005 IEEE

Proof: Using the stock price solution (4), the IID property of Q_k given a jump in $N(t)$ and iterated expectations,

$$\begin{aligned}
\mathrm{E}[S(t)] &= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}\left[e^{\sigma W(t)} e^{\sum_{i=1}^{N(t)} Q_i}\right] \\
&= S_0 e^{(\mu-\sigma^2/2)t}\mathrm{E}_W\left[e^{\sigma W(t)}\right]\mathrm{E}_{N,Q}\left[\prod_{i=0}^{N(t)} e^{Q_i}\right] \\
&= S_0 e^{(\mu-\sigma^2/2)t} e^{\sigma^2 t/2}\mathrm{E}_{N,Q}\left[\prod_{i=1}^{N(t)} e^{Q_i}\right] \\
&= S_0 e^{\mu t}\mathrm{E}_N\left[\mathrm{E}_{Q|N}\left[\prod_{i=1}^{N(t)} e^{Q_i}\middle| N(t)\right]\right] \\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty} p_k \mathrm{E}\left[\prod_{i=1}^{k} e^{Q_i}\right] = S_0 e^{\mu t}\sum_{k=0}^{\infty} p_k \prod_{i=1}^{k}\mathrm{E}\left[e^{Q}\right] \\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty} p_k \mathrm{E}^k[J(Q)+1] \\
&= S_0 e^{\mu t}\sum_{k=0}^{\infty} e^{-\lambda t}\frac{\left(\lambda t\left(\bar{J}+1\right)\right)^k}{k!} = S_0 e^{(\mu+\lambda\bar{J})t},
\end{aligned}$$

where the Poisson distribution $p_k(\lambda t) \equiv e^{-\lambda t}(\lambda t)^k/k!$ has been used. □

Assume the source of the jumps is due to extraordinary changes in the firm's specifics, such as the loss of a court suit or bankruptcy, but not from external events such as war. Thus, such jump components in the jump-diffusion model represent only non-systematic risks. The correlation *beta* of the portfolio for non-systematic risk is constructed by *delta* hedging as in Black-Scholes and is zero (see [16]). In the risk-neutral world, $\mathrm{E}[S(t)] = S_0 e^{rt}$, so $S_0 e^{(\mu+\lambda\bar{J})t} = S_0 e^{rt}$ and solving for μ, yields the risk-neutral appreciation rate, $\mu = \mu_{\mathrm{rn}} = r - \lambda\bar{J}$. In the more general case with time-dependent coefficients, the expected instant rate is the risk-free rate and $\mathrm{E}[dS(t)/S(t)] = (\mu(t) + \mathrm{E}[J(Q,t)]\lambda(t))dt = r(t)dt$, leading to the risk-neutral mean rate relationship $\mu(t) = \mu_{\mathrm{rn}}(t) = r(t) - \mathrm{E}[J(Q,t)]\lambda(t)$.

Back to the constant coefficient case and substituting $\mu = r - \lambda\bar{J}$ into (1), we get the risk-neutral SDE under the risk-neutral measure $\mathcal{M}$ as the following:

$$\begin{aligned}
dS(t)/S(t) &= \left(r-\lambda\bar{J}\right)dt + \sigma dW(t) + \sum_{k=1}^{dN(t)} J(Q_k) \\
&= r dt + \sigma dW(t) + \sum_{k=1}^{dN(t)}\left(J(Q_k)-\bar{J}\right) \\
&\quad + \bar{J}\left(dN(t)-\lambda dt\right),
\end{aligned}$$

where the jump terms are separated into the zero-mean forms of the compound Poisson process.

III. Risk-Neutral Option Price Solutions

Using risk-neutral valuation of the payoff for the European call option in (5) with the stock price solution (4) and risk-neutral drift,

$$\begin{aligned}
\mathcal{C}(S_0,T) &\equiv e^{-rT}\hat{\mathrm{E}}[\max(S(T)-K,0)] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\int_{ka}^{kb}\int_{Z_0(s_k)}^{\infty}\left(S_0 e^{DJ(z,s_k)} - K\right) \\
&\quad \cdot e^{-z^2/2}\phi_{\widetilde{\mathcal{S}}_k}(s_k)\,dz\,ds_k \\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[\int_{Z_0(\widetilde{\mathcal{S}}_k)}^{\infty}\left(S_0 e^{DJ(z,s_k)-rT}\right.\right. \\
&\quad \left.\left. -Ke^{-rT}\right)e^{-z^2/2}dz\right],
\end{aligned}$$

where $DJ(z,s_k) \equiv (r-\lambda\bar{J}-\sigma^2/2)T + \sigma\sqrt{T}z + s_k$, $Z_0(s) \equiv (\ln(K/S_0) - (r-\lambda\bar{J}-\sigma^2/2)T - s)/(\sigma\sqrt{T})$ is the *at-the-money* value of the normal variable of integration z and $\widetilde{\mathcal{S}}_k = \sum_{i=1}^{k} Q_i$ is the sum of k jump amplitudes, such that Q_i are uniformly distributed IID random variables over the interval $[a,b]$ but $\widetilde{\mathcal{S}}_0 = \sum_{i=1}^{0} Q_i \equiv 0$. Splitting up the integral term, let

$$\begin{aligned}
A(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty} S_0 e^{-(\lambda\bar{J}+\sigma^2/2)T+\sigma\sqrt{T}z+s}e^{-z^2/2}dz \\
&= S_0 e^{s-\lambda\bar{J}T}\Phi\left(d_1\left(S_0 e^{s-\lambda\bar{J}T}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
B(s) &\equiv \frac{1}{\sqrt{2\pi}}\int_{Z_0(s)}^{\infty} Ke^{-rT}e^{-z^2/2}dz \\
&= Ke^{-rT}\Phi\left(d_2\left(S_0 e^{s-\lambda\bar{J}T}\right)\right),
\end{aligned}$$

where $d_1(x) \equiv (\ln(x/K) + (r+\sigma^2/2)T)/(\sigma\sqrt{T})$ and $d_2(x) \equiv d_1(x) - \sigma\sqrt{T}$ are the usual Black-Scholes normal distribution argument functions, while $\Phi(y) \equiv \int_{-\infty}^{y} e^{-z^2/2}dz/\sqrt{2\pi}$ is the standardized normal distribution. Therefore,

$$\begin{aligned}
\mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}[A(\widetilde{\mathcal{S}}_k) - B(\widetilde{\mathcal{S}}_k)] \\
&= \sum_{k=0}^{\infty} p_k(\lambda T)\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\Phi\left(d_1\left(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\right)\right)\right. \\
&\quad \left. -Ke^{-rT}\Phi\left(d_2\left(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}\right)\right)\right].
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mathcal{C}(S_0,T) &= \sum_{k=0}^{\infty} p_k(\lambda T) \\
&\quad \cdot\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[\mathcal{C}^{(BS)}\left(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}, T; K, \sigma^2, r\right)\right],
\end{aligned} \tag{7}$$

where

$$\mathcal{C}^{(BS)}(x,T;K,\sigma^2,r) \equiv x\Phi(d_1(x)) - Ke^{-rT}\Phi(d_2(x))$$

or briefly $\mathcal{C}^{(BS)}(x,T)$, is the Black-Scholes formula [2], but with the stock price argument shifted by a jump factor $\exp(\widetilde{\mathcal{S}}_k - \lambda\bar{J}T)$. The above equation agrees with Merton's formula (16) in [16].

The next step is to compute

$$\mathrm{E}_{\widetilde{\mathcal{S}}_k}\left[\mathcal{C}^{(BS)}\left(S_0 e^{\widetilde{\mathcal{S}}_k-\lambda\bar{J}T}, T, K, \sigma^2, r\right)\right].$$

However, producing a simple analytical solution is difficult, since the probability density of the partial sums $\widetilde{\mathcal{S}}_k$ for the log-uniform model is very complicated, so this problem will be solved by high-level simulation techniques.

A. Put-Call Parity

Put-call parity is founded on basic maximum function properties (Merton [15], Hull [11] and Higham [10]), so is independent of the particular process and

$$\mathcal{C}(S_0,T)+Ke^{-rT}=\mathcal{P}(S_0,T)+S_0 \tag{8}$$

or solving for the European put option price,

$$\mathcal{P}(S_0,T)=\mathcal{C}(S_0,T)+Ke^{-rT}-S_0, \tag{9}$$

in absence of dividends.

IV. A Monte Carlo Algorithm

From (7), the European call option price formulae can be equivalently written as

$$\mathcal{C}(S_0,T)=\mathrm{E}_{\gamma(T)}\Big[\mathcal{C}^{(BS)}\Big(S_0e^{\gamma(T)-\lambda\bar{J}T},T\Big)\Big], \tag{10}$$

where $\gamma(T)=\sum_{i=1}^{N(T)}Q_i$, Q_i are uniformly distributed IID random variables from $[a,b]$. Note if $\widehat{\gamma}(T)\equiv\gamma(T)-\lambda T\bar{J}$, then $\exp(\widehat{\gamma}(T))$ is an exponential compound Poisson process with the exponential martingale property on $[0,T]$ that $\mathrm{E}[\exp(\widehat{\gamma}(T))]=\exp(\widehat{\gamma}(0))=1$. The Monte Carlo method may be a good choice to compute it numerically. For the treatment of Monte Carlo methods, see, e.g., [4], [5] or [10].

Let N_i be a sample point taken from the same Poisson distribution as $N(T)$, so that N_i for $i=1:n$ sample points form a set of IID Poisson variates. Given an N_i jumps, let the $U_{i,j}$ for $j=1:N_i$ be jump amplitude sample points, so that they are IID uniformly generated on [0, 1], then

$$\gamma_i=\sum_{j=1}^{N_i}(a+(b-a)U_{i,j})=aN_i+(b-a)\sum_{j=1}^{N_i}U_{i,j}$$

for $i=1:n$ will be a set of IID random variables on [a, b] having the same compound Poisson distribution with uniformly distributed jump amplitudes as $\gamma(T)$. Based upon (10), an elementary Monte Carlo estimate (EMCE) for $\mathcal{C}(S_0,T)$ is

$$\widehat{\mathcal{C}}_n=v\frac{1}{n}\sum_{i=1}^{n}\mathcal{C}^{(BS)}\left(S_0e^{\gamma_i-\lambda\bar{J}T},T\right)\equiv\frac{1}{n}\sum_{i=1}^{n}\mathcal{C}_i^{(BS)},$$

such that the $\mathcal{C}_i^{(BS)}$ are IID random variables based on γ_i. Then, by the strong law of large numbers,

$$\widehat{\mathcal{C}}_n\to\mathcal{C}(S_0,T)\quad\text{with probability one as}\quad n\to\infty,$$

and by the IID property of $\mathcal{C}_i^{(BS)}$, the standard deviation $\sigma_{\widehat{\mathcal{C}}_n}$ is equal to $\sigma^{(BS)}/\sqrt{n}$, where

$$\sigma^{(BS)}=\sqrt{\mathrm{Var}\left[\mathcal{C}^{(BS)}(S_0e^{\gamma(T)-\lambda\bar{J}T},T)\right]}=\sqrt{\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]},$$

but may be estimated by the unbiased sample variance

$$s^{(BS)}=\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\mathcal{C}_i^{(BS)}-\widehat{\mathcal{C}}_n\right)^2}.$$

In order to reduce the standard deviation $\sigma_{\widehat{\mathcal{C}}_n}$ by a factor of ten, the number of simulations n has to be increased one hundredfold. However, there are alternative Monte Carlo methods which can have smaller variance than that of EMCE by variance reduction techniques.

If $U_{i,j}$ is uniformly distributed on $[0,1]$, then the $\widehat{Q}_{i,j}=a+(b-a)U_{i,j}$ are uniformly distributed (thetic) random variables from $[a,b]$ and so also are the antithetic counterparts $\widehat{Q}_{i,j}^{(a)}=a+(b-a)(1-U_{i,j})$. Hence,

$$\gamma_i^{(a)}\quad=\quad(b+a)N_i-\gamma_i, \tag{11}$$

for $i=1:n$, are IID random variables having the same compound Poisson distribution with uniformly distributed jump amplitudes as $\gamma(T)$. So, the antithetic variates method, first applied to finance by Boyle [3] (see also [4] and [5] for more recent and expanded treatments), can be used. Furthermore, we notice that the variable $\exp(\gamma(T))$ has the expectation $\exp(\lambda T\bar{J})$ known from the proof of Theorem 2.1 and has positive correlation with $\mathcal{C}^{(BS)}\left(S_0e^{\gamma(T)-\lambda\bar{J}T},T;K,\sigma^2,r\right)$. Therefore, the control variates technique can be used to further reduce the variance of Monte Carlo estimation since it works faster the higher the correlation between the paired target and control variates, provided that the mean of the control variate is known [5]. The control variates technique was also first used by Boyle [3] for financial applications.

Thus, the Monte Carlo simulations will be used with antithetic variate and control variate variance reduction techniques. Let

$$X_i=\frac{1}{2}\Big(\mathcal{C}^{(BS)}\Big(S_0e^{\gamma_i-\lambda\bar{J}T},T\Big)+\mathcal{C}^{(BS)}\Big(S_0e^{\gamma_i^{(a)}-\lambda\bar{J}T},T\Big)\Big)$$
$$\equiv 0.5\left(\mathcal{C}_i^{(BS)}+\mathcal{C}_i^{(aBS)}\right),$$

for $i=1:n$ is the thetic-antithetic averaged, Black-Scholes risk-neutral, discounted payoff and

$$Y_i=0.5\left(\exp(\gamma_i)+\exp\left(\gamma_i^{(a)}\right)\right).$$

is the thetic-antithetic averaged jump factors and a variance reducing control variate. The control adjusted payoff is

$$Z_i(\alpha)=X_i-\alpha\cdot(Y_i-\exp(\lambda T\bar{J})),$$

where $(Y_i-\exp(\lambda T\bar{J}))$ is the control deviation and α is an adjustable control parameter. The sample mean of $Z_i(\alpha)$ produces the Monte Carlo estimator for $\mathcal{C}(S_0,T)$: $\overline{Z}_n(\alpha)=\sum_{i=1}^{n}Z_i(\alpha)/n=\sum_{i=1}^{n}X_i/n-\alpha\sum_{i=1}^{n}(Y_i-\exp(\lambda T\bar{J}))/n=\overline{X}_n-\alpha(\overline{Y}_n-\exp(\lambda T\bar{J}))$, an unbiased estimation since $\mathrm{E}[\overline{Z}_n(\alpha)]=\mathcal{C}(S_0,T)$ using IID mean properties $\mathrm{E}[\overline{X}_n]=\mathrm{E}[X_i]=\mathcal{C}(S_0,T)$ by (10) and $\mathrm{E}[\overline{Y}_n]=\mathrm{E}[Y_i]=\exp(\lambda T\bar{J})$ from the proof of Thm. 2.1.

The variance of the sample mean $\overline{Z}_n(\alpha)$ is

$$\sigma^2_{\overline{Z}_n(\alpha)}\equiv\mathrm{Var}\left[\overline{Z}_n(\alpha)\right]=\mathrm{Var}[Z_i(\alpha)]/n,$$

following from IID property of the $Z_i(\alpha)$. However,

$$\mathrm{Var}[Z_i(\alpha)]=\mathrm{Var}[X_i]-2\alpha\mathrm{Cov}[X_i,Y_i]+\alpha^2\mathrm{Var}[Y_i].$$

So, the optimal parameter α^* to minimize $\mathrm{Var}[Z_i(\alpha)]$ is

$$\alpha^*=\mathrm{Cov}[X_i,Y_i]/\mathrm{Var}[Y_i]. \tag{12}$$

Using this optimal parameter α^*,

$$\begin{aligned}\mathrm{Var}[Z_i^*] &\equiv \mathrm{Var}[Z_i(\alpha^*)]=\mathrm{Var}[X_i]-\frac{\mathrm{Cov}^2[X_i,Y_i]}{\mathrm{Var}[Y_i]}\\ &\equiv \left(1-\rho^2_{X_i,Y_i}\right)\mathrm{Var}[X_i],\end{aligned}$$

where ρ_{X_i,Y_i} is the correlation coefficient between X_i and Y_i. We also know that

$$\begin{aligned}\mathrm{Var}[X_i] &= \frac{1}{4}\bigg(\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]+2\mathrm{Cov}\left[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}\right]\\ &\quad+\mathrm{Var}\left[\mathcal{C}_i^{(aBS)}\right]\bigg)\\ &= \frac{1}{2}\left(1+\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\right)\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\end{aligned}$$

because $\mathrm{Var}\left[\mathcal{C}_i^{(aBS)}\right]=\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]$. Therefore,

$$\begin{aligned}\mathrm{Var}[Z_i^*] &= \frac{1}{2}\left(1-\rho^2_{X_i,Y_i}\right)\left(1+\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\right)\\ &\quad\cdot\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\le\frac{1}{2}\mathrm{Var}\left[\mathcal{C}_i^{(BS)}\right]\end{aligned} \tag{13}$$

because $\rho^2_{X_i,Y_i}\ge 0$ and provided $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\le 0$. From (13), $\sigma^2_{\widehat{Z}_n}\le\mathrm{Var}[\mathcal{C}_i^{(BS)}]/(2n)=(\sigma_{\widehat{\mathcal{C}}_n})^2/2$. This says the variance of the Monte Carlo estimate with antithetic and control variates techniques is at most the half as the variance of the elementary Monte Carlo estimate if $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}\le 0$.

Remark: In a real market, the ratio a/b will be close to -1, that is $b+a$ will be very small since the skewness of the daily return distribution is not far away from 0 and the skewness is generated by the jump part of the jump-diffusion model. For example, the skewness is -0.1952 for 1988-2003 S&P 500 daily return market data and $a/b=-1.08$ and $a+b=-0.002$ [17]. In fact, in our Monte-carlo algorithm, the $\rho_{\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}}$ is about -0.83. So, we can get a lot of benefit from the antithetic variate variance reduction method by equation (13). In fact, our simulations using uniformly distributed jump amplitudes confirms that $\mathrm{Cov}[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}]<0$ in the range of the ratio $-3.75<a/b<-0.25$ with $a=-0.028$ which is well within the range of market data. However, if b/a is far away from -1, the correlation coefficient $\mathrm{Cov}[\mathcal{C}_i^{(BS)},\mathcal{C}_i^{(aBS)}]$ can be positive which will worsen the variance, though this range is not realistic.

In general, we do not know the parameter α^* exactly, so some estimation is needed for it and we need the following Lemma.

Lemma 4.1:

$$\mathrm{Var}\left[e^{\gamma_i}+e^{\gamma_i^{(a)}}\right]=2\left(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)}\right),$$

where $\hat{J}=(\exp(2b)-\exp(2a))/(2(b-a))-1$ and $\bar{J}=(\exp(b)-\exp(a))/(b-a)-1$ from (3).

Proof: Using the properties of the antithetic pair $(\gamma_i,\gamma_i^{(a)})$,

$$\begin{aligned}\mathrm{Cov}\left[e^{\gamma_i},e^{\gamma_i^{(a)}}\right] &= \mathrm{E}\left[e^{\gamma_i}e^{\gamma_i^{(a)}}\right]-\mathrm{E}\left[e^{\gamma_i}\right]E\left[e^{\gamma_i^{(a)}}\right]\\ &= \mathrm{E}\left[e^{(a+b)N(T)}\right]-\mathrm{E}^2\left[e^{\gamma_i}\right]\\ &= e^{\lambda T(e^{a+b}-1)}-e^{2\lambda T\bar{J}}\end{aligned}$$

and $\mathrm{Var}[e^{\gamma_i}]=\mathrm{E}[e^{2\gamma_i}]-\mathrm{E}^2[e^{\gamma_i}]=e^{\lambda T\hat{J}}-e^{2\lambda T\bar{J}}=\mathrm{Var}[e^{\gamma_i^{(a)}}]$. Thus, $\mathrm{Var}[e^{\gamma_i}+e^{\gamma_i^{(a)}}]=\mathrm{Var}[e^{\gamma_i}]+2\mathrm{Cov}[e^{\gamma_i},e^{\gamma_i^{(a)}}]+\mathrm{Var}[e^{\gamma_i^{(a)}}]=2\mathrm{Var}[e^{\gamma_i}]+2\mathrm{Cov}[e^{\gamma_i},e^{\gamma_i^{(a)}}]=2(e^{\lambda T\hat{J}}-2e^{2\lambda T\bar{J}}+e^{\lambda T(e^{a+b}-1)})$. □

From Lemma 4.1, $\sigma_Y^2\equiv\mathrm{Var}[Y_i]=\mathrm{Var}[0.5(\exp(\gamma_i)+\exp(\gamma_i^{(a)}))]=0.5(\exp(\lambda T\hat{J})-2\exp(2\lambda T\bar{J})+\exp(\lambda T(\exp(a+b)-1)))$.

Proposition 4.1: An unbiased estimator for α^* is

$$\begin{aligned}\widehat{\alpha} &= \left(\frac{1}{n-1}\sum_{i=1}^{n}X_iY_i-\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}X_iY_j\right)\frac{1}{\sigma_Y^2}\\ &= \frac{n}{n-1}\frac{\overline{XY}_n-\overline{X}_n\overline{Y}_n}{\sigma_Y^2},\end{aligned} \tag{14}$$

where $\overline{X}_n=\sum_{i=1}^{n}X_i/n$ is the sample mean, $\overline{XY}_n$ and $\overline{Y}_n$ have the similar meaning.

Proof: It is necessary to show the condition for an unbiased estimate $E[\widehat{\alpha}]=\alpha^*$ is true. Splitting the common part out of the double sum and the IID property of the random variables at different compound Poisson sample points for $i=1\!:\!n$,

$$\begin{aligned}\mathrm{E}[\widehat{\alpha}] &= \mathrm{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i-\frac{1}{n}\sum_{j=1}^{n}X_iY_j\right)\frac{1}{\sigma_Y^2}\right]\\ &= \frac{1}{n-1}\sum_{i=1}^{n}\mathrm{E}\left[\left(1-\frac{1}{n}\right)X_iY_i-\frac{1}{n}\sum_{j=1,j\ne i}^{n}X_iY_j\right]\frac{1}{\sigma_Y^2}\\ &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\left((n-1)\mathrm{E}[X_iY_i]-\sum_{j=1,j\ne i}^{n}\mathrm{E}[X_i]\mathrm{E}[Y_j]\right)\frac{1}{\sigma_Y^2}\\ &= \left(\mathrm{E}[XY]-\mathrm{E}[X]\mathrm{E}[Y]\right)/\sigma_Y^2=\mathrm{Cov}[X,Y]/\sigma_Y^2=\alpha^*.\end{aligned}$$

□

Since $\widehat{\alpha}$ depends on Y_i for $i=1\!:\!n$, the estimate $\widehat{\alpha}$ of α^* introduces a bias into the estimate

$$\widehat{Z}_n=\frac{1}{n}\sum_{i=1}^{n}X_i-\widehat{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n}Y_i-e^{\lambda T\bar{J}}\right). \tag{15}$$

Fortunately, we can compute the bias which asymptoticly goes to zero at the rate $O(1/n)$ as shown in the following theorem.

Theorem 4.1: The estimate $\widehat{Z}_n$ of $\mathcal{C}(S_0,T)$ has bias

$$b\equiv\mathrm{E}[\widehat{Z}_n]-\mathcal{C}(S_0,T)=\mathrm{Cov}[X,(2\mu_Y-Y])Y]]/(n\sigma_Y^2),$$

where $\mu_Y=\mathrm{E}[Y_i]=E[Y]=\exp(\lambda T\bar{J})$, $\sigma_Y^2=\mathrm{Var}[Y_i]=\mathrm{Var}[Y]$, Y has the same distribution as Y_i, for $i=1\!:\!n$.

Proof: Set $\eta_k = \sigma_Y^2 \widehat{\alpha}(Y_k - \mu_Y)$. Then,

$$\begin{aligned}\eta_k &= \left(\frac{\sum_{i=1}^n X_i Y_i}{n-1} - \frac{\sum_{i=1}^n \sum_{j=1}^n X_i Y_j}{n(n-1)}\right)(Y_k - \mu_Y) \\ &= \frac{1}{n}\sum_{i=1}^n X_i Y_i Y_k - \frac{\sum_{i=1}^n \sum_{j\neq i} X_i Y_j Y_k}{n(n-1)} \\ &\quad - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n} + \frac{\mu_Y \sum_{i=1}^n \sum_{j\neq i} X_i Y_j}{n(n-1)} \\ &= \frac{X_k Y_k^2 + \sum_{i\neq k} X_i Y_i Y_k}{n} - \\ &\frac{\sum_{j\neq k} X_k Y_j Y_k + \sum_{i\neq k} X_i Y_k^2 + \sum_{i\neq k}\sum_{j\neq i,k} X_i Y_j Y_k}{n(n-1)} \\ &\quad - \frac{\mu_Y \sum_{i=1}^n X_i Y_i}{n} + \frac{\mu_Y \sum_{i=1}^n \sum_{j\neq i} X_i Y_j}{n(n-1)}.\end{aligned}$$

By the independence of $\{X_i, Y_i\}$ and $\{X_j, Y_j\}$ for $j \neq i$ but with identical distributions, $\mathrm{E}[\eta_k] = (\overline{XY^2} + (n-1)\overline{XY}\mu_Y)/n - ((n-1)\overline{XY}\mu_Y + (n-1)\mu_X\overline{Y^2} + (n-1)(n-2)\mu_X {\mu_Y}^2)/(n(n-1)) - \mu_Y\overline{XY} + \mu_Y^2\mu_X = (\overline{XY^2} - 2\overline{XY}\mu_Y - \mu_X\overline{Y^2} + 2\mu_X\mu_Y^2)/n = \mathrm{Cov}[X, Y^2] - 2\mu_Y \mathrm{Cov}[X, Y]/n = \mathrm{Cov}[X, Y(Y - 2\mu_Y)]/n$, where $\mu_X = \mathrm{E}[X_i]$, $\mu_Y = \mathrm{E}[Y_i]$, $\overline{XY} = \mathrm{E}[X_i Y_i]$, $\overline{Y^2} = \mathrm{E}[Y_i^2]$ and $\overline{XY^2} = \mathrm{E}[X_i Y_i^2]$. Therefore, the bias $b \equiv \mathrm{E}[\widehat{Z}_n] - \mathcal{C}(S_0, T) = \mathrm{E}[-\widehat{\alpha}(Y_k - \mu_Y)] = -\mathrm{E}[\sigma_Y^2 \widehat{\alpha}(Y_k - \mu_Y)]/\sigma_Y^2 = -\mathrm{E}[\eta_k]/\sigma_Y^2 = \mathrm{Cov}[X, Y(2\mu_Y - Y)]/(n\sigma_Y^2)$. □

Remark: From Theorem 4.1, the corrected estimate to $\widehat{Z}_n$ is $\widehat{\theta} \equiv \widehat{Z}_n - \widehat{b}$, where $\widehat{b}$ is an estimate of b similar to $\widehat{\alpha}$ in (14) ,

$$\begin{aligned}\widehat{b} &= \left(\frac{1}{n(n-1)}\sum_{i=1}^n X_i Y_i' - \frac{1}{n^2(n-1)}\sum_{i=1}^n\sum_{j=1}^n X_i Y_j'\right)\frac{1}{\sigma_Y^2} \\ &= \frac{1}{n-1}\frac{\overline{XY'_n} - \overline{X}_n \overline{Y'}_n}{\sigma_Y^2}, \qquad (16)\end{aligned}$$

where $Y_i' = Y_i(2\mu_Y - Y_i)$, for $i = 1 : n$, $\overline{XY'_n}$, $\overline{X}_n$ and $\overline{Y'}_n$ are sample means. Then, the estimate $\widehat{\theta}$ is an unbiased estimate of $\mathcal{C}(S_0, T)$.

Finally, our Monte Carlo algorithm with antithetic and control variates variance reduction techniques is:

The Monte Carlo Algorithm:

for $i = 1 : n$
 Randomly generate N_i;
 Randomly generate IID $U_{i,j}$, $j = 1 : N_i$;
 Set $\gamma_i = aN_i + (b-a)\sum_{j=1}^{N_i} U_{i,j}$;
 Set $\gamma_i^{(a)} = (a+b)N_i - \gamma_i$;
 Set $\mathcal{C}_i^{(BS)} = \mathcal{C}^{(BS)}\left(S_0 \exp\left(\gamma_i - \lambda T \bar{J}\right), T\right)$;
 Set $\mathcal{C}_i^{(aBS)} = \mathcal{C}^{(BS)}\left(S_0 \exp\left(\gamma_i^{(a)} - \lambda T \bar{J}\right), T\right)$;
 Set $X_i = 0.5\left(\mathcal{C}_i^{(BS)} + \mathcal{C}_i^{(aBS)}\right)$;
 Set $Y_i = 0.5\left(\exp(\gamma_i) + \exp\left(\gamma_i^{(a)}\right)\right)$;
end for i
Compute $\widehat{\alpha}$ according to (14);
Set $\widehat{Z}_n = \frac{1}{n}\sum_{i=1}^n X_i - \widehat{\alpha}(\frac{1}{n}\sum_{i=1}^n Y_i - e^{\lambda T \bar{J}})$;
Estimate bias $\widehat{b}$ according to (16);
Get European call $\widehat{\theta} = \widehat{Z}_n - \widehat{b}$;
Get European put $\widehat{\mathcal{P}}$ by (9).

V. Numerical Results and Discussions

In this section, some numerical results and discussions are given to illustrate the Monte Carlo algorithm. First of all, the elementary Monte Carlo method and the Monte Carlo method with antithetic and control variates techniques (abbreviated as AOCV) are compared. The compound Poisson process is simulated by first using the inverse transform method given by Glasserman ([5]) for the jump counting component process N_i and then the N_i jump amplitude antithetic pairs $(\gamma_i, \gamma_i^{(a)})$ are simulated by a standard uniform random number generator to get the $U_{i,j}$. These are implemented using MATLAB 6.5 and run them on the PC with a Pentium4@1.6GHz CPU. The numerical test results for elementary Monte Carlo method are listed in Table I and the Monte carlo with AOCV's are listed in Table II.

TABLE I

Numerical Results of elementary Monte Carlo Method

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.76	0.67	0.055	2.640	0.090
	1.0	5.26	3.28	0.035	2.578	0.056
	1.1	1.38	8.49	0.014	2.562	0.022
0.4	0.9	15.99	2.90	0.048	2.562	0.077
	1.0	8.45	6.47	0.033	2.578	0.053
	1.1	4.07	11.18	0.020	2.531	0.032
0.6	0.9	19.15	6.03	0.044	2.454	0.069
	1.0	11.79	9.81	0.033	2.500	0.052
	1.1	7.09	14.21	0.023	2.500	0.036

Option parameters: $K = 100$, $r = 0.1$, $T = 0.2$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation number $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{C}_n} = \sigma^{(BS)}/\sqrt{n}$.

TABLE II

Numerical Results of Improved Monte Carlo with AOCV

σ	K/S_0	$\mathcal{C}$	$\mathcal{P}$	ϵ	t (sec.)	$\epsilon\sqrt{t}$
0.2	0.9	13.73	0.64	0.004	6.875	0.011
	1.0	5.23	3.25	0.008	6.828	0.021
	1.1	1.38	8.49	0.006	6.781	0.016
0.4	0.9	16.03	2.94	0.004	7.031	0.011
	1.0	8.42	6.44	0.004	6.922	0.011
	1.1	4.06	11.17	0.004	7.218	0.011
0.6	0.9	19.11	6.02	0.003	6.797	0.008
	1.0	11.81	9.83	0.003	6.859	0.008
	1.1	7.12	14.23	0.003	6.812	0.008

Option parameters: $K = 100$, $r = 0.1$, $T = 0.2$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation number $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$.

The results in Table I and Table II show that the antithetic variates combined with control variates can reduce the standard error by a factor ranging from 2 to about 14, but also increase the computing time by 2 to 3 times. Therefore, we use standard error multiplying square root of computng time $\epsilon\sqrt{t}$ as a benchmark for the trade-off between the estimated variance and computing time, for a detailed explanation, see Boyle, Broadie and Glasserman [4]. Seen from these results, the Monte Carlo method with AOCV is an overall the better estimate than the elementary Monte Carlo method. Also, these results show that the European call option price is an increasing function of S_0 and the European put option is a decreasing function of it. Both the call and put option prices are increased as the volatility σ of stock price is increased. The estimated model parameters used are $\mu = 0.1626$, $\sigma = 0.1074$, $\lambda = 64.16$, $a = -0.028$, $b = 0.026$ from our double-unform distribution paper [17] to compute the Standard & Poor 500 index option prices. Also, we compute Black-Scholes call price $\mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r)$ and the put price $\mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r) = \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r) + K\exp(-rT) - S_0$ as a rough estimation of the true values. The numerical results are listed in Table III.

TABLE III

NUMERICAL RESULTS FOR S&P 500 OPTION PRICES

$\frac{K}{S_0}$	$\mathcal{C}$	$\mathcal{P}$	ϵ	$\mathcal{C}^{(BS)}$	$\mathcal{P}^{(BS)}$	$\mathcal{C}^*$	$\mathcal{P}^*$
0.8	269.81	0.01	2.e-3	269.80	2.e-6	269.82	0.02
0.9	132.36	1.45	0.03	130.98	0.07	132.39	1.47
1.0	40.07	20.27	0.11	30.49	10.69	40.05	20.25
1.1	5.49	76.60	0.06	1.13	72.24	5.50	76.61
1.2	0.31	147.17	0.01	4.e-3	146.87	0.32	147.19

Option parameters: $K = 1000$, $r = 0.1$, $T = 0.2$, $\sigma = 0.1074$, $\lambda = 64$, $a = -0.028$, $b = 0.026$. Simulation number $n = 10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$. The call and put values are estimated by the Monte Carlo method with AOCV. The $\mathcal{C}^*$ and $\mathcal{P}^*$ values are obtained by more simulations, say $n = 400,000$ sample points.

The numerical results in Table III show that the estimated call $\mathcal{C}$ and put $\mathcal{P}$ values by the Monte Carlo method with AOCV are within the 95% confidence interval of the true call $\mathcal{C}^*$ and put $\mathcal{P}^*$ values, i.e., $\mathcal{C} \in [\mathcal{C}^* - 1.96\epsilon, \mathcal{C}^* + 1.96\epsilon]$ or $\mathcal{P} \in [\mathcal{P}^* - 1.96\epsilon, \mathcal{P}^* + 1.96\epsilon]$ by the central limit theorem. Also, we observe that the estimated European call and put option prices are bigger than the Black-Scholes call and put option prices, respectively. This is a fact stated in the following theorem.

Theorem 5.1: The European call and put option prices based on the jump-diffusion model in (1) are bigger than the Black-Scholes call and put option prices, respectively, i.e., $\mathcal{C}(S_0, T; K, \sigma^2, r) > \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r)$, and $\mathcal{P}(S_0, T; K, \sigma^2, r) > \mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r)$.

Proof: Since the Black-Scholes call option pricing formula $\mathcal{C}^{(BS)}(S, T; K, \sigma^2, r)$ is a strictly convex function about S. By Jensen's inequality (see [8] for instance), we have

$$\begin{aligned}\mathcal{C}(S_0, T; K, \sigma^2, r) &\overset{(10)}{=} E_{\gamma(T)}\left[\mathcal{C}^{(BS)}\left(S_0 e^{\gamma(T) - \lambda\bar{J}T}, T\right)\right] \\ &> \mathcal{C}^{(BS)}\left(E_{\gamma(T)}[S_0 e^{\gamma(T) - \lambda\bar{J}T}], T\right) \\ &= \mathcal{C}^{(BS)}\left(S_0, T\right).\end{aligned}$$

By put-call parity and the above proven inequality,

$$\begin{aligned}\mathcal{P}(S_0, T; K, \sigma^2, r) &= \mathcal{C}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 \\ &> \mathcal{C}^{(BS)}(S_0, T; K, \sigma^2, r) + Ke^{-rT} - S_0 \\ &= \mathcal{P}^{(BS)}(S_0, T; K, \sigma^2, r).\end{aligned}$$

□

Remark: In the proof of the Theorem 5.1, no special distribution of Q in the Jump-Diffusion model (1) is used. Hence, this is a general result also suitable for log-normal [16], log-double-exponential [14] and log-double-uniform [17] jump amplitude models for jump-diffusions.

VI. CONCLUSION

In the paper, we developed a quite general option pricing framework to all the jump-diffusion models. The original SDE has been transformed to a risk-neutral SDE by setting the stock price increases at the risk-free interest rate. Based on this risk-neutral SDE, a reduced European call option pricing formula is derived. Then, a Monte Carlo algorithm with both antithetic and control variate variance reduction techniques are applied. This algorithm is easy to implement and the numerical results show that it is also efficient, taking less than 8 seconds per case to get the practical accuracy. Finally, we show that the European call and put option prices based on the jump-diffusion model in (1) are bigger than the Black-Scholes call and put option prices, respectively.

REFERENCES

[1] C. A. Ball and W. N. Torous, "On Jumps in Common Stock Prices and Their Impact on Call Option Prices," *J. Finance*, vol. 40 (1), 1985, pp. 155-173.

[2] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *J. Political Economy*, vol. 81, 1973, pp. 637-659.

[3] P. Boyle, "Options: A Monte Carlo Approach," *J. Financial Economics*, vol. 21, 1997, pp. 1267-1321.

[4] P. Boyle, M. Broadie and P. Glasserman, "Monte Carlo methods for security pricing," *J. Economic Dynamics and Control*, vol. 4, 1977, pp. 323-338.

[5] P. Glasserman, *Monte Carlo methods in Financial Engineering*, Springer, New York, 2004.

[6] F. B. Hanson and J. J. Westman, "Jump-Diffusion Stock Return Models in Finance: Stochastic Process Density with Uniform-Jump Amplitude," *Proc. 15th International Symposium on Mathematical Theory of Networks and Systems*, 7 pages, August 2002.

[7] F. B. Hanson, J. J. Westman and Z. Zhu, "Multinomial Maximum Likelihood Estimation of Market Parameters for Stock Jump-Diffusion Models," *AMS Contemporary Mathematics*, vol. 351, 2004, pp. 155-169.

[8] F. B. Hanson, *Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation,* SIAM Books, Philadelphia, PA, to appear 2006; `http://www2.math.uic.edu/~hanson/math574/#Text`.

[9] F. B. Hanson and Z. Zhu, "Comparison of Market Parameters for Jump-Diffusion Distributions Using Multinomial Maximum Likelihood Estimation," *Proc. 43rd IEEE Conference on Decision and Control*, pp. 3919-3924, December 2004.

[10] D. J. Higham, *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*, Cambridge University Press, Cambridge, UK, 2004.

[11] J. C. Hull, *Options, Futures, & Other Derivatives*, 4th Edition, Prentice-Hall, Englewood Cliffs, NJ, 2000.

[12] R. A. Jarrow and E. R. Rosenfeld, "Jump Risks and the Intertemporal Capital Asset Pricing Model," *J. Business*, vol. 57 (3), 1984, pp. 337-351.

[13] P. Jorion, "On Jump Processes in the Foreign Exchange and Stock Markets," *Rev. Fin. Studies*, vol. 88 (4), 1989, pp. 427-445.

[14] S. G. Kou and H. Wang, "Option Pricing Under a Double Exponential Jump Diffusion Model," *Management Science*, vol. 50 (9), 2004, pp. 1178-1192.

[15] R. C. Merton, "Theory of Rational Option Pricing," *Bell J. Econ. Mgmt. Sci.*, vol. 4, 1973, pp. 141-183.

[16] R. C. Merton, "Option Pricing when Underlying Stock Returns are Discontinuous," *J. Financial Economics*, vol. 3, 1976, pp. 125-144.

[17] Z. Zhu and F. B. Hanson, "A Log-Double-Uniform Jump-Diffusion Model," *working paper, Dept. Math., Stat., and Comp. Sci., U. Illinois at Chicago*, 2005, pp. 1-29; `http://www.math.uic.edu/~hanson/pub/ZZhu/dbuniformfree2.pdf`.