Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

QUICKEST DETECTION OF A MINIMUM OF DISORDER TIMES

Erhan Bayraktar and H. Vincent Poor

***Abstract*— A multi-source quickest detection problem is considered. Assume there are two independent Poisson processes X^1 and X^2 with disorder times θ_1 and θ_2, respectively: that is the intensities of X^1 and X^2 change at random unobservable times θ_1 and θ_2, respectively. θ_1 and θ_2 are independent of each other and are exponentially distributed. Define $\theta \triangleq \theta_1 \wedge \theta_2 = \min\{\theta_1, \theta_2\}$. For any stopping time τ that is measurable with respect to the filtration generated by the observations define a penalty function of the form**

$$R_\tau = \mathbb{P}(\tau < \theta) + c\mathbb{E}\left[(\tau - \theta)^+\right],$$

where $c > 0$ and $(\tau - \theta)^+$ is the positive part of $\tau - \theta$. It is of interest to find a stopping time τ that minimizes the above performance index. Since both observations X^1 and X^2 reveal information about the disorder time θ, even this simple problem is more involved than solving the disorder problems for X^1 and X^2 separately. This problem is formulated in terms of a two dimensional sufficient statistic, and the corresponding optimal stopping problem is examined. Using a suitable single jump operator, this problem is solved explicitly.

I. INTRODUCTION

Consider two independent Poisson processes $X^i = \{X^i_t : t \geq 0\}$ $i \in \{1, 2\}$ with the same rate α. At some random unobservable times θ_1 and θ_2, which have exponential distribution $\mathbb{P}(\theta_i > t) = (1 - \pi_i)e^{-\lambda_i}$ for $t \geq 0$, the arrival rates of the Poisson processes X^1 and X^2 change from α to β. Here α and β are known positive constants. We seek a stopping rule τ that detects the instant $\theta = \theta_1 \wedge \theta_2$ of the first regime change as accurately as possible given the past and the present observations of the processes X^1 and X^2. More precisely, we will try to choose a stopping time of the history of the processes X^1 and X^2 that minimizes the following penalty function

$$R_\tau = \mathbb{P}(\tau < \theta) + c\mathbb{E}\left[(\tau - \theta)^+\right]. \qquad \text{(I.1)}$$

The first term in (I.1) penalizes the losses due to the false alarms, and the second term penalizes the detection delay. The disorder time demarcates two regimes, and in each of these regimes the decision maker uses distinctly different strategies. Therefore, it is in the decision maker's interest to detect the disorder time as accurately as possible from its observations.

Quickest detection problems arise in a variety of applications such as seismology, machine monitoring, finance, health, and surveillance, among others (see e.g. [1], [9], [7] , [10] and [11]). Because Poisson processes are often used to model abrupt changes, Poisson disorder problems have potential applications e.g. to the effective control and prevention of infectious diseases, quickest detection of quality and reliability problems in industrial processes, and surveillance of internet traffic to protect the network servers from the attacks of malicious users. This is because the number of patients infected, number of defected items produced and number of packets arriving at a network node are usually modeled by Poisson processes. In these examples the disorder time corresponds to the time when an outbreak occurs, when a machine in an assembly line breaks down or when a router is under attack, respectively.

The one dimensional Poisson disorder problem, i.e., the problem of detecting θ_1 as accurately as possible given the observations from the Poisson process X^1 has recently been solved (see [2], [4] and the references therein). The two-dimensional disorder problem we have introduced could not be reduced to solving the corresponding one-dimensional disorder problems since both X^1 and X^2 reveal some information about θ whenever these processes jump. That is if we take the minimum of the optimal stopping times that solve the one Poisson dimensional disorder problems, then we obtain a stopping time that is a sub-optimal solution to (I.1) (see Remark 4.1).

We will show that the quickest detection problem of (I.1) can be reduced to an optimal stopping problem for a two-dimensional piece-wise deterministic Markov process. The optimal stopping problems are usually solved by formulating them as free boundary problems associated with the infitesimal generator of the Markov process. The infitesimal generator however contains differential delay operators. Solving the free boundary problems involving differential delay operators prove to be a challenge even in the one dimensional case (see see [2], [4] and the references therein). Instead as in [3] and [6] we work with an integral operator, iteration of which generates a monotone increasing sequence of functions converging to the value function of the optimal stopping problem. Using this approach we are able construct an optimal stopping time as a limit of a sequence optimal stopping times that are solutions of a sequence of related optimal stopping problems. Using the structure of the paths of the piece-wise deterministic Markov process we also provide a non-trivial bound on the optimal stopping time.

The remainder of this paper is organized as follows. In Sections II and III, we state the problem under a suitable reference measure $\mathbb{P}_0$ that is equivalent to $\mathbb{P}$. Working under this reference measure reduces the computations considerably. Here we show that the quickest detection problem reduces to solving an optimal stopping problem for a two-

E. Bayraktar is with the Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA erhan@umich.edu

H. V. Poor is with the Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, USA poor@princeton.edu

dimensional sufficient statistic. In Section IV, we analyze the path behaviour of this sufficient statistic. In Section V, we convert the optimal stopping problem into a sequence of deterministic optimal stopping problems using an integral operator. In Section VI, we provide a non-trivial bound on the continuation region. In Section VII, we construct an optimal stopping time from a sequence of stopping times that react before the processes X^1 and X^2 jump a certain number of times.

II. Problem Description

Let us start with a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ that hosts two independent Poisson processes X^1 and X^2, both of which have rate α, as well as two independent random variables θ_1 and θ_2 independent of the Poisson processes with distributions

$$\mathbb{P}_0(\theta_i = 0) = \pi_i \quad \text{and} \quad \mathbb{P}_0(\theta_i > t) = (1-\pi_i)e^{-\lambda_i t}, \tag{II.1}$$

for $0 \le t < \infty$, $i \in \{1,2\}$ and for some known constants $\pi_i \in [0,1)$ and $\lambda_i > 0$ for $i \in \{1,2\}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t<\infty}$ the filtration generated by X^1 and X^2, i.e., $\mathcal{F}_t = \sigma(X^1_s, X^2_s, 0 \le s \le t)$, and denote by $\mathbb{G} = \{\mathcal{G}_t\}_{0\le t<\infty}$ the initial enlargement of $\mathbb{F}$ by θ_1 and θ_2, i.e., $\mathcal{G}_t \triangleq \sigma(\theta_1, \theta_2, X^2_s, X^2_s : 0 \le s \le t)$. Let us introduce the $\mathbb{G}$ adapted processes

$$h_i(t) = \beta 1_{\{t<\theta_i\}} + \alpha 1_{\{t\ge\theta_i\}}, \quad i \in \{1,2\}. \tag{II.2}$$

The processes $X^i_t - \int_0^t h_i(s)ds$, $i \in \{1,2\}$, $t \ge 0$ are martingales under a new probability measure $\mathbb{P}$, which is characterized by

$$\left.\frac{d\mathbb{P}}{d\mathbb{P}_0}\right|_{\mathcal{G}_t} \triangleq Z_t \triangleq Z^1_t Z^2_t, \tag{II.3}$$

where

$$Z^i_t \triangleq \exp\left(\int_0^t \log\left(\frac{h_i(s-)}{\beta}\right) dX^i_s - \int_0^t [h_i(s) - \beta]ds\right), \tag{II.4}$$

for $t \ge 0$ and $i \in \{1,2\}$ are exponential martingales (see e.g. [5]). In terms of the exponential likelihood processes

$$L^i_t \triangleq \left(\frac{\alpha}{\beta}\right)^{X^i_t} \exp(-(\alpha-\beta)t),\ t \ge 0,\ i \in \{1,2\}, \tag{II.5}$$

we can write

$$Z^i_t = 1_{\{\theta_i > t\}} + 1_{\{\theta_i \le t\}} \frac{L^i_t}{L^i_{\theta_i}} \tag{II.6}$$

Note that $\mathbb{P}$ and $\mathbb{P}_0$ coincide on $\mathcal{G}_0$, and θ_1, θ_2 are $\mathcal{G}_0$ measurable. Therefore θ_1 and θ_2 have the same distribution under both $\mathbb{P}$ and $\mathbb{P}_0$.

Let us introduce the posterior probability process

$$\Pi_t \triangleq \mathbb{P}\left(\theta \le t \middle| \mathcal{F}_t\right) = \frac{\mathbb{E}_0\left[Z_t 1_{\{\theta \le t\}} \middle| \mathcal{F}_t\right]}{\mathbb{E}_0\left[Z_t \middle| \mathcal{F}_t\right]}, \tag{II.7}$$

where the second equality follows from the Bayes formula (see e.g. [8]). Then it follows that from (II.6) and (II.7) that

$$1 - \Pi_t = \frac{(1-\pi)e^{-\lambda t}}{\mathbb{E}_0\left[Z_t \middle| \mathcal{F}_t\right]}, \tag{II.8}$$

where $\lambda \triangleq \lambda_1 + \lambda_2$ and $\pi \triangleq 1 - (1-\pi_1)(1-\pi_2)$. Let us now introduce the odds-ratio process

$$\Phi_t \triangleq \frac{\Pi_t}{1-\Pi_t},\ 0 \le t < \infty. \tag{II.9}$$

Then observe from (II.8) that

$$\mathbb{E}_0\left[Z_t 1_{\{\theta \le t\}}\right] = (1-\pi)e^{-\lambda t}\Phi_t, \tag{II.10}$$

$t \ge 0$. Now, we will write the penalty function of (I.1) in terms of the odds-ratio process.

$$\begin{aligned}\mathbb{E}\left[(\tau-\theta)^+\right] &= \mathbb{E}\left[\int_0^\infty 1_{\{\tau>t\}} 1_{\{\theta\le t\}} dt\right] \\ &= \int_0^\infty \mathbb{E}_0\left[1_{\{\tau>t\}} \mathbb{E}_0\left[Z_t 1_{\{\theta\le t\}} \middle| \mathcal{F}_t\right]\right] dt \\ &= (1-\pi)\mathbb{E}_0 \int_0^\tau e^{-\lambda t}\Phi_t dt.\end{aligned} \tag{II.11}$$

Similarly, it can be shown that

$$\mathbb{P}(\tau < \theta) = (1-\pi)\left(1 - \lambda\mathbb{E}_0\left[\int_0^\tau e^{-\lambda t} dt\right]\right), \tag{II.12}$$

and therefore the penalty function can be written as

$$R_\tau(\pi_1, \pi_2) = 1 - \pi + c(1-\pi)\mathbb{E}_0\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right) dt\right]. \tag{II.13}$$

Denoting

$$\Phi^i_t \triangleq \frac{e^{\lambda_i t}}{1-\pi_2}\mathbb{E}_0\left[1_{\{\theta_1 \le t\}} \frac{L^i_t}{L^i_{\theta_i}} \middle| \mathcal{F}^i_t\right], \tag{II.14}$$

for $t \ge 0$ and $i \in \{1,2\}$, we can write the odds-ratio process Φ as

$$\Phi_t = \Phi^1_t + \Phi^2_t + \Phi^1_t\Phi^2_t, \quad t \ge 0 \tag{II.15}$$

using (II.6) and (II.10). Using the fact that the likelihood ratio process L^i is the unique solution of the equation

$$dL^i_t = [(\alpha/\beta) - 1]L_{t-}(dX^i - \alpha\, dt), \tag{II.16}$$

and by means of the chain-rule we obtain

$$d\Phi^i_t = (\lambda + (\lambda - \alpha + \beta)\Phi^i_t)dt + [(\alpha/\beta) - 1]\Phi^i_t dX^i_t, \tag{II.17}$$

for $t \ge 0$ and $i \in \{1,2\}$ (see [4]). If we let

$$\Phi^+_t \triangleq \Phi^1_t + \Phi^2_t,\ \ \Phi^\times_t \triangleq \Phi^1_t\Phi^2_t,\ \ t \ge 0, \tag{II.18}$$

then using a change of variable formula for jump processes gives

$$\begin{aligned}d\Phi^\times_t &= [\lambda\Phi^+_t + a\Phi^\times_t]dt + (\alpha/\beta) - 1)\Phi^\times_t d(X^1_t + X^2_t), \\ d\Phi^+_t &= [2\lambda + a\Phi^+_t]dt + ((\alpha/\beta) - 1)[\Phi^1_t dX^1_t + \Phi^2_t dX^2_t]\end{aligned} \tag{II.19}$$

with $\Phi^\times_0 = \pi_1\pi_2/[(1-\pi_1)(1-\pi_2)]$, and $\Phi^+_0 = \pi_1/(1-\pi_1) + \pi_2/(1-\pi_2)$, where $a \triangleq \lambda - \alpha + \beta$. Note that $X_t \triangleq X^1_t + X^2_t$, $t \ge 0$ is a Poisson process with rate 2β under $\mathbb{P}_0$. It is clear from this equation that

$$\Psi \triangleq (\Phi^\times, \Phi^+) \tag{II.20}$$

is a piece-wise deterministic Markov process; therefore the original change detection problem with penalty function (I.1) has been reformulated as (II.13) and (II.19), which is an optimal stopping problem for a two dimensional Markov process. Here the pair $\Psi = (\Phi^{\times}, \Phi^{+})$ is a sufficient statistic for the multi-source Poisson disorder problem.

Let us denote

$$\mathbb{B}_{+}^{2} \triangleq \{(x, y) \in \mathbb{B}_{+}^{2} : y \geq 2\sqrt{x}\}. \tag{II.21}$$

Now, for every $(\phi^{+}, \phi^{\times}) \in \mathbb{B}_{+}^{2}$, let us denote denote by $x(t, \phi^{+})$ and $y(t, \phi^{\times})$, $t \in \mathbb{R}$ the solutions of

$$\begin{aligned} \frac{d}{dt}x(t, \phi^{\times}) &= [\lambda y(t, \phi^{+}) + a x(t, \phi^{\times})]dt \\ \frac{d}{dt}y(t, \phi^{+}) &= [2\lambda + a y(t, \phi^{+})]dt. \end{aligned} \tag{II.22}$$

The solutions of (II.22) are explicitly given by

$$\begin{aligned} x(t, \phi^{\times}) &= \frac{2\lambda^2}{a^2} + e^{at}\left[\phi^{\times} - \frac{2\lambda^2}{a^2} + \lambda\left(\phi^{+} + \frac{2\lambda}{a}\right)t\right], \\ y(t, \phi^{+}) &= -\frac{2\lambda}{a} + e^{at}\left(\phi^{+} + \frac{2\lambda}{a}\right). \end{aligned} \tag{II.23}$$

Using (II.19) and (II.22) for $\sigma_n \leq t < \sigma_{n+1}$ we can write

$$\begin{aligned} &\Phi_t^{\times} = x(t - \sigma_n, \Phi_{\sigma_n}^{\times}),\ \Phi_t^{+} = y(t - \sigma_n, \Phi_{\sigma_n}^{+}), \\ &\Phi_{\sigma_{n+1}}^{\times} = \frac{\alpha}{\beta}\Phi_{\sigma_n -}^{\times}, \\ &\Phi_{\sigma_{n+1}}^{+} = \begin{cases} \Phi_{\sigma_n -}^{+} + \frac{1}{2}\left(\frac{\alpha}{\beta} - 1\right)\left[\Phi_{\sigma_n -}^{+} + \sqrt{(\Phi_{\sigma_n -}^{+})^2 - 4\Phi_{\sigma_n -}^{\times}}\right] \\ \qquad \text{with } \frac{1}{2} \text{ probability}, \\ \Phi_{\sigma_n -}^{+} + \frac{1}{2}\left(\frac{\alpha}{\beta} - 1\right)\left[\Phi_{\sigma_n -}^{+} - \sqrt{(\Phi_{\sigma_n -}^{+})^2 - 4\Phi_{\sigma_n -}^{\times}}\right] \\ \qquad \text{with } \frac{1}{2} \text{ probability}, \end{cases} \end{aligned} \tag{II.24}$$

where σ_n denotes the n^{th} jump time of the process X, $n \in \mathbb{N}$. We will use the notation $\sigma_0 = 0$.

III. An Optimal Stopping Problem

The value function of the quickest detection problem

$$U(\pi_1, \pi_2) \triangleq \inf_{\tau \in \mathcal{S}} R_\tau(\pi_1, \pi_2) \tag{III.1}$$

can be written as

$$U(\pi_1, \pi_2) = (1 - \pi)\left[1 + cV\left(\frac{\pi_1\pi_2}{1 - \pi}, \frac{\pi_1 + \pi_2 - 2\pi_1\pi_2}{1 - \pi}\right)\right], \tag{III.2}$$

where V is the value function of the optimal stopping problem

$$V(\phi_0, \phi_1) \triangleq \inf_{\tau \in \mathcal{S}} \mathbb{E}_0^{\phi_0, \phi_1}\left[\int_0^{\tau} e^{-\lambda t} h\left(\Phi_t^{\times}, \Phi_t^{+}\right) dt\right], \tag{III.3}$$

in which $(\phi_0, \phi_1) \in \mathbb{B}_{+}^{2}$, and $h(x, y) \triangleq x + y - \lambda/c$. It is clear from this equation that it is not optimal to stop before Ψ leaves

$$\mathbb{C}_0 \triangleq \{(\phi^{\times}, \phi^{+}) \in \mathbb{B}_{+}^{2} : \phi^{\times} + \phi^{+} < \lambda/c\}. \tag{III.4}$$

IV. Sample Paths of $\Psi = (\Phi^{\times}, \Phi^{+})$

It is illustrative to look at the sample paths of the sufficient statistic Ψ, to understand the nature of the problem. Indeed, this way, for a certain parameter range, we will be able to identify the optimal stopping time.

From (II.23), we see that, if $a > 0$, then the paths of the processes $\Phi^{\times}$ and Φ^{+} increase between the jumps, and otherwise the paths of the processes $\Phi^{\times}$ and Φ^{+} mean-revert to the levels $2\lambda^2/a^2$ and $-2\lambda/a$ respectively. Also observe that $\Phi^{\times}$, Φ^{+} increase (decrease) with a jump if $\alpha > \beta$ ($\beta > \alpha$).

Case I: $\alpha > \beta$, $a > 0$. The following theorem follows from the description of the behavior of the paths above.

Theorem 4.1: If $\alpha > \beta$ and $a > 0$, then the stopping rule

$$\tau_0 \triangleq \inf\{t \geq 0 : \Phi_t^{\times} + \Phi_t^{+} > \lambda/c\} \tag{IV.1}$$

is optimal for (III.3).

Remark 4.1: Let $\kappa_i \triangleq \inf_{\tau \in \mathcal{S}}\{t : \Phi_t^{i} > \lambda/c\}$. If $\alpha > \beta$ and $a > 0$ then κ_i is the optimal stopping time for the one dimensional disorder problem with disorder time θ_i ([4]). Let us define $\kappa \triangleq \kappa_1 \wedge \kappa_2$. Then it follows that $\tau_0 < \kappa$ almost surely. This illustrates that solving the two one dimensional quickest detection problems in order to minimize the penalty function of (I.1) is suboptimal.

In what follows we will consider the remaining cases: $\alpha > \beta$ and $a < 0$; $\alpha < \beta$.

V. Optimal stopping with time Horizon σ_n

In this section will approximate the optimal stopping problem (III.3) by a sequence of optimal stopping problems. Let us denote

$$V_n(\phi_0, \phi_1) \triangleq \inf_{\tau \in \mathcal{S}} \mathbb{E}_0^{\phi_0, \phi_1}\left[\int_0^{\tau \wedge \sigma_n} e^{-\lambda t} h\left(\Phi_t^{\times}, \Phi_t^{+}\right) dt\right] \tag{V.1}$$

where $(\phi_0, \phi_1) \in \mathbb{R}_{+}$, $n \in \mathbb{N}$, and σ_n is the n^{th} jump time of the process X. Observe that $(V_n)_{n \in \mathbb{N}}$ is decreasing and satisfies $-1/c < V_n < 0$. Therefore its limit $\lim_n V_n$ exists. It can be shown that more is true using the fact that the function h is bounded from below and σ_n is a sum of independent exponential random variables.

Lemma 5.1: For any $(\phi_0, \phi_1) \in \mathbb{B}_{+}^{2}$

$$\frac{1}{c}\left(\frac{2\beta}{2\beta + \lambda}\right)^n \geq V_n(\phi_0, \phi_1) - V(\phi_0, \phi_1) \geq 0. \tag{V.2}$$

As in [3] and [6] to calculate the value functions V_n we introduce the following functional operators defined on bounded functions $f : \mathbb{B}_{+}^{2} \to \mathbb{R}$

$$\begin{aligned} Jf(t, \phi_0, \phi_1) \triangleq \mathbb{E}_0^{\phi_0, \phi_1}\Bigg[&\int_0^{t \wedge \sigma_1} e^{-\lambda s} h(\Phi_s^{\times}, \Phi_s^{+}) ds \\ &+ 1_{\{t \geq \sigma_1\}} e^{-\lambda \sigma_1} f(\Phi_{\sigma_1}^{\times}, \Phi_{\sigma_1}^{+})\Bigg] \end{aligned} \tag{V.3}$$

$$J_t f(\phi_0, \phi_1) \triangleq \inf_{s \in [t, \infty]} Jf(s, \phi_0, \phi_1), \quad t \in [0, \infty].$$

Using the fact that σ_1 has exponential distribution with rate 2β and Fubini's theorem we can write

$$Jf(t,\phi_0,\phi_1) = \int_0^t e^{-(\lambda+2\beta)s}(h+\beta\cdot f\circ(F_1+F_2))(x(s,\phi_0),y(s,\phi_1))ds, \tag{V.4}$$

where

$$F_i(\phi_0,\phi_1) = \left(\frac{\alpha}{\beta}\phi_0, \left[\frac{\alpha}{\beta}+\frac{1}{2}\right]\phi_1 + (-1)^i\sqrt{\phi_1^2-4\phi_0}\right). \tag{V.5}$$

Lemma 5.2: For every bounded function f, the mapping $J_0 f$ is bounded. If f is a concave function of its variables, then $J_0 f$ is also a concave function. If $f_1 \le f_2$, then $J_0 f_1 \le J_0 f_2$.

Proof: The third assertion of the lemma follows from the representation (V.4). The first assertion holds since h is bounded from below and $J_0 f(\phi_0,\phi_1) \le Jf(0,\phi_0,\phi_1) = 0$. The second assertion follows from the linearity of the functions $x(t,\cdot)$ and $y(t,\cdot)$. ■

Using Lemma 5.2 we can prove the following corollary:

Corollary 5.1: Let us define a sequence of function (v_n) by

$$v_0 = 0 \quad \text{and } v_n = J_0 v_{n-1}. \tag{V.6}$$

Then every $n\in\mathbb{N}$, v_n is bounded and concave, and $v_{n+1} \le v_n$. Therefore $v = \lim_n v_n$, exists, and is bounded and concave.

We will need the following lemma to give a characterization of the stopping times of $\mathbb{F}$ (see [5]).

Lemma 5.3: For every $\tau\in\mathcal{S}$, there are $\mathcal{F}_{\sigma_n}$ measurable random variables $\xi_n:\Omega\to\infty$ such that $\tau\wedge\sigma_(n+1) = (\sigma_n+\xi_n)\wedge\sigma_{n+1}$ $\mathbb{P}_0$ almost surely on $\{\tau\ge\sigma_n\}$.

The main theorem of this section can be proven using induction using Lemma 5.3 and the strong Markov property.

Theorem 5.1: For every $n\in\mathbb{N}$ v_n defined in Corollary 5.1 is equal to V_n of (V.1). For $\varepsilon>0$, let us denote

$$r_n^\varepsilon(\phi_0,\phi_1) \triangleq \inf\{t\in(0,\infty] : Jv_n(t,(\phi_0,\phi_1)) \le J_0 v_n(\phi_0,\phi_1)+\varepsilon\}. \tag{V.7}$$

And let us define a sequence of stopping times by $S_1^\varepsilon \triangleq r_0^\varepsilon(\Phi)\wedge\sigma_1$ and

$$S_{n+1}^\varepsilon \triangleq \begin{cases} r_n^{\varepsilon/2}(\Psi) & \text{if } \sigma_1 \ge r_n^{\varepsilon/2}(\Psi) \\ \sigma_1 + S_n^{\varepsilon/2}\circ\theta_{\sigma_1}. \end{cases} \tag{V.8}$$

Here θ_s is the shift operator on Ω, i.e., $X_t\circ\theta_s = X_{s+t}$. Then S_n^ε is ε optimal, i.e.,

$$\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^{S_n^\varepsilon} e^{-\lambda t}h(\psi_t)dt\right] \le v_n(\phi_0,\phi_1)+\varepsilon. \tag{V.9}$$

This theorem shows that the functions V^n in (V.1) and the functions v^n introduced in Corollary 5.1 by an iterative application of the operator J_0 are equal. Therefore the value functions can be found by solving a sequence of deterministic minimization problems.

Let us denote the optimal stopping regions and optimal continuation regions by

$$\mathbf{\Gamma_n} \triangleq \{(\phi_0,\phi_1)\in\mathbb{B}_+^2 : v_n(\phi_0,\phi_1)=0\}, \quad \mathbf{C}_n \triangleq \mathbb{B}_+^2 - \mathbf{\Gamma_n},$$
$$\mathbf{\Gamma} \triangleq \{(\phi_0,\phi_1)\in\mathbb{B}_+^2 : v(\phi_0,\phi_1)=0\}, \quad \mathbf{C} \triangleq \mathbb{B}_+^2 - \mathbf{\Gamma}. \tag{V.10}$$

VI. Construction of a Bound on the Continuation Region

Case II: $\alpha>\beta$ and $a<0$. Observe that $\Phi^\times = y(t,\phi_1)$ for $0<t<\sigma_1$ and $\Phi_{\sigma_1}^+ = (\alpha/\beta)\Phi_{\sigma_1-}^+ = (\alpha/\beta)y(\sigma_1,\phi_) \ge y(\sigma_1,\phi_1)$. Now, by (II.23) and (II.24) we can conclude that $\Phi_t^+ \ge y(t,\phi_1)$ for $\sigma_1\le t<\sigma_2$. Carrying out an induction argument we conclude that

$$\Phi_t^+ \ge y(t,\phi_1),\ t\ge 0, \tag{VI.1}$$

almost surely if $\Phi_0^+ = \phi_1$. Now using (VI.1), (II.23) and (II.24) it can be seen by following a similar line of arguments that

$$\Phi_t^\times \ge x(t,\phi_0),\ t\ge 0, \tag{VI.2}$$

almost surely if $\Phi_0^\times = \phi_0$. From this observation we obtain the following inequality

$$\inf_{\tau\in\mathcal{S}}\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s}h(\Phi_s^\times,\Phi_s^+)ds\right] \tag{VI.3}$$
$$\ge \inf_{\tau\in\mathcal{S}}\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s}h(x(s,\phi_0),y(s,\phi_1))ds\right] \tag{VI.4}$$
$$= \inf_{t\in[0,\infty]}\int_0^t e^{-\lambda s}h(x(s,\phi_0),y(s,\phi_1))ds. \tag{VI.5}$$

Note that if for a given (ϕ_0,ϕ_1) (VI.5) is equal to zero, then the infimum in (VI.3) is attained by setting $\tau=0$, i.e., (ϕ_0,ϕ_1) is in the stopping region.

Case II-a: $2\lambda/a^2 - 2/a < 1/c$. In this case the mean reversion level $(2\lambda^2/a^2, -2\lambda/a)$ is inside the region $\mathbb{C}_0$. In this case for any $(\phi_0,\phi_1)\in\mathbb{B}_+^2$ the minimizer t_{opt} of (VI.5) is either 0 or ∞. We shall find the pairs (ϕ_0,ϕ_1) for which $t_{\text{opt}}=0$. Using (II.23) we can write

$$\int_0^\infty e^{-\lambda s}h(x(s,\phi_0),y(s,\phi_1))ds = \phi_0\left(-\frac{1}{\alpha-\beta}\right) + \phi_1\left(\frac{a}{(\alpha-\beta)^2}\right) + k, \tag{VI.6}$$

where k is given by

$$k \triangleq \frac{4\lambda}{a^2} - \frac{2}{a} + \frac{2\lambda^2}{a(\alpha-\beta)^2} - \frac{1}{c}. \tag{VI.7}$$

Let us denote

$$\mathbb{D}_0 \triangleq \left\{(x,y)\in\mathbb{B}_+^2 : x\left(-\frac{1}{\alpha-\beta}\right) + y\left(\frac{a}{(\alpha-\beta)^2}\right) + k > 0\right\}, \tag{VI.8}$$

then it can be verified that $\mathbb{B}_+^2 - \mathbb{D}_0 \supset \mathbb{C}_0$. Note that if $(\phi_0,\phi_1)\in\mathbb{D}_0$, then the infimum in (VI.5) is equal to 0. Therefore $\mathbb{B}_+^2 - \mathbb{D}_0$ is a superset of the optimal continuation region of (III.3).

Case II-b: $2\lambda/a^2 - 2/a \ge 1/c$. In this case the mean reversion level is outside $\mathbb{C}_0$. Therefore the minimizer of (VI.5) is $t_{\text{opt}}(\phi_0,\phi_1)\in\{0,t_c(\phi_0,\phi_1),\infty\}$ where $t_c(\phi_0,\phi_1)$ is the exit time $(x(t,\phi_0),y(t,\phi_1))$ from $\mathbb{C}_0$. The derivative

$$\frac{d}{dt}[x(t,\phi_0)+y(t,\phi_1)] = (\lambda+a)y(t,\phi_1) + ax(t,\phi_1) + 2\lambda \tag{VI.9}$$

vanishes if $(x(t,\phi_0),y(t,\phi_1))$ meets the line

$$L : (\lambda+a)y + ax + 2\lambda = 0. \tag{VI.10}$$

Note that the mean reversion level is in L.

Case II-b-i: $\lambda + a < 0$. In this case L is decreasing. Using (VI.9) it is easy to see that if $(\phi_0, \phi_1) \notin \mathbb{C}_0$ then $(x(t,\phi_0), y(t,\phi_1)) \notin \mathbb{C}_0$ for any $t \geq 0$. Therefore the minimizer t_{opt} is equal to 0 if $(\phi_0, \phi_1) \notin \mathbb{C}_0$, and is equal to $t_c(\phi_0, \phi_1)$ if $(\phi_0, \phi_1) \in \mathbb{C}_0$. Therefore $\mathbb{C}_0$ is equal to the optimal stopping region of (III.3).

Case II-b-ii: $\lambda + a > 0$. In this case L is increasing. If $(x(t,\phi_0), y(t,\phi_1))$ meets the line L at $t_L(\phi_0), \phi_1$ then $(x(t,\phi_0) + y(t,\phi_1))$ is increasing (decreasing) on $[0, t_L]$ ($[t_L, \infty]$). Otherwise, $(x(t,\phi_0) + y(t,\phi_1))$ is increasing on $[0,\infty]$. If $-a/c - 2 < 0$, then L does not intersect $\mathbb{C}_0$. In this case the minimizer of (VI.5) is 0 for $(\phi_0, \phi_1) \notin \mathbb{C}_0$. Therefore the optimal continuation region is equal to $\mathbb{C}_0$ of (III.3).

If $-a/c - 2 < 0$, then L intersects $\mathbb{C}_0$. Let us denote by (ϕ_0^*, ϕ_1^*) the intersection of L and the boundary of the region $\mathbb{C}_0$, $x + y - \lambda/c = 0$. Then running the paths backwards from (ϕ_0^*, ϕ_1^*), we can find a finite ξ^* and t^* such that $(0, \xi^*) = (x(-t^*, \phi_0^*), t(-t^*, \phi_1^*))$. Now using the semi-group property of the paths $(x(t,\phi_0), y(t,\phi_1))$ we can argue that if $(\phi_0, \phi_1) \notin \mathbb{D}_1$, then $(x(t,\phi_0), y(t,\phi_1)) \notin \mathbb{C}_0$ where

$$\mathbb{D}_1 \triangleq \{(x,y) \in \mathbb{B}_+^2 : x + y \geq \xi^*\}. \tag{VI.11}$$

Finally we can conclude that $t_{\text{opt}}(\phi_0, \phi_1) = 0$ if $(\phi_0, \phi_1) \notin \mathbb{C}_0$. Therefore $\mathbb{B}_+^2 - \mathbb{D}_1$ is a superset of the optimal continuation region of (III.3).

Case III: $\alpha < \beta$. Note that in this case $a > 0$. The paths of the processes $\Phi^\times$, Φ^+ increase between the jumps and decrease with a jump. If $\tau \in \mathcal{S}$ then there is a constant $t \geq 0$ such that $\tau \wedge \sigma_1 = t \wedge \sigma_1$ almost surely. Hence we can write

$$\begin{aligned}
&\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds\right] = \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^{\tau\wedge\sigma_1} e^{-\lambda s} h(\Psi_s) ds\right] + \\
&\qquad + \mathbb{E}_0^{\phi_0,\phi_1}\left[1_{\{\tau \geq \sigma_1\}} \int_{\sigma_1}^\tau e^{-\lambda s} h(\Psi_s) ds\right] \\
&= \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^{t\wedge\sigma_1} e^{-\lambda s} h(\Psi_s) ds\right] + \mathbb{E}_0^{\phi_0,\phi_1}\left[1_{\{t \geq \sigma_1\}} \int_{\sigma_1}^\tau e^{-\lambda s} h(\Psi_s) ds\right] \\
&\geq \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^{\tau\wedge\sigma_1} e^{-\lambda s} h(\Psi_s) ds\right] - \frac{1}{c}\mathbb{E}_0^{\phi_0,\phi_1}\left[1_{\{t\geq\sigma_1\}} e^{-\lambda\sigma_1}\right] \\
&= \int_0^t e^{-(\lambda+2\beta)s}\left[h(x(s,\phi_0), y(s,\phi_1)) - \frac{2\beta}{c}\right] ds,
\end{aligned} \tag{VI.12}$$

using also the fact that σ_1 has exponential distribution with rate 2β. From (VI.12) it follows that if $x(s,\phi_0) + y(s,\phi_1) - (\lambda + 2\beta)/c > 0$, then $\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds\right] > 0$ for every stopping time $\tau \neq 0$. Since the paths $x(t,\phi_0)$, $y(t,\phi_1)$ are increasing we can conclude that stopping immediately is optimal, i.e., $\tau = 0$ is optimal if $(\phi_0, \phi_1) \in \mathbb{D}_2$ where

$$\mathbb{D}_2 \triangleq \left\{(x,y) \in \mathbb{B}_+^2 : x + y \geq \frac{\lambda + 2\beta}{c}\right\}. \tag{VI.13}$$

Therefore $\mathbb{B}_+^2 - \mathbb{D}_2$ is a superset of the optimal continuation region of (III.3).

The results of the next section can be used to determine approximate detection rules besides helping us to determine the location and the shape of the continuation region. As we have seen, in some cases these approximate rules turn out to be tight.

VII. Optimal Stopping Time

Theorem 7.1:

$$\tau^*(\phi_0, \phi_1) \triangleq \inf\{t \geq 0 : V(\Psi_t) = 0\} \tag{VII.1}$$

is an optimal stopping time for (III.3).

This theorem shows that Γ defined in (V.10) is indeed an optimal stopping region. We will divide the proof of this theorem into several lemmas. The following lemma shows that if there exists an optimal stopping time it necessarily greater than or equal to τ^*.

Lemma 7.1:

$$V(\phi_0, \phi_1) = \inf_{\tau \geq \tau^*} \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds\right]. \tag{VII.2}$$

Proof: Let us define

$$\tilde{\tau} \triangleq \begin{cases} \tau, & \text{if } \tau \geq \tau^*, \\ \tau + \tau^* \circ \theta_\tau, & \text{if } \tau < \tau^*. \end{cases} \tag{VII.3}$$

Then the stopping time $\tilde{\tau}$

$$\begin{aligned}
&\mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^{\tilde{\tau}} e^{-\lambda s} h(\Phi_s) ds\right] \\
&= \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds + \int_\tau^{\tilde{\tau}} e^{-\lambda s} h(\Psi_s) ds\right] \\
&= \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds + e^{-\lambda\tau}\int_0^{\tau^*\circ\theta_\tau} e^{-\lambda s} h(\Psi_{s+\tau}) ds\right] \\
&= \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds + e^{-\lambda\tau}\int_0^{\tau^*} e^{-\lambda s} h(\Psi_s) ds\right] \\
&\leq \mathbb{E}_0^{\phi_0,\phi_1}\left[\int_0^\tau e^{-\lambda s} h(\Psi_s) ds\right].
\end{aligned} \tag{VII.4}$$

Here the third equality follows from the strong Markov property. This concludes the proof of the lemma. ■

The following dynamic programming principle can be proven by the special representation of the stopping times of a jump process, Lemma 5.3 and strong Markov property.

Lemma 7.2: For any bounded function f we have

$$J_t f(\phi_0, \phi_1) = Jf(t, \phi_0, \phi_1) + e^{-(\lambda+2\beta)t} J_0 f(x(t,\phi_0), y(t,\phi_1)). \tag{VII.5}$$

Let us denote $r_n(\phi_0, \phi_1) \triangleq r_n^0(\phi_0, \phi_1)$ (see (V.7)) and

$$r(\phi_0, \phi_1) \triangleq \inf\{t \geq 0 : JV(t, \phi_0, \phi_1) = J_0 V(\phi_0, \phi_1)\}. \tag{VII.6}$$

Corollary 7.1:

$$\begin{aligned} r_n &= \{t \geq 0 : v_{n+1}(x(t,\phi_0), y(t,\phi_1)) = 0\} \\ r &= \{t \geq 0 : v(x(t,\phi_0), y(t,\phi_1)) = 0\}. \end{aligned} \tag{VII.7}$$

Therefore, since $\Gamma \subset \Gamma_n$, $r_n \uparrow r$.

Proof: Since the continuation region $\mathbb{C}_n \subset \mathbb{C}$ is bounded, it follows that $r_n < \infty$.

$$\begin{aligned} &Jv_n(r_n, \phi_0, \phi_1) = J_{r_n} v_n(\phi_0, \phi_1) \\ &= Jv_n(r_n, \phi_0, \phi_1) + e^{-(\lambda+2\beta)r_n} v_{n+1}(x(r_n, \phi_0), y(r_n, \phi_1)), \end{aligned} \tag{VII.8}$$

where the second equality follows from Lemma 7.2 and the fact that $J_0 v_n = v_{n+1}$. Hence, $v_{n+1}(x(r_n,\phi_0), y(r_n,\phi_1)) = 0$. For $t \in (0, r_n)$ we have $Jv_n(t, \phi_0, \phi_1) > J_0 v_n(\phi_0, \phi_1) =$

$J_{r_n} v_n(\phi_0, \phi_1) = J_t v_n(\phi_0, \phi_1)$, since the function $s \to J_s v_n(\phi_0, \phi_1)$ is non-decreasing. Using Lemma 7.2 we can write

$$\begin{aligned} &J_0 v_n(\phi_0, \phi_1) = J_t v_n(\phi_0, \phi_1) \\ &= J v_n(\phi_0, \phi_1) + e^{-(\lambda+2\beta)} v_{n+1}(x(t, \phi_0), y(t, \phi_1)), \end{aligned} \tag{VII.9}$$

which implies that $v_{n+1}(x(t, \phi_0), y(t, \phi_1)) < 0$ for all $t < r_n$.

The proof for the representation of r can be proven using the same line of arguments and the fact that $J_0 V = V$. The fact that $J_0 V = V$ can be proven by the dominated convergence theorem, since the sequences $(v_n(\phi_0, \phi_1))$ and $(J v_n(t, \phi_0, \phi_1))$ are decreasing, and since v_n is bounded. ■

In the next lemma we construct optimal stopping times for the family of problems introduced in (V.1).

Lemma 7.3: Let us denote $S_n \triangleq S_n^0$, where S_n^{ε} is defined in Theorem 5.1 for $\varepsilon \geq 0$. Then the sequence $(S_n)_{n \in \mathbb{N}}$ is an almost surely increasing sequence. Moreover $S_n < \tau^*$ almost surely for all n.

Proof: Since $r_1 > 0$, using Corollary 7.1 we can write

$$S_2 - S_1 = \left\{ \begin{array}{ll} r_1 - r_0, & \text{if } \sigma_1 > r_1 \\ \sigma_1 - r_0 + S_1 \circ \theta_{\sigma_1}, & \text{if } r_0 < \sigma_1 \leq r_1 \\ S_1 \circ \theta_{\sigma_1} & \text{if } \sigma_1 \leq r_0 \end{array} \right\} > 0. \tag{VII.10}$$

Now let us assume $S_n - S_{n-1} > 0$ almost surely. From Lemma 7.1 we have that $r_n > r_{n+1}$. Then we can write

$$S_{n+1} - S_n = \left\{ \begin{array}{ll} r_n - r_{n-1}, & \text{if } \sigma_1 > r_n \\ \sigma_1 - r_{n-1} + S_n \circ \theta_{\sigma_1}, & \text{if } r_{n-1} < \sigma_1 \leq r_n \\ (S_n - S_{n-1}) \circ \theta_{\sigma_1} & \text{if } \sigma_1 \leq r_{n-1} \end{array} \right\} > 0, \tag{VII.11}$$

which proves the first assertion of the lemma.

From Corollary 7.1 it follows that $\tau^* \wedge \sigma_1 = r \wedge \sigma_1$. Therefore $\tau^* \wedge \sigma_1 > r_0 \wedge \sigma_1 = S_1$, since $r_0 < r$. Now we will assume that $S_n \leq \tau^*$ and show that $S_{n+1} < \tau^*$. On $\{\sigma_1 \leq r_n\}$ we have that

$$S_{n+1} = \sigma_1 + S_n \circ \theta_{\sigma_1} < \sigma_1 + \tau^* \circ \theta_{\sigma_1}. \tag{VII.12}$$

Since $\tau^* \wedge \sigma_1 = r \wedge \sigma_1$ and $r > r_n$, if $\sigma_1 \leq r_n$, then $\tau^* \wedge \sigma_1 = \sigma_1$. Because τ^* is a hitting time, on the set $\{\sigma_1 \leq r_n\}$ we can write

$$S_{n+1} \leq \sigma_1 + \tau^* \circ \theta_{\sigma_1} = \tau^*.$$

On the other hand if $\sigma_1 > r_n$, then $\tau^* \wedge \sigma_1 = r \wedge \sigma_1 > r_n$. Therefore on $\{\sigma_1 > r_n\}$, $S_{n+1} = r_n < \tau^*$. Which concludes the proof of the second assertion. ■

Lemma 7.4: Let us define $S^* \triangleq \lim_n S_n$; then $S^* \leq \tau$ almost surely. Moreover S^* is an optimal stopping time, i.e.,

$$V(\phi_0, \phi_1) = \mathbb{E}_0^{\phi_0, \phi_1} \left[\int_0^{S^*} e^{-\lambda s} h(\Psi_s) ds \right],$$

Proof:

$$\begin{aligned} &\mathbb{E}_0^{\phi_0, \phi_1} \left[\lim_n \int_0^{S_n} e^{-\lambda t} h(\Psi_t) dt \right] \\ &\quad \leq \liminf_n \mathbb{E}_0^{\phi_0, \phi_1} \left[\int_0^{S_n} e^{-\lambda t} h(\Psi_t) dt \right] \\ &\quad = \lim_n V_n(\phi_0, \phi_1) = V(\phi_0, \phi_1). \end{aligned} \tag{VII.13}$$

The first inequality follows from Fatou's inequality, which we can apply since

$$\int_0^{S_n} e^{-\lambda t} h(\Phi_t) dt \geq \int_0^{\infty} e^{-\lambda t} h(\Phi_t) dt \geq -\frac{\sqrt{2}}{c}, \quad \text{a.s.}$$

The first equality in (VII.13) follows from Theorem 5.1. Now it can be seen from (VII.13) that S^* is an optimal stopping time. ■

Proof of Theorem 7.1. S^* introduced in Lemma 7.4 is an optimal stopping time such that $S^* \leq \tau^*$. On the other hand Lemma 7.1 tell us that all the optimal stopping times are greater than equal to τ^*. These arguments imply that $\tau^* = S^*$ is the smallest optimal stopping time of (III.3). □

VIII. Conclusion

We have solved a multi-source quickest detection problem in which the aim is to detect the minimum of two disorder times. Our approach can easily be generalized to problems including several dimensions. In the future, using the techniques developed here, we would like to solve a multi-source detection problem where the observations come from correlated sources.

IX. Acknowledgment

This work was supported in part by the U.S. Army Pantheon Project.

References

[1] M. Basseville and I. V. Nikiforov, *Detection of Abrupt Changes: Theory and Application*. Englewood Cliffs, N.J: Prentice Hall, 1993.
[2] E. Bayraktar and S. Dayanik, "Poisson disorder problem with exponential penalty for delay," submitted, 2003.
[3] E. Bayraktar, S. Dayanik, and I. Karatzas, "Adaptive poisson disorder problem," submitted, 2005.
[4] ——, "The standard Poisson disorder problem revisited," *to appear in Stochastic Processes and Their Applications*, 2005.
[5] P. Brémaud, *Point Processes and Queues*. Berlin: Springer-Verlag, 1981.
[6] M. H. A. Davis, *Markov Models and Optimization*. London: Chapman & Hall, 1993.
[7] P. Dube and R. Mazumdar, "A framework for quickest detection of traffic anomalies in networks," preprint, Purdue University, 2001.
[8] R. S. Lipster and A. N. Shiryaev, *Statistics of Random Processes*. Berlin: Springer-Verlag, 2001.
[9] S. N. Neftci, "Optimal prediction of cyclical downturns," *Journal of Economic Dynamics and Control*, vol. 4, pp. 225–241, 1982.
[10] H. V. Poor, "Quickest detection with exponential penalty for delay," *Annals of Statistics*, pp. 2179–2205, 1998.
[11] C. Sonesson and D. Bock, "A review and discussion of prospective statistical surveilance in public health," *Journal of Royal Statistical Society, A.*, vol. 166, pp. 5–21, 2003.