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A Simple Class of Functions for the Analysis/Optimization of Robotic Motions

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Abstract— **The paper deals with the problem of analyzing and eventually optimizing robotic motions using a simple class of feasible trajectories for the joints. The basic idea stems from the analysis of motions in sports. Many tasks in sport are not specified in term of equivalent joint trajectories but mainly with boundary conditions of joint position and velocity (e.g. weight lifting, punching, swinging in golf). In such tasks even if the 'optimal' trajectory is not a-priori known, some properties can be however inferred and therefore translated in the shape of the representing functions.**

I. Introduction

The analysis and optimization of robotic motions, even with reference to human tasks, can be performed with a variety of techniques. Among those, a well established approach is that of using some set of parameterized functions (polynomial, harmonic, neural networks, different kind of spline) able to represent the unknown control and or joint signals and then finding the optimal set of parameters by means of some procedure (direct optimization, learning) [1],[2],[3],[4],[6],[7]. What is generally left out from the direct approach is the use of some information about the trajectories to be found. There are a number of tasks for which, even if the optimal time-trajectory is not known, it is possible to safely assume some property (e.g. monotonic behavior). Consider, for example, the javelin launch. If we consider the system from the shoulder to the wrist, it is very unlikely that the time trajectories of the joints located in the shoulder, elbow and wrist can have non monotonic behavior from the beginning to the exact moment of the launch (not considering the run). So, if we would analyze such motion using any 'standard' set of functions, we should have to introduce a sufficient number of parameters and to trust the results of the optimization, with little (if any) possibility of forcing the monotonic property in other way than introducing additional constraints to the optimization procedure itself. As an example, with nevertheless very remarkable results, see [6].

In this paper the basic approach is the classic one, i.e. parameterize the time-trajectories and then find the set of optimal parameters. The difference is first in a trivial but useful decomposition of the trajectory in term of component functions and second to characterize such functions with a structure where the parameters influence in a predictable and controllable way their shape. This aspect is most useful for reducing the nominal number of parameters to a subset of them depending on the characteristics of the task to be analyzed.

The paper is organized as follows: Section II is devoted to the definition of the proposed basis functions starting first with a very simple (albeit representative) structure and then making it more complex (basically 'nesting' the simple structure onto itself) in order to account for a broader class of boundary conditions. Section III will validate the proposed functions, even with a reduced set of parameters, with the analysis of the golf swing.

II. Function decomposition and structure of the proposed class

Consider a robotic structure with N degrees of freedom, in which each joint variable q_i has to start from $q_i(0)$ and to $q_i(T)$ at time T. Then whatever will be the trajectory $q_i(t)$ it will can be written as

$$q_i(t) = q_i(0) + [q_i(T) - q_i(0)]f_{i,01}(t) + \epsilon f_{i,00}(t) \quad t \in [0, T] \tag{1}$$

for some constant ϵ and with functions $f_{i,01}(t)$ and $f_{i,00}(t)$ such that

$$f_{i,01}(0) = 0 \; ; \; f_{i,01}(T) = 1 \tag{2}$$

$$f_{i,00}(0) = 0 \; ; f_{i,00}(T) = 0 \tag{3}$$

$$\tag{4}$$

Defining

$$q(t) = col[q_i(t), i = 1, N] \tag{5}$$

in general, to find the optimal motion with respect to the cost functional $J(q(t), u(t), T)$, one has to represent in a parametric way either $f_{i,01}(t)$ and $f_{i,00}(t)$ and proceed to find the optimal parameters. An usual issue about this approach is the relationship between the number of parameters and the control on the shape of the solution.

A. A very simple structure

As a first step a very simplified structure of functions $f_{i,01}$ and $f_{i,00}$ is proposed just to illustrate the related properties. The very basic element of the proposed decomposition is the following (dropping the subscript i):

$$f_{01}(x, t, T) = \frac{1 - e^{xt/T}}{1 - e^{x}} \tag{6}$$

Such a function, for different values of x represents different shapes that are, even at this early stage, very usable since with a single parameter can span a family that may very well represent a monotonic time behavior. Figure 1 report a set of such shapes for different values of x.

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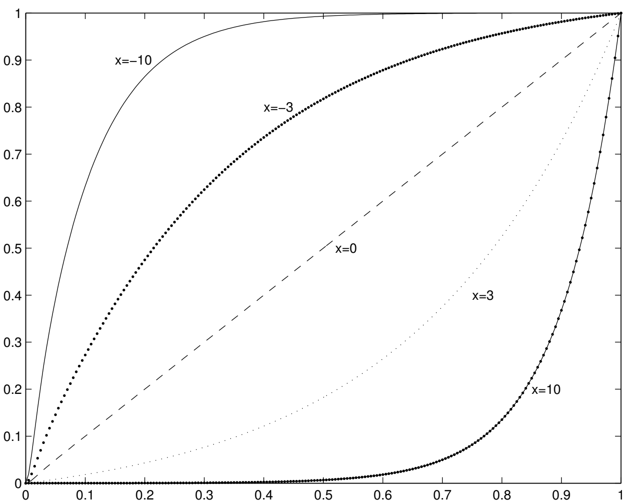

Fig. 1. Set of $f_{01}(x,t,T)$ for $t \in [0,1]$ and different values of x

On such a basis it is also possible to express $f_{00}(t)$ as

$$f_{00}(y,t,T) = \frac{-e^{y} + e^{y(1-t/T)} + e^{yt/T} - 1}{1-e^{y}} \tag{7}$$

whose shape, with varying y, is reported in Figure 2

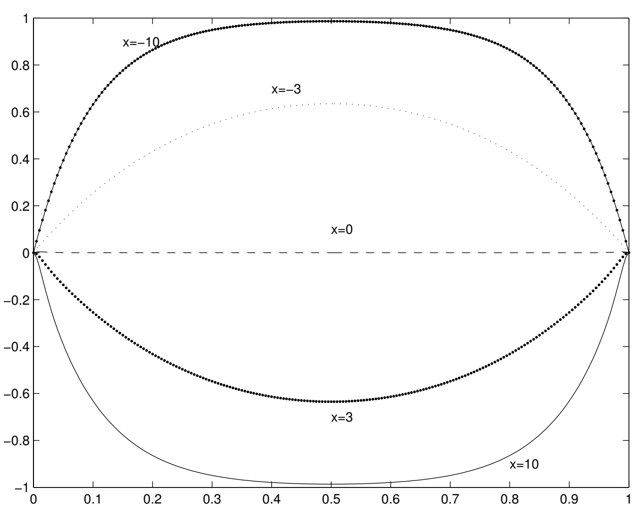

Fig. 2. Set of $f_{00}(y,t,T)$ for $t \in [0,1]$ and different values of y

As a consequence a three parameter expression of a possible trajectory is readily built:

$$\begin{aligned} q(x,y,t,T) = q(0) + [q(T)-q(0)]\frac{1-e^{xt/T}}{1-e^{x}} + \\ +\epsilon\frac{-e^{y} + e^{y(1-t/T)} + e^{yt/T} - 1}{1-e^{y}} \end{aligned} \tag{8}$$

A set of shapes corresponding to $q(0) = 0$, $q(T) = 1$, $\epsilon = 1$, and for some couple x, y is reported in the following Figure 3.

It is quite evident that for fixed $q(0)$ and $q(T)$ even such a simple structure can represent a quite useful family of functions. Just as an example consider the following trivial:

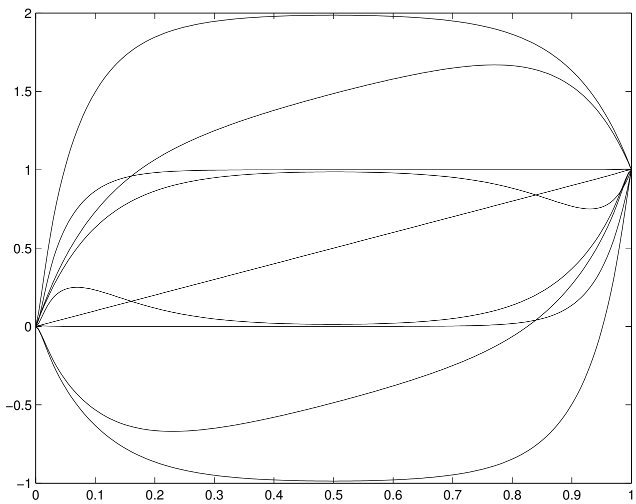

Fig. 3. Set of $q(x,y,t)$ for $t \in [0,1]$ and different values of (x,y)

Problem 1: given the system

$$\dot{\theta} = u$$

find the control command $u(t), t \in [0,T]$ such that starting from $\theta(0) = 0$ it minimizes

$$\int_0^T u^2(\tau)d\tau$$

s.to

$$\theta(T) = 1$$

△

From standard control theory (or variational calculus) it is well known that the optimal control and time trajectory are given by

$$u(t) = 1/T \quad \theta(t) = t/T$$

Such functions are generated by (8) by choosing $q(0) = \theta(0) = 0$, $q(T) = \theta(T) = 1$, $x = 0$, $\epsilon = 0$.

□

B. The proposed structure

In order to justify the more complex structure that will be presented, first ask how far can one go with (8) if some other boundary conditions are given. If for instance one is given not only the initial and final postures but also initial and final velocities, such specifications translate immediately into constraints on the values of the parameters eventually preventing the possibility of finding a feasible set of them. In this case it will be apparent the need of increasing the number of parameters and hence the complexity of the basis functions. Consider for example the following:

Problem 2: With $T = 1$, $q(0) = 0$ and $q(1) = 1$, find a triple (x, y, ϵ) such that

$$\dot{q}(0) = 0 \ \text{ and } \ \dot{q}(1) = \dot{q}^*$$

△

For such a problem, differentiating (8) with respect to time and computing the time derivatives in $t = 0$ and $t = T$ respectively lead to the following set of conditions:

$$-\frac{x}{1-e^x} + \epsilon y = 0 \tag{9}$$

$$-\frac{xe^x}{1-e^x} - \epsilon y = \dot{q}^* \tag{10}$$

which imply to find x such that

$$-\frac{x(1+e^x)}{1-e^x} = \dot{q}^* \tag{11}$$

Now, since $-\frac{x(1+e^x)}{1-e^x} \geq 2$ the above problem can be solved only if $\dot{q}^* \geq 2$.

□

Having put into evidence the limits of (6) consider now:

$$f_{01}(x(t), t) = \frac{1 - e^{x(t)t/T}}{1 - e^{x(T)}} \tag{12}$$

Obviously, (12) gives little indications since $x(t)$ may be almost any function. For this reason the following particular structure for $x(t)$ is proposed:

$$x(t) = \alpha_{in} + [\alpha_{fin} - \alpha_{in}]\frac{1 - e^{\gamma t/T}}{1 - e^{\gamma}} \tag{13}$$

so that, defining

$$\alpha = [\alpha_{in}, \alpha_{fin}]$$

the complete form of f_{01} becomes:

$$f_{01}(\alpha, \gamma, t, T) = \frac{1 - e^{\left[\alpha_{in} + [\alpha_{fin} - \alpha_{in}]\frac{1-e^{\gamma t/T}}{1-e^{\gamma}}\right]t/T}}{1 - e^{\alpha_{fin}}} \tag{14}$$

Basically, the proposed structure just introduces a time-varying 'time constant'. In this case, the welcome property of being monotonically increasing is not guaranteed any more and depends on the new parameter introduced. However, in the special case of $\alpha_{in} = 0$, differentiating (14) with respect to time, we get (with some trivial but cumbersome computation)

$$\dot{f}_{01}(\alpha_{fin}, \gamma, t, T) > 0 \quad \forall (t \in [0, T], \alpha_{fin}, \gamma) \tag{15}$$

In figure 4 some plots of (14) are reported corresponding to $\alpha_{in} = 0$, $T = 1$ and various values of (α_{fin}, γ). Exactly in the same way we can modify f_{00} introducing a similar time-varying behavior for $y(t)$.

$$y(t) = \beta_{in} + [\beta_{fin} - \beta_{in}]\frac{1 - e^{\delta t/T}}{1 - e^{\delta}} \tag{16}$$

which is to be used in the following form of $f_{00}(t)$

$$f_{00}(\beta, \delta, t, T) = \frac{-e^{\beta_{fin}} + e^{y(T-t)(1-t/T)} + e^{y(t)t/T} - 1}{1 - e^{\beta_{fin}}} \tag{17}$$

So, the complete trajectory can be expressed as (dropping all the arguments from q to simplify the notation

$$q(\cdot) = q(0) + [q(T) - q(0)]f_{01}(\alpha, \gamma, t, T) + \\ + \epsilon f_{00}(\beta, \delta, t, T) \tag{18}$$

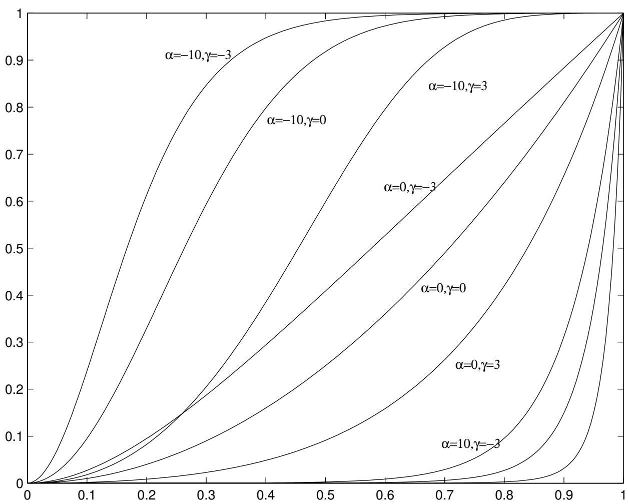

Fig. 4. Set of $f_{01}(\cdot)$ for $t \in [0, 1]$

with $f_{01}(\alpha, \gamma, t, T)$, $f_{00}(\beta, \delta, t, T)$, and $y(t)$ given by (14), (17) and (16) respectively.

Just for completeness, figure 5 reports some plots of the function $q(\alpha, \beta, \gamma, \delta, \epsilon, t, T)$ for $T = 1$, $\alpha_{in} = \beta_{in} = 0$, $\gamma = \delta$, $\epsilon = 1$, and for some values of $(\alpha_{fin}, \beta_{fin}, \gamma)$ The new number of parameters are now 7. It is not a small number but there is a fact to consider: each of them has an apparent effect. In this respect one has a great control on the parameters themselves as regards different aspects to be taken into account. For instance both α_{fin} and β_{fin} control the shape of the functions f_{01} and f_{00} in the sense that control the speed at which the final value (1 or 0 respectively) is reached. In the same way γ and δ just modulate the speed at which the relevant exponents, namely $x(t)$ and $y(t)$ approach their final values α_{fin} and β_{fin} respectively. With this in mind is not difficult to fix a-priori some of such values on the basis of a qualitative analysis of the problem to be solved as will be done in the following section.

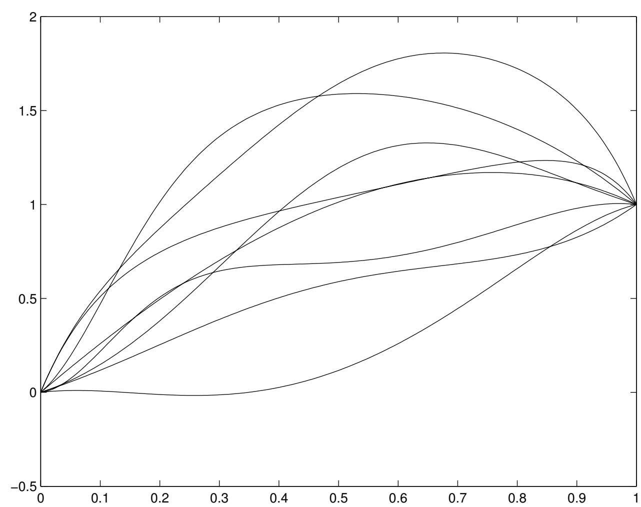

Fig. 5. Set of $f_{01}(\cdot)$ for $t \in [0, 1]$

III. ANALYSIS OF A GOLF SWING

In this section we analyze the golf swing by using a triple-pendulum model of the upper/lower arms and golf club.The first link represents the upper arms plus some contribution of the torso, the second link is for the lower arms, whereas the third link represents the hands and the golf club (a driver). The lengths, masses and moments of inertia of the links are determined in order to replicate the experiments as reported in [8] and [9].

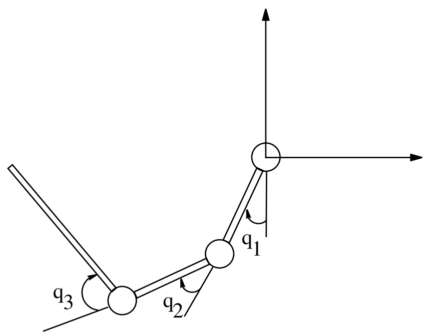

Fig. 6. Triple link model

TABLE I
TRIPLE PENDULUM PARAMETERS

	Length	Mass	M.o.I
Link 1	0.3 m	5.168 Kg	0.155 Kg$\cdot m^2$
Link 2	0.315 m	4.1 Kg	0.123 Kg$\cdot m^2$
Link 3	1.105 m	0.394 Kg	0.77 Kg$\cdot m^2$

The aim of this section is not that of suggesting a way to manage the different parameters but instead to show that, even dropping some of them, we can accurately analyze some kind of motion. In general, after the structure has been validated, one can rely on standard optimization routines in order to find the optimal parameters.

With this in mind note that since the motion to be analyzed has its own characteristics we can try to eliminate most parameters to research for just on the basis of some physical considerations. First, let us consider only the down-swing, and assume that the starting and impact position correspond to

$$(q_1(0), q_2(0), q_3(0)) = (5\pi/6, \pi/12, \pi/2)$$

and

$$(q_1(T), q_2(T), q_3(T)) = (0, 0, 0)$$

Moreover fix the boundary conditions about initial joint velocities and the down-swing time to:

$$\dot{q}_i(0) = 0 \; \forall i \;\; ; \;\; T = 0.34$$

since at the top of the back-swing we have no velocity (for the specifications on the down-swing time they are taken from real measures as reported in [9]).

Before going on, we assume that

1) All the joints have a monotonic time behavior from the top of the back-swing to the impact position since it seems unlikely that during the down-swing a joint has to invert its velocity;
2) The impact position should be reached with null torques in order to let the follow-through go by itself.

All those specifications immediately cut the number of the parameters to be considered. Consider first assumption 1. In order to fulfill the monotonic requirements, with the aim of reducing the set of parameters to be searched for, we may set for any joint $\epsilon = 0$ therefore dropping the search for β and δ and leaving to functions f_{01} the tasks of generating monotonic behaviors subject to null initial velocities. Go now with assumption 2. The triple link is subject to the usual dynamic equations

$$A(q)\ddot{q} + B(q, \dot{q})\dot{q} + C(q) = m \tag{19}$$

where $A(q)$ is the inertia matrix, $B(q, \dot{q})\dot{q}$ takes into account centrifugal and Coriolis terms , $C(q)$ represents the effects of gravity and m is the vector of applied torques. We consider the swing on a vertical plane dropping also any friction effects (these assumption are made just to compare the obtained results with the ones obtained in literature since it would not make the problem under analysis any more complex). In these conditions, the requirement about ”zero torque at impact” together with $q(T) = (0, 0, 0)$ translate immediately to

$$\ddot{q}(T) = 0$$

Summing up the boundary conditions to be satisfied are

$$q(0) = (5\pi/6, \pi/12, \pi/2)' \tag{20}$$
$$q(T) = (0, 0, 0)' \tag{21}$$
$$\dot{q}(0) = (0, 0, 0)' \tag{22}$$
$$\ddot{q}(T) = (0, 0, 0)' \tag{23}$$

The first two are actually obvious as they fix only the initial and final conditions on the trajectory and can be directly substituted in (18). Remember now that we have fixed $\epsilon = 0$, therefore we have to work only with (14). More specifically we shall have for a generic joint:

$$\dot{q}(t) = -q(0)\dot{f}_{01}(\alpha, \gamma, t, T) \tag{24}$$
$$\ddot{q}(t) = -q(0)\ddot{f}_{01}(\alpha, \gamma, t, T) \tag{25}$$

which, computed in $t = 0$ and $t = T$ respectively have to fulfill the relevant (22) and (23). Condition (22) leads to

$$\dot{f}_{01}(\alpha, \gamma, 0, T) = 0$$

To this end compute $\dot{f}_{01}(\alpha, \gamma, t, T)$. Resorting to expression (12), and taking into account that $x(T) = \alpha_{fin}$, we have

$$\dot{f}_{01} = \frac{-e^{x(t)t/T}}{T(1 - e^{\alpha_{fin}})}(\dot{x}(t)t + x(t)) \tag{26}$$

which, computed in $t = 0$ leads to

$$\frac{-1}{T(1 - e^{\alpha_{fin}})}(\alpha_{in}) = 0 \tag{27}$$

that in turn gives, for all joints $\alpha_{in} = 0$. Such a result gives also the solution to the problem of fulfilling Assumption 1. In fact, as we have already seen in the previous section, if $\alpha_{fin} = 0$ then $\dot{f}_{01} > 0$ (see (15)).

So far, consider now the requirements about the second time-derivative computed in $t = T$ (Assumption 2). Considering $\alpha_{in} = 0$ and with some computation we get the condition:

$$\frac{-\alpha_{fin} e^{\alpha_{fin}}}{T^2(1 - e^{\alpha_{fin}})} \left[\alpha_{fin} \left(1 - \frac{\gamma e^{\gamma}}{1 - e^{\gamma}} \right)^2 - (\gamma + 2) \frac{\gamma e^{\gamma}}{1 - e^{\gamma}} \right] = 0 \tag{28}$$

Therefore, to satisfy (28), for each joint $\alpha_{i,fin}$ must be chosen as follows

$$\alpha_{i,fin} = \frac{(\gamma_i + 2) \frac{\gamma_i e_i^{\gamma}}{1 - e_i^{\gamma}}}{\left(1 - \frac{\gamma_i e_i^{\gamma}}{1 - e_i^{\gamma}} \right)^2} \tag{29}$$

Note that the range of $\alpha_{i,fin}$ for varying γ_i is $-1 \leq \alpha_{i,fin} \leq 0.134$.

At this point we are left with a single parameter per joint, namely $\gamma_i, i = 1, 2, 3$, and we have to deal with the 'real' problem that is how to fix such parameters in order to make a 'good' shot without using 'unfeasible' joint torques. First recall that the flight on the ball is a function of the club-head speed at impact [5]. Since the club-head speed is the speed of the end effector we can easily compute such quantity using the Jacobian matrix at the impact position. Due to the assumption on the impact position (21), the Jacobian matrix turns out to be singular with only one row being meaningful for the computation. Namely, (reporting the only non-zero row) the Jacobian at impact results $J(q(T)) = [-1.72 - 1.42 - 1.105]$ and therefore the club-head speed v at impact can be computed as:

$$v(T) = J(q(T))\dot{q}(T) = [-1.72 - 1.42 - 1.105]\dot{q}(T) \tag{30}$$

which is apparently a function of all γ_i through $\dot{q}(T)$. The computation of $\dot{q}(T)$ yields (recalling that $\alpha_{i,in} = 0, \forall i$)

$$\dot{q}_i(T) = -q_i(0) \frac{-\alpha_{i,fin} e^{\alpha_{i,fin}}}{T(1 - e^{\alpha_{i,fin}})} \left(1 - \frac{\gamma_i e^{\gamma_i}}{1 - e^{\gamma_i}} \right) \tag{31}$$

with each $\alpha_{i,fin}$ given by (29).

Moreover, since the human body can generate different maximum torques at different joints we consider the following constraints to be satisfied for $t \in [0, 0.34]$, (see [8], [9])

$$\begin{array}{lcl} |m_1(t)| & \leq & 160 \;\; Nm \\ |m_2(t)| & \leq & 90 \;\;\; Nm \\ |m_3(t)| & \leq & 30 \;\;\; Nm \end{array} \tag{32}$$

where all the torques are computed by mean of (19). At this point can state the following optimization problem:

Problem 3: find

$$\max_{\gamma} \;\; [-1.72 - 1.42 - 1.105]\dot{q}(T) \tag{33}$$

s.to (31) and (29) for $i = 1, 3$, and conditions (32).

△

The solution of the above problem has been carried out basically using a numerical gradient approach (not the best but sufficient to show the effectiveness of the proposed structure of joint trajectories). The final results are

$$\gamma = \begin{bmatrix} 0.58 \\ 0.1 \\ 5.73 \end{bmatrix} \quad q^*(T) = \begin{bmatrix} -12.7907 \\ -1.1589 \\ -18.4202 \end{bmatrix} \quad v = 44.81 \; m/s$$

The time behaviors are plotted on the following figures

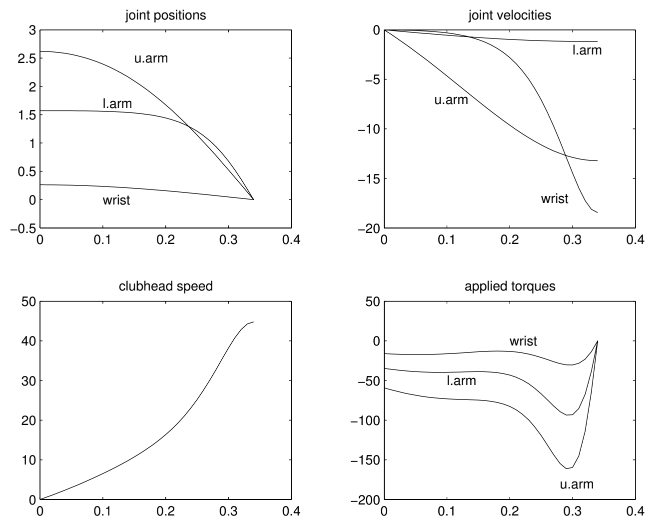

Fig. 7. Position, velocities club-head speed and applied torques

It is interesting to note that as reported in many studies (see for instance [9]) the wrist angle is kept steady for most of the down-swing. What follows is a pictorial representation of the swing with the duration of the main phases (here again in agreement with the literature).

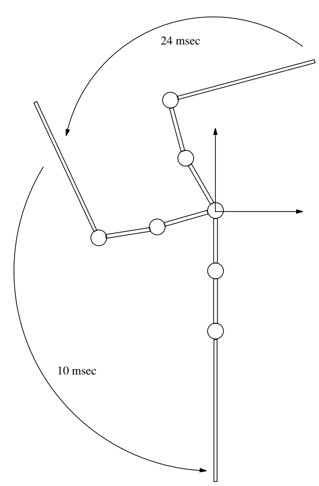

Fig. 8. Swing and phases duration

To end this section it is worth noting that the very same kind of computation could be extended to find for instance the best initial position (actually fixed in the previous study but possibly matter of optimization), or the set of parameters that guarantee 'robustness' of the impact with respect to wrong initial posture.

IV. CONCLUSIONS AND FUTURE WORKS

The paper has presented a structure for functions to be used in the analysis of robot motion with particular reference to simulated human tasks. For such problems, only boundary conditions on the trajectories are generally given whereas the actual time-behaviors are unknown and matter of optimization. The proposed structure has the advantage of not using fixed basis functions but functions whose shape in influenced by the parameters involved. Moreover each parameter has an apparent effect on the shape so, for particular tasks it is possible to override safely some of them on the basis of qualitative considerations. The structure has been used for the analysis of the golf swing giving results that are in agreement to what reported in the literature. Future works can be twofold: use of the reported structure to analyze motions in which even some boundary conditions are left as a matter of optimization, and study of the properties of the structure in order to evaluate a tradeoff between complexity and representativeness. As to the last point, it is possible to infer a 'fractal' way to increase complexity as already done from subsection II-A to II-B.

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