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A Generalized Lyapunov Stability Theorem for Discrete-time Systems based on Quadratic Difference Forms

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Abstract— **In this paper, we consider the generalized Lyapunov stability analysis for a discrete-time system described by a high order difference-algebraic equation. In the behavioral approach, a Lyapunov function is characterized in terms of a quadratic difference form. As a main result, we derive a generalized Lyapunov stability theorem that the asymptotic stability of a behavior is equivalent to the solvability of the two-variable polynomial Lyapunov equation (TVPLE) whose solution induces the Lyapunov function. Moreover, we derive another asymptotic stability condition by using a polynomial matrix solution of the one-variable dipolynomial Lyapunov equation which is reduced from the TVPLE.**

I. Introduction

The Lyapunov stability theory plays an important role in the stability analysis of a dynamical system. In this paper, we consider the generalized Lyapunov stability analysis for a linear discrete-time (time-invariant) system represented by a high order difference-algebraic equation as an extension of the traditional Lyapunov stability theorem based on the state space representation.

In the behavioral approach, a quadratic differential/difference form (QDF) is used for describing a quadratic functional such as a storage function, supply rate and Lyapunov function which play important roles in the dissipativity and stability theory [4][9]. Since there is an one-to-one correspondence between a QDF and a two-variable polynomial matrix, a QDF is useful as an algebraic tool.

Willems and Trentelman [9] derived a generalized Lyapunov stability theorem for the continuous-time system based on QDFs. In their theorem, a Lyapunov function is constructed using the solution of the (one-variable) polynomial Lyapunov equation (PLE) obtained from the two-variable polynomial equation which is referred to as the two-variable polynomial Lyapunov equation (TVPLE) in this paper. Based on the theorem in [9], Kaneko and Fujii [3], Cotroneo and Willems [1] derived asymptotic stability conditions in terms of linear matrix inequalities obtained from the TVPLE. Moreover, Peeters and Rapisarda [6] have developed an algorithm to solve the PLE by a symbolic computation based on the Faddeev's method.

In the discrete-time system, a generalized Lyapunov stability analysis based on a QDF has never been studied so far. The purpose of this paper is to prove the generalized Lyapunov stability theorem for discrete-time systems. In the discrete-time case, the equation corresponding to the continuous-time PLE becomes by the dipolynomial Lyapunov equation (DLE). Since the DLE allows negative power terms, it is not useful to reduce the proof using the DLE. Thus, we complete the proof only based on the TVPLE. Moreover, we show how the conditions of the theorem can be rewritten using the polynomial matrix solution of the DLE.

We give the notations used in this paper in the following.
$\mathbb{R}_s^{q\times q}$: the set of $q\times q$ real symmetric matrices
$\mathbb{R}[\xi]$: the set of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}^{p\times q}[\xi]$: the set of $p\times q$ polynomial matrices in the indeterminate ξ
$\mathbb{R}^{p\times q}[\zeta,\eta]$: the set of $p\times q$ polynomial matrices in the indeterminates ζ and η
$\mathbb{R}_s^{q\times q}[\zeta,\eta]$: the set of $q\times q$ real symmetric polynomial matrices in the indeterminates ζ and η
$\mathbb{R}[\xi^{-1},\xi]$: the set of dipolynomials in the indeterminate ξ
$\mathbb{R}^{p\times q}[\xi^{-1},\xi]$: the set of $p\times q$ dipolynomial matrices in the indeterminate ξ
$\mathbb{W}^{\mathbb{T}}$: the set of maps from $\mathbb{T}$ to $\mathbb{W}$
$R(\xi)^\sim := R(\xi^{-1})^\top$
diag $\begin{bmatrix} a_1 & a_2 & \cdots & a_q \end{bmatrix}$: $q\times q$ (block) diagonal matrix with (block) diagonal elements $\{a_1, a_2, \cdots, a_q\}$
$\operatorname{rank} R$: the rank of polynomial matrix $R(\xi)$
$\operatorname{rank} R(\lambda)$: the rank of constant matrix $R(\lambda)$

II. Preliminaries

In this section, we will review the basic definitions and results from the behavioral system theory.

A. Linear discrete-time system

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the set of all possible trajectories. In this paper, we will consider a *linear* time-invariant *discrete-time system* whose time axis is $\mathbb{T} = \mathbb{Z}$ and signal space is $\mathbb{W} = \mathbb{R}^q$. Such a Σ is represented by a system of linear difference-algebraic equation as

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_L w(t+L) = 0 \quad (1)$$

where $R_i \in \mathbb{R}^{\bullet\times q}$ $(i = 0, 1, \cdots, q)$ and $L \geq 0$. The variable $w \in (\mathbb{R}^q)^{\mathbb{Z}}$ is called the *manifest variable*. We call the representation of (1) a *kernel representation* of $\mathfrak{B}$. A short hand notation for (1) is

$$R(\sigma)w = 0,$$

where $R(\xi) := R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{\bullet\times q}[\xi]$. The operator σ is the shift operator defined by $(\sigma w)(t) := w(t+1)$

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and $(\sigma^T w)(t) := w(t+T)$ for all $T \in \mathbb{Z}$. Then, the behavior is given by

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} \middle| \ R(\sigma)w = 0 \right\}. \tag{2}$$

Consider another representation with an auxiliary variable described by

$$A(\sigma)\ell = 0, \ w = C(\sigma)\ell, \tag{3}$$

where $A \in \mathbb{R}^{m\times m}[\xi]$, $C \in \mathbb{R}^{p\times m}[\xi]$ and $\ell \in (\mathbb{R}^m)^{\mathbb{Z}}$ is an auxiliary variable called *latent variable*. This is called a *latent variable representation* of $\mathfrak{B}$ if

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} \,\middle|\, \exists \ \ell \in (\mathbb{R}^m)^{\mathbb{Z}} \text{ such that (3) holds.} \right\}.$$

A latent variable representation of $\mathfrak{B}$ is called *observable* if $A(\sigma)\ell = 0$ and $w = C(\sigma)\ell = 0$ imply $\ell = 0$. The representation (3) is observable if and only if the constant matrix $\begin{bmatrix} A(\lambda) & C(\lambda) \end{bmatrix}^\top$ is of full column rank for all $\lambda \in \mathbb{C}$. If this is the case, we call the pair $(A(\xi), C(\xi))$ observable.

B. Quadratic difference form

Consider a two-variable polynomial matrix in $\mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ described by

$$\Phi(\zeta,\eta) = \sum_{i=0}^{N}\sum_{j=0}^{N} \Phi_{ij}\zeta^i\eta^j,$$

where $\Phi_{ij} \in \mathbb{R}^{q\times q}$ $(i,j = 0,1,\cdots,N)$, $N \geq 0$ and $\Phi(\zeta,\eta)^\top = \Phi(\eta,\zeta)$. This $\Phi(\zeta,\eta)$ induces a quadratic difference form (QDF)

$$Q_\Phi : (\mathbb{R}^q)^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, \ Q_\Phi(\ell)(t) := \sum_{i=0}^{N}\sum_{j=0}^{N} \ell(t+i)^\top \Phi_{ij}\ell(t+j).$$

This means that ζ and η correspond to the shift operations on $\ell(t)^\top$ and $\ell(t)$, respectively. The rate of change of the QDF $Q_\Phi(\ell)(t)$, i. e. $\nabla Q_\Phi(\ell)(t) = Q_\Phi(\ell)(t+1) - Q_\Phi(\ell)(t)$ is also a QDF. Let $\nabla\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ denote the two-variable polynomial matrix which induces $\nabla Q_\Phi(\ell)(t)$, namely $\nabla Q_\Phi(\ell)(t) = Q_{\nabla\Phi}(\ell)(t)$. Then, it is given by $\nabla\Phi(\zeta,\eta) = (\zeta\eta - 1)\Phi(\zeta,\eta)$.

A two-variable polynomial matrix $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ is called *nonnegative definite*, denoted by $\Phi \geq 0$, if $Q_\Phi(\ell)(t) \geq 0$ for all $\ell \in (\mathbb{R}^q)^{\mathbb{Z}}$ and $t \in \mathbb{Z}$. If $\Phi \geq 0$, and if $Q_\Phi(\ell) = 0$ implies $\ell = 0$, then $\Phi(\zeta,\eta)$ is said to be *positive definite*, denoted by $\Phi > 0$. We have $\Phi \geq 0$ if and only if $\Phi(\zeta,\eta)$ is factored as $\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta)$ for some $D \in \mathbb{R}^{\bullet\times q}[\xi]$.

The mapping ∂ associates a two-variable polynomial matrix and a dipolynomial matrix by

$$\partial : \mathbb{R}^{p\times q}[\zeta,\eta] \to \mathbb{R}^{p\times q}[\xi^{-1},\xi], \ \partial\Phi(\xi) := \Phi(\xi^{-1},\xi).$$

Proposition 1: [4] Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be given. Then, the following statements are equivalent.

(i) There exists a two-variable polynomial matrix $\Psi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ satisfying $\nabla\Psi(\zeta,\eta) = \Phi(\zeta,\eta)$, or equivalently $\nabla Q_\Psi(\ell)(t) = Q_\Phi(\ell)(t)$ for all $t \in \mathbb{Z}$ and $\ell \in (\mathbb{R}^q)^{\mathbb{Z}}$.

(ii) $\partial\Phi(\xi) = 0$ holds for all $\xi \in \mathbb{C}\backslash\{0\}$.

A two-variable polynomial matrix $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ is called *nonnegative definite along* $\mathfrak{B}$, if $Q_\Phi(w)(t) \geq 0$ for all $w \in \mathfrak{B}$ and $t \in \mathbb{Z}$. We denote this by $\Phi \overset{\mathfrak{B}}{\geq} 0$. Moreover, if $\Phi \overset{\mathfrak{B}}{\geq} 0$ and $\{Q_\Phi(w) = 0 \implies w = 0\}$, we call $\Phi(\zeta,\eta)$ is *positive definite along* $\mathfrak{B}$, denoted $\Phi \overset{\mathfrak{B}}{>} 0$.

C. $\mathfrak{B}$-equivalence of polynomial matrices

We introduce the notion of $\mathfrak{B}$-*equivalence* of polynomial matrices.

Definition 1: [9]

(i) Polynomial matrices $D_1 \in \mathbb{R}^{p\times q}[\xi]$ and $D_2 \in \mathbb{R}^{p\times q}[\xi]$ are $\mathfrak{B}$-equivalent if $D_1(\sigma)w = D_2(\sigma)w$ for all $w \in \mathfrak{B}$ and $t \in \mathbb{Z}$. We denote this by $D_1 \overset{\mathfrak{B}}{=} D_2$.

(ii) Two-variable polynomial matrices $\Phi_1 \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ and $\Phi_2 \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ are $\mathfrak{B}$-equivalent if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \mathfrak{B}$ and $t \in \mathbb{Z}$. We denote this by $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$.

The $\mathfrak{B}$-equivalence is characterized by $R \in \mathbb{R}^{p\times q}[\xi]$ which induces the kernel representation of $\mathfrak{B}$.

Proposition 2: [9]

(i) $D_1 \overset{\mathfrak{B}}{=} D_2$ if and only if there exists an $F \in \mathbb{R}^{p\times p}[\xi]$ satisfying $D_1(\xi) - D_2(\xi) = F(\xi)R(\xi)$.

(ii) $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$ if and only if there exists a $G \in \mathbb{R}^{p\times q}[\zeta,\eta]$ satisfying

$$\Phi_2(\zeta,\eta) = \Phi_1(\zeta,\eta) + R(\zeta)^\top G(\zeta,\eta) + G(\eta,\zeta)^\top R(\eta).$$

The next proposition gives a necessary and sufficient condition for the nonnegative definiteness of $\Phi(\zeta,\eta)$ along $\mathfrak{B}$.

Proposition 3: [9] Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be given.

(i) $\Phi \overset{\mathfrak{B}}{\geq} 0$ if and only if there exist $\Phi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times q}[\xi]$ satisfying $\Phi \overset{\mathfrak{B}}{=} \Phi'$ and $\Phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$.

(ii) $\Phi \overset{\mathfrak{B}}{>} 0$ if and only if there exist $\Phi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times q}[\xi]$ satisfying $\Phi \overset{\mathfrak{B}}{=} \Phi'$ and $\Phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$ with $(R(\xi), D(\xi))$ observable.

D. R-canonical polynomial matrices

In this section, we restrict our attention to the case where a kernel representation of $\mathfrak{B}$ is induced by a nonsingular square matrix $R \in \mathbb{R}^{q\times q}[\xi]$. We define the R-*canonicity* of polynomial matrices in the following.

Definition 2: [9]

(i) A polynomial matrix $D \in \mathbb{R}^{\bullet\times q}[\xi]$ is called R-canonical if $D(\xi)R(\xi)^{-1}$ is strictly proper.

(ii) A two-variable polynomial matrix $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ is called R-canonical if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ is strictly proper.

The next lemma ensures the uniqueness of an R-canonical polynomial matrix up to $\mathfrak{B}$-equivalent polynomial matrices.

Lemma 1: [9] For any $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$, there exists a unique R-canonical $\Phi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ which is $\mathfrak{B}$-equivalent to $\Phi(\zeta,\eta)$.

Proposition 4: [9] Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be R-canonical. Then, the following statements hold.

(i) $\Phi \overset{\mathfrak{B}}{\geq} 0$ if and only if $\Phi \geq 0$.

(ii) $\Phi \overset{\mathfrak{B}}{>} 0$ if and only if there exists a $D \in \mathbb{R}^{\bullet\times q}[\xi]$ satisfying $\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta)$ with $(R(\xi), D(\xi))$ observable. Such a $D(\xi)$ is R-canonical.

III. Generalized Lyapunov Stability Theorem

In this section, we give the generalized Lyapunov stability theorem based on QDFs. Suppose that a system is given by $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ in the following. We first define the asymptotic stability of behavior.

Definition 3: [9]

(i) A behavior $\mathfrak{B}$ is said to be *autonomous* if $w_1, w_2 \in \mathfrak{B}$ and $w_1(t) = w_2(t)$ $(t < 0)$ imply $w_1 = w_2$.

(ii) A behavior $\mathfrak{B}$ is said to be *asymptotically stable* if $w \in \mathfrak{B}$ implies $\lim_{t\to\infty} w(t) = 0$.

The autonomy of $\mathfrak{B}$ is a necessary condition for the asymptotic stability of $\mathfrak{B}$.

Suppose that a polynomial matrix $R \in \mathbb{R}^{p\times q}[\xi]$ induces a kernel representation of $\mathfrak{B}$. A polynomial matrix $R(\xi)$ is called *Schur* if $\operatorname{rank} R = q$ and $\operatorname{rank} R(\lambda) = q$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$. The behavior $\mathfrak{B}$ in (2) is autonomous if and only if $\operatorname{rank} R = q$ [8]. We give a necessary and sufficient condition for the asymptotic stability of the behavior which is a well-known fact.

Lemma 2: [5] The behavior $\mathfrak{B}$ in (2) is asymptotically stable if and only if $R(\xi)$ is Schur.

By Proposition 3 (ii) and Lemma 2, we obtain Lemma 3 which gives a sufficient condition for the asymptotic stability of $\mathfrak{B}$.

Lemma 3: The behavior $\mathfrak{B}$ is asymptotically stable if there exists a two-variable polynomial matrix $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ satisfying $\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\nabla\Psi \overset{\mathfrak{B}}{<} 0$.

Proof: The proof is omitted because the lemma can be proved in the same way as the continuous-time case [9]. ■

The QDF $Q_\Psi(w)$ satisfying the conditions of Lemma 3 is called a *Lyapunov function for* $\mathfrak{B}$.

In order to derive a necessary condition, we will see how to construct a Lyapunov function under the assumption of the asymptotic stability of $\mathfrak{B}$.

Assume that $\mathfrak{B}$ is asymptotically stable. Let $\Phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ be an arbitrary two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Consider the two-variable polynomial matrix equation with unknown matrices $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and $Y \in \mathbb{R}^{p\times q}[\zeta,\eta]$

$$\nabla\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - Y(\eta,\zeta)^\top R(\eta) - R(\zeta)^\top Y(\zeta,\eta). \quad (4)$$

We refer to this equation as the *two-variable polynomial Lyapunov equation (TVPLE)*. From Propositions 2 (ii) and 3, there holds

$$\nabla\Psi \overset{\mathfrak{B}}{=} \Phi \overset{\mathfrak{B}}{<} 0 \quad (5)$$

for $\Psi(\zeta,\eta)$ and $Y(\zeta,\eta)$ satisfying the TVPLE (4). If there exists a solution of (4) satisfying $\Psi \overset{\mathfrak{B}}{\geq} 0$, then such a $\Psi(\zeta,\eta)$ induces a Lyapunov function for $\mathfrak{B}$.

Lemma 4: Assume that $\mathfrak{B}$ is asymptotically stable. Let $\Phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ be an arbitrary two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, there exist $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and $Y(\zeta,\eta) \in \mathbb{R}^{p\times q}[\zeta,\eta]$ satisfying the TVPLE (4). In this case, we have $\Psi \overset{\mathfrak{B}}{\geq} 0$.

Proof: We first show the solvability of the TVPLE (4). Without loss of generality, we assume that $R(\xi)$ is in the Smith form, i.e. $R(\xi) = \operatorname{diag}\begin{bmatrix} r_1(\xi) & r_2(\xi) & \cdots & r_q(\xi)\end{bmatrix}$, where $r_i \in \mathbb{R}[\xi]$ $(i = 1, 2, \cdots, q)$. Then, the TVPLE (4) reduces to q^2 scalar equations

$$\begin{aligned}(\zeta\eta - 1)&\psi_{ij}(\zeta,\eta) \\ &= \phi_{ij}(\zeta,\eta) - y_{ji}(\eta,\zeta)r_j(\eta) - r_i(\zeta)y_{ij}(\zeta,\eta)\end{aligned} \quad (6)$$

for $i, j = 1, 2, \cdots, q$, where $\psi_{ij}, \phi_{ij}, y_{ij} \in \mathbb{R}[\zeta,\eta]$ are the (i,j)-th elements of $\Psi(\zeta,\eta)$, $\Phi(\zeta,\eta)$ and $Y(\zeta,\eta)$, respectively. Thus, we have only to prove the existence of $\psi_{ij}(\zeta,\eta)$, $y_{ij}(\zeta,\eta)$ and $y_{ji}(\eta,\zeta)$ satisfying (6). The asymptotic stability of $\mathfrak{B}$ and Lemma 2 imply that $R(\xi)$ is Schur, i.e. both $r_i(\xi)$ and $r_j(\xi)$ are Schur. Thus, $r_i(\zeta)$, $r_j(\eta)$ and $\zeta\eta - 1$ have no common zeros when viewed as two-variable polynomials. It follows from Lemma A that there exist two-variable polynomials $\hat{\psi}_{ij}(\zeta,\eta)$, $\hat{y}_{ij}(\zeta,\eta)$ and $\hat{y}_{ji}(\eta,\zeta)$ satisfying

$$\hat{\psi}_{ij}(\zeta,\eta)(\zeta\eta - 1) + \hat{y}_{ij}(\zeta,\eta)r_i(\zeta) + \hat{y}_{ji}(\eta,\zeta)r_j(\eta) = 1.$$

By multiplying this by $\phi_{ij}(\zeta,\eta)$, we see that $\psi_{ij}(\zeta,\eta) := \phi_{ij}(\zeta,\eta)\hat{\psi}_{ij}(\zeta,\eta)$, $y_{ij}(\zeta,\eta) := \phi_{ij}(\zeta,\eta)\hat{y}_{ij}(\zeta,\eta)$ and $y_{ji}(\eta,\zeta) := \phi_{ij}(\zeta,\eta)\hat{y}_{ji}(\eta,\zeta)$ satisfies (6).

Next, we show $\Psi \overset{\mathfrak{B}}{\geq} 0$. The equation (5) is equivalent to $Q_\Psi(w)(t+1) - Q_\Psi(w)(t) = Q_\Phi(w)(t)$. Summing this up from $t = 0$ to $t = T$ yields

$$Q_\Psi(w)(T+1) - Q_\Psi(w)(0) = \sum_{t=0}^{T} Q_\Phi(w)(t).$$

Taking $T \to \infty$ in the above equation, we obtain $Q_\Psi(w)(0) = -\sum_{t=0}^{\infty} Q_\Phi(w)(t)$ from the asymptotic stability of $\mathfrak{B}$. Together with $\Phi \overset{\mathfrak{B}}{<} 0$, this implies $Q_\Psi(w)(0) \geq 0$. There holds $\Psi \overset{\mathfrak{B}}{\geq} 0$, since $w \in \mathfrak{B}$ is arbitrary. This completes the proof ■

We obtain the next theorem as a main result of this paper from Proposition 2 and Lemmas 3, 4.

Theorem 1: Let $\Phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ be an arbitrary two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, the following statements are equivalent.

(i) The behavior $\mathfrak{B}$ is asymptotically stable.

(ii) There exists a two-variable polynomial matrix $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi \overset{\mathfrak{B}}{\geq} 0 \text{ and } \nabla\Psi \overset{\mathfrak{B}}{<} 0. \quad (7)$$

(iii) There exists a two-variable polynomial matrix $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ satisfying (5) and $\Psi \overset{\mathfrak{B}}{\geq} 0$.

(iv) There exist two-variable polynomial matrices $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and $Y \in \mathbb{R}^{p\times q}[\zeta,\eta]$ satisfying the TVPLE (4) and $\Psi \overset{\mathfrak{B}}{\geq} 0$.

In the remainder of this section, we derive a necessary and sufficient condition for the asymptotic stability of $\mathfrak{B}$ in terms of the R-canonical solution of the TVPLE (4) based on Theorem 1.

Suppose that a kernel representation of $\mathfrak{B}$ is induced by the nonsingular square polynomial matrix $R \in \mathbb{R}^{q\times q}[\xi]$. Moreover, we assume that the two-variable polynomial matrix $\Phi \overset{\mathfrak{B}}{<} 0$ is R-canonical.

By Theorem 1 (i)⇔(iii), $\mathfrak{B}$ is asymptotically stable if and only if there exists $\Psi \overset{\mathfrak{B}}{\geq} 0$ satisfying (5). For this $\Psi(\zeta,\eta)$, there exists an R-canonical $\Psi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ such that $\Psi' \overset{\mathfrak{B}}{=} \Psi$. It thus follows from Proposition 2 (ii) that $\mathfrak{B}$ is asymptotically stable if and only if there exist a $\Psi' \overset{\mathfrak{B}}{\geq} 0$ and a $Y' \in \mathbb{R}^{q\times q}[\zeta,\eta]$ satisfying the TVPLE

$$\nabla\Psi'(\zeta,\eta) = \Phi(\zeta,\eta) - Y'(\eta,\zeta)^\top R(\eta) - R(\zeta)^\top Y'(\zeta,\eta). \tag{8}$$

Pre- and post-multiplying (8) by $R(\zeta)^{-\top}$ and $R(\eta)^{-1}$ yield

$$\begin{aligned}&(\zeta\eta-1)R(\zeta)^{-\top}\Psi'(\zeta,\eta)R(\eta)^{-1}\\ &\quad=R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1} - R(\zeta)^{-\top}Y'(\eta,\zeta)^\top\\ &\qquad - Y'(\zeta,\eta)R(\eta)^{-1}.\end{aligned}$$

Since the first term on the right-hand side is strictly proper in the sense of two-variable rational matrices, $\Psi'(\zeta,\eta)$ is R-canonical, namely the left-hand side is proper, if and only if $Y'(\zeta,\eta)R(\eta)^{-1}$ is proper. The latter condition is equivalent to the existence of a one-variable polynomial matrix $X' \in \mathcal{X}_R$ such that $X'(\eta) = Y'(\zeta,\eta)$, where the set $\mathcal{X}_R$ is defined by

$$\mathcal{X}_R = \left\{X \in \mathbb{R}^{q\times q}[\xi] \,\middle|\, X(\xi)R(\xi)^{-1} : \text{proper}\right\}. \tag{9}$$

Hence, the TVPLE (8) is reduced to the TVPLE

$$\nabla\Psi'(\zeta,\eta) = \Phi(\zeta,\eta) - X'(\zeta)^\top R(\eta) - R(\zeta)^\top X'(\eta) \tag{10}$$

with the unknown matrices $\Psi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ and $X' \in \mathcal{X}_R$. Moreover, since $\Psi'(\zeta,\eta)$ is R-canonical, $\Psi' \overset{\mathfrak{B}}{\geq} 0$ if and only if $\Psi' \geq 0$ from Proposition 4 (i).

As a result, we obtain the following corollary.

Corollary 1: Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be an arbitrary R-canonical two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, the behavior $\mathfrak{B}$ in (2) is asymptotically stable if and only if there exist an R-canonical $\Psi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ and $X' \in \mathbb{R}^{q\times q}[\xi]$ satisfying the TVPLE (10) and $\Psi' \geq 0$. Moreover, for $\Psi'(\zeta,\eta)$ and $X'(\eta)$ satisfying (10), $\Psi'(\zeta,\eta)$ is R-canonical if and only if $X' \in \mathcal{X}_R$.

In the next example, we explain that Corollary 1 is the generalization of Lyapunov stability theorem based on the state space equation.

Example 1: Consider the behavior $\mathfrak{B}$ described by the state space equation

$$w(t+1) = Aw(t). \tag{11}$$

Define $R(\xi) = A - \xi I_q$ and $\Phi(\zeta,\eta) = \Phi_0 = -D_0^\top D_0$, where (A, D_0) is observable in the sense of state space system. This is equivalent to the observability of $(R(\xi), D_0)$ in the sense of the definition in section II-A. In the following, we show that the TVPLE (10) is solvable if and only if so is the state-space Lyapunov equation.

Assume that there exist an R-canonical $\Psi' \geq 0$ and $X' \in \mathcal{X}_R$ satisfying the TVPLE (8). Since $R(\xi) = A - \xi I_q$, from Lemma 6.3-10 of [2], $\Psi'(\zeta,\eta)$ and $X'(\eta)$ are a constant symmetric matrix $\Psi_0 \in \mathbb{R}_{\mathrm{s}}^{q\times q}$ and a polynomial matrix with degree less than one, denoted as $X'(\eta) = X_0 + X_1\eta$, $X_0, X_1 \in \mathbb{R}^{q\times q}$, respectively. Substituting these into (8) yields

$$\begin{aligned}0 =&\nabla\Psi'(\zeta,\eta) - \Phi(\zeta,\eta) + X'(\zeta)^\top R(\eta) + R(\zeta)^\top X'(\eta)\\ =&\zeta\eta\left(\Psi_0 - X_1^\top - X_1\right) + \zeta\left(X_1^\top A - X_0\right)\\ &+ \eta\left(A^\top X_1 - X_0^\top\right) - \left(\Psi_0 + \Phi_0 - X_0^\top A - A^\top X_0\right).\end{aligned}$$

Comparing the coefficients in the above equation, there holds

$$\Psi_0 - X_1^\top - X_1 = 0,\ X_1^\top A - X_0 = 0 \tag{12}$$

$$\Psi_0 + \Phi_0 - X_0^\top A - A^\top X_0 = 0. \tag{13}$$

All pairs (X_0,X_1) satisfying (12) are parametrized by $X_0 = \Psi_0 A/2 + GA$ and $X_1 = \Psi_0/2 - G$, respectively, where $G \in \mathbb{R}^{q\times q}$ is an arbitrary skew symmetric matrix. Hence, we have $X'(\eta) = \Psi_0\left(\eta I_q + A\right)/2 + GR(\eta)$. Substituting X_0 into (13), we see that Ψ_0 must satisfy the well-known Lyapunov equation

$$A^\top\Psi_0 A - \Psi_0 = \Phi_0. \tag{14}$$

The positive definiteness of Ψ_0 follows from the observability of $(R(\xi), D_0)$.

Conversely, if Ψ_0 is the positive definite solution of (14), it is not difficult to see that $\Psi'(\zeta,\eta) := \Psi_0$ and $X'(\eta) := \Psi_0(\eta I_q + A)/2$ satisfy the TVPLE (8) by tracing back the above calculation.

Consequently, the equivalence of the solvabilities of the TVPLE (8) and the Lyapunov equation (14) has been proved.

It is well-known that the system described by (11) is asymptotically stable if and only if (14) has a positive definite solution [10]. Therefore, the above discussion shows that Corollary 1 includes the Lyapunov stability theorem based on the state space equation as the special case. ■

IV. Dipolynomial Lyapunov Equation

In Section III, we have seen that a necessary and sufficient condition for the asymptotic stability of $\mathfrak{B}$ is characterized in terms of the TVPLE (4). On the other hand, in the continuous-time case, the asymptotic stability condition is derived using the one-variable PLE as well as the TVPLE (4). In this section, we will consider the discrete-time counterpart of the results on the continuous-time PLE.

Again, let $\Phi \overset{\mathfrak{B}}{<} 0$ be an arbitrary two-variable polynomial matrix. By Proposition 1, the TVPLE (4) is equivalent to the *dipolynomial Lyapunov equation (DLE)*

$$R(\xi)^\sim X(\xi) + X(\xi)^\sim R(\xi) = \partial\Phi(\xi), \tag{15}$$

where $X \in \mathbb{R}^{p\times q}[\xi^{-1},\xi]$ is the unknown dipolynomial polynomial matrix. For the solution $X(\xi)$ to the DLE (15), let $Y \in \mathbb{R}^{p\times q}[\zeta,\eta]$ satisfy $\partial Y(\xi) = X(\xi)$. Then, we can compute a Lyapunov function for $\mathfrak{B}$ by substituting $Y(\zeta,\eta)$ into the TVPLE (4).

Lemma 5: Assume that $\mathfrak{B}$ in (2) is asymptotically stable. Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be an arbitrary two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, the following statements hold from Corollary 1 and Lemmas 5, 6.

(i) The DLE (15) has a solution.

(ii) Let $X_0 \in \mathbb{R}^{p\times q}[\xi^{-1},\xi]$ be one solution of the DLE (15). Then, all solutions of the DLE (15) are parametrized by

$$X(\xi) = X_0(\xi) + F(\xi)R(\xi),$$

where $F \in \mathbb{R}^{p\times p}[\xi^{-1},\xi]$ is the free parameter satisfying $F(\xi)^{\sim} = -F(\xi)$.

(iii) Let $Y \in \mathbb{R}^{p\times q}[\zeta,\eta]$ be such that $\partial Y(\xi) = X(\xi)$ for $X \in \mathbb{R}^{p\times q}[\xi^{-1},\xi]$ satisfying the DLE (15). Then, a Lyapunov function for $\mathfrak{B}$ is induced by

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - Y(\eta,\zeta)^{\top}R(\eta) - R(\zeta)^{\top}Y(\zeta,\eta)}{\zeta\eta - 1}. \tag{16}$$

(iv) Let $X_i \in \mathbb{R}^{p\times q}[\xi^{-1},\xi]$ $(i = 1,2)$ be solutions of the DLE (15). Let $\Psi_i \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be computed from (16) for $X_i(\xi)$. Then, we have

$$\Psi_1 \overset{\mathfrak{B}}{=} \Psi_2.$$

Proof: (i) There exists a $Y(\zeta,\eta)$ satisfying TVPLE (4) from Lemma 4. By taking $X(\xi) = \partial Y(\xi)$ for this $Y(\zeta,\eta)$, $X(\xi)$ satisfies the DLE (15).

(ii) Define $\bar{X}(\xi) := X(\xi) - X_0(\xi)$. Then, we obtain $R(\xi)^{\sim}\bar{X}(\xi) + \bar{X}(\xi)^{\sim}R(\xi) = 0$. Using $R(\xi)$ in the Smith form as in the proof of Lemma 4, the (i,j)-th element of this equation is written by

$$r_i(\xi^{-1})\bar{x}_{ij}(\xi) + \bar{x}_{ji}(\xi^{-1})r_j(\xi) = 0, \tag{17}$$

where $\bar{x}_{ij} \in \mathbb{R}[\xi^{-1},\xi]$ is the (i,j)-th element of $\bar{X}(\xi)$. Multiplying this equation by $\xi^{\deg r_i(\xi)}$, we get

$$\xi^{\deg r_i(\xi)} r_i(\xi^{-1})\bar{x}_{ij}(\xi) = -\xi^{\deg r_i(\xi)}\bar{x}_{ji}(\xi^{-1})r_j(\xi).$$

Since $r_j(\xi)$ and $\xi^{\deg r_i(\xi)}r_i(\xi^{-1})$ have no common zeros, there exists $f_{ij} \in \mathbb{R}[\xi^{-1},\xi]$ satisfying

$$\bar{x}_{ij}(\xi) = f_{ij}(\xi)r_j(\xi). \tag{18}$$

By rewriting (18) in terms of a matrix, we have $\bar{X}(\xi) = R(\xi)F(\xi)$.

Next, we show $F(\xi)^{\sim} = -F(\xi)$. Substituting (18) into (17) yields $r_i(\xi^{-1})\left\{f_{ji}(\xi^{-1}) + f_{ij}(\xi)\right\}r_j(\xi) = 0$. Since $r_i(\xi^{-1}), r_j(\xi) \neq 0$, we have $f_{ji}(\xi^{-1}) + f_{ij}(\xi) = 0$. This is equivalent to $F(\xi)^{\sim} + F(\xi) = 0$.

(iii) The proof is clear from Lemma 4.

(iv) It follows from (16) that $\nabla\Psi_1 \overset{\mathfrak{B}}{=} \nabla\Psi_2$. Then, we have

$$\begin{aligned} 0 &= \sum_{t=0}^{T-1} \nabla Q_{\Psi_1}(w)(t) - \sum_{t=0}^{T-1} \nabla Q_{\Psi_2}(w)(t) \\ &= Q_{(\Psi_1-\Psi_2)}(w)(T) - Q_{(\Psi_1-\Psi_2)}(w)(0) \end{aligned} \tag{19}$$

for all $w \in \mathfrak{B}$ and $T \in \mathbb{Z}$. Since $\mathfrak{B}$ is asymptotically stable, we have $\lim_{T\to\infty} Q_{(\Psi_1-\Psi_2)}(w)(T) = 0$. Hence, taking $T \to \infty$ in (19) yields $Q_{(\Psi_1-\Psi_2)}(w)(0) = 0$. By substituting this into (19), we get $Q_{(\Psi_1-\Psi_2)}(w)(T) = 0$ for all $w \in \mathfrak{B}$ and $T \in \mathbb{Z}$. This implies $\Psi_1 - \Psi_2 \overset{\mathfrak{B}}{=} 0$. ■

Remark 1: Since the solution $X(\xi)$ of the DLE (15) is not unique from Lemma 5 (ii), $\Psi(\zeta,\eta)$ in (16) is not unique. In contrast, Lemma 5 (iv) shows that the Lyapunov functions $Q_\Psi(w)$ is unique over $\mathfrak{B}$ for $\Phi(\zeta,\eta)$ independently of $X(\xi)$.

We derive a necessary and sufficient DLE condition for the asymptotic stability of $\mathfrak{B}$ based on Corollary 1. Suppose that a kernel representation of $\mathfrak{B}$ is induced by the nonsingular square polynomial matrix $R \in \mathbb{R}^{q\times q}[\xi]$. Moreover, we assume that the two-variable polynomial matrix $\Phi \overset{\mathfrak{B}}{<} 0$ is R-canonical. Substituting $\zeta = \xi^{-1}$ and $\eta = \xi$ into TVPLE (10), we obtain

$$X'(\xi)^{\sim}R(\xi) + R(\xi)^{\sim}X'(\xi) = \partial\Phi(\xi). \tag{20}$$

It follows that $X'(\eta)$ satisfying the TVPLE (10) is a polynomial matrix solution of the DLE (15). Thus, we have the following lemma.

Lemma 6: Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be an arbitrary R-canonical two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, the following statements are equivalent for $X' \in \mathbb{R}^{q\times q}[\xi]$.

(i) $X' \in \mathcal{X}_R$ is a polynomial matrix solution of the DLE (15).

(ii) There exists an R-canonical two-variable polynomial matrix $\Psi' \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\Psi'(\zeta,\eta)$ and $X'(\eta)$ satisfy the TVPLE (10).

Proof: (ii) ⇒ (i) Substituting $\zeta = \xi^{-1}$ and $\eta = \xi$ into (10) yields (20). Thus, the proof follows immediately.

(i) ⇒ (ii) By Propositions 1 (i)⇔(ii) and 2 (ii), there exists $\Psi' \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ satisfying

$$\nabla\Psi'(\zeta,\eta) = \Phi(\zeta,\eta) - X'(\zeta)^{\top}R(\eta) - R(\zeta)^{\top}X'(\eta),$$

which shows that $\Psi'(\zeta,\eta)$ and $X'(\eta)$ satisfies the TVPLE (10). Since $\Phi(\zeta,\eta)$ is R-canonical and $X'(\xi)R(\xi)^{-1}$ is proper, $\Psi'(\zeta,\eta)$ is R-canonical. ■

Lemma 5 is rewritten by the following lemma using the polynomial matrix solution of the DLE (15).

Lemma 7: Assume that $\mathfrak{B}$ in (2) is asymptotically stable. Let $\Phi \in \mathbb{R}_{\mathrm{s}}^{q\times q}[\zeta,\eta]$ be an arbitrary R-canonical two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Let $\mathcal{X}_R$ be the set given by (9). Then, the following statements hold.

(i) Let $X' \in \mathcal{X}_R$ be a solution of the DLE (15). Then, the R-canonical two-variable polynomial matrix which induces a Lyapunov function for $\mathfrak{B}$ is given by

$$\Psi'(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - X'(\zeta)^{\top}R(\eta) - R(\zeta)^{\top}X'(\eta)}{\zeta\eta - 1}. \tag{21}$$

(ii) Let $X'_0 \in \mathcal{X}_R$ be a solution of the DLE (15). Then, all polynomial matrix solutions $X' \in \mathcal{X}_R$ are parametrized by $X'(\xi) = X'_0(\xi) + F_0R(\xi)$, where $F_0 \in \mathbb{R}^{q\times q}$ is the free parameter satisfying $F_0^{\top} = -F_0$.

(iii) Let $\Psi'(\zeta,\eta)$ be computed by (21). Then, $\Psi'(\zeta,\eta)$ is determined uniquely for $\Phi(\zeta,\eta)$ independently of the solution $X' \in \mathcal{X}_R$ of the DLE (15).

Remark 2: As explained in Remark 1, there does not hold the uniqueness of $\Psi'(\zeta,\eta)$ in general. But, Lemma 7 (iii) claims that $\Psi'(\zeta,\eta)$ is unique if it is restricted to be an R-canonical polynomial matrix.

From Corollary 1 and Lemmas 5, 6, we obtain a necessary and sufficient condition for the asymptotic stability of $\mathfrak{B}$ in terms of a polynomial matrix solution of the DLE (15).

Theorem 2: The behavior $\mathfrak{B}$ is asymptotically stable if and only if there exists a polynomial matrix solution $X' \in \mathcal{X}_R$ of the DLE (15) such that $\Psi'(\zeta,\eta)$ in (21) satisfies $\Psi' \geq 0$.

Example 1 (continued): We show the relationship between DLE (15) and the asymptotic stability of the behavior $\mathfrak{B}$ described by the state space equation (11). Let $X'(\xi) = \Psi_0\left(\xi I_q + A\right)/2$. Then, we have

$$R(\xi)^{\sim}X'(\xi) + X'(\xi)^{\sim}R(\xi) = A^{\top}\Psi_0 A - \Psi_0 = \Phi_0.$$

This implies that $X' \in \mathcal{X}_R$ is the polynomial matrix solution of the DLE (15). Since $\Psi'(\zeta,\eta)$ computed by (21) is nonnegative definite, $\mathfrak{B}$ is asymptotically stable by Theorem 2. ■

V. Numerical Example

Example 2: Consider the behavior $\mathfrak{B}$ whose kernel representation is induced by the polynomial matrix

$$R(\xi) = \begin{bmatrix} 3\xi - 1 & 4\xi^2 - 2 \\ 0 & 2\xi^2 - 1 \end{bmatrix}.$$

Define the (R-canonical) two-variable polynomial matrix

$$\Phi(\zeta,\eta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 - \zeta\eta \end{bmatrix} \overset{\mathfrak{B}}{<} 0.$$

Then, a solution of the TVPLE (4) is given by

$$\Psi(\zeta,\eta) = \begin{bmatrix} \frac{9}{8} & 0 \\ 0 & \frac{3}{8}\zeta\eta + \frac{5}{3} \end{bmatrix},$$
$$Y(\zeta,\eta) = \begin{bmatrix} -\frac{1}{16}(3\eta+1) & 0 \\ \frac{1}{8}(3\eta+1) & -\frac{1}{3}(2\eta^2+1) \end{bmatrix}.$$

For $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^{\top} \in \mathfrak{B}$, the QDFs $Q_\Psi(w)$ and $\nabla Q_\Psi(w)$ are given by

$$Q_\Psi(w)(t) = \frac{9}{8}w_1(t)^2 + \frac{5}{3}w_2(t)^2 + \frac{8}{3}w_2(t+1)^2,$$
$$\nabla Q_\Psi(w)(t) = -w_1(t)^2 - w_2(t)^2 - w_2(t+1)^2.$$

Since $\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\nabla\Psi \overset{\mathfrak{B}}{<} 0$, $\mathfrak{B}$ is asymptotically stable, and $Q_\Psi(w)$ is a Lyapunov function for $\mathfrak{B}$.

Also, a polynomial matrix solution of the DLE (15) is given by

$$X'(\xi) = \begin{bmatrix} -\frac{1}{16}(3\xi+1) & 0 \\ \frac{1}{8}(3\xi+1) & -\frac{1}{3}(2\xi^2+1) \end{bmatrix} \in \mathcal{X}_R.$$

Computing $\Psi'(\zeta,\eta)$ for this $X'(\xi)$ by (21) yields

$$\Psi'(\zeta,\eta) = \Psi(\zeta,\eta) = \begin{bmatrix} \frac{9}{8} & 0 \\ 0 & \frac{3}{8}\zeta\eta + \frac{5}{3} \end{bmatrix} \geq 0.$$

Therefore, the asymptotic stability is proved from Theorem 2, too. ■

VI. Conclusions and Future Works

In this paper, we have derived a generalized Lyapunov stability theorem for a system represented by a high order difference-algebraic equation (kernel representation) based on QDFs. In the theorem, we have shown that the asymptotic stability of the behavior is equivalent to the solvability of the TVPLE (4) whose solution induces the Lyapunov function. Moreover, we have clarified the relationship between the TVPLE (4) and the DLE (15). This results in another asymptotic stability condition of the behavior in terms of a polynomial matrix solution of the DLE (15). As future works, we have to study the algorithm for solving the TVPLE (4) using linear matrix inequalities of the coefficient matrices, or a symbolic computation.

Appendix

A. Hilbert's Nullstellensatz

Lemma A: [7] n-variable polynomials $r_i(\xi_1,\cdots,\xi_n)$ $(i = 1,2,\cdots,l)$ have no common zeros if and only if there exist n-variable polynomials $g_i(\xi_1,\cdots,\xi_n)$ $(i = 1,2,\cdots,l)$ satisfying the following n-variable polynomial equation

$$\sum_{i=1}^{l} g_i(\xi_1,\cdots,\xi_n)r_i(\xi_1,\cdots,\xi_n) = 1.$$

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