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A Robust Nonlinear Controller with Application to a Magnetic Bearing System

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***Abstract*— In this paper, a robust nonlinear controller for a nonlinear system subject to model uncertainties is proposed. This controller consists in the association of a "robust feedback linearization" with a robust linear H_∞ controller. The robust feedback linearization yields a linear system equal to the linear approximation of the original nonlinear system around a nominal operating point. The robustness of the resulting overall nonlinear controller is proved by theoretical arguments and illustrated through an application example. The advantage of the robust feedback linearization with respect to the classical one is emphasized.**

I. Introduction

Feedback linearizing laws are applied to nonlinear systems to obtain a system that can be regulated using a linear controller. This is the main advantage of the feedback linearization, because for linear systems the choice of techniques is wider and the design is easier.

However, the classical feedback linearization [1] has the disadvantage of simplifying the nonlinearities of the system, which might result in a closed-loop that is not robust in the presence of uncertainties. This simplification also causes the loss of the physical meaning for the linearized system, which is in the Brunovsky form.

A new form of feedback linearization, called robust feedback linearization, was proposed in [2]. This method gives a linearizing control law that transforms the nonlinear system in its linear approximation around a nominal operating point. Thus, it causes only a small transformation in the natural behavior of the original system, which is desired in order to obtain robustness.

In this paper, the robustness properties of the robust feedback linearization when associated with a Glover-McFarlane H_∞ controller [3] are demonstrated by theory and illustrated through an application example. Furthermore, by the definition of a nonlinear robustness gain, it became possible to measure how robust the resulting closed-loop system is, thus giving mathematical substantiation to what was, until now, an intuitive result.

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The paper is organized as follows. In Section II, both the classical and the robust feedback linearizing methods are recalled. In Section III, the dual case (normalized right coprime factorization) of the Glover-McFarlane method for the design of an H_∞ controller is presented. The robustness properties of the association of the robust feedback linearization with a Glover-McFarlane H_∞ controller are demonstrated in Section IV. Finally, the theory is illustrated by the application of the two feedback linearizing methods to a magnetic bearing system, in Section V, where the design of both controllers is presented and results of simulations are given.

II. Feedback Linearization

Consider the nonlinear system with n states and m inputs described by the state-space equation

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ is the control input and $f(x), g_1(x), \cdots, g_m(x)$ are smooth vector fields defined on an open subset of $\mathbb{R}^n$. Suppose that this system satisfies the well-known conditions for feedback linearization [1]. A vector $\lambda(x) = \begin{bmatrix}\lambda_1(x) & \cdots & \lambda_m(x)\end{bmatrix}^{\mathrm{T}}$, formed by functions $\lambda_i(x)$ with relative degree r_i (such that $r_1 + \cdots + r_m = n$), is chosen as the output of the system (1), that is,

$$y(x) = \lambda(x) \tag{2}$$

Thus, the system composed by (1) and (2) is square.

The objective here is to linearize this system by feedback in a neighborhood of an operating point x_0 chosen, with no loss of generality, as $x_0 = 0$. Two different forms of feedback linearization are presented next. It is assumed that the state is available for control purposes.

A. Classical Feedback Linearization

The classical feedback linearization [1] is accomplished by using a linearizing control law of the form

$$u_{\mathrm{c}}(x, w) = \alpha_{\mathrm{c}}(x) + \beta_{\mathrm{c}}(x)w \tag{3}$$

where w is a linear control, and the change of coordinates (diffeomorphism)

$$x_{\mathrm{c}} = \phi_{\mathrm{c}}(x) \tag{4}$$

The linearized system is

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}w \tag{5}$$

where A_{c} and B_{c} are the matrices in the Brunovsky canonical form [1]. The expressions for $\alpha_{\mathrm{c}}(x)$, $\beta_{\mathrm{c}}(x)$ and $\phi_{\mathrm{c}}(x)$ are recalled in Appendix I-A.

B. Robust Feedback Linearization

The main difference between the robust feedback linearization [2] and the classical one is that the linearized system has the form

$$\dot{x}_{\mathrm{r}} = A_{\mathrm{r}}x_{\mathrm{r}} + B_{\mathrm{r}}v \tag{6}$$

with

$$A_{\mathrm{r}} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} \quad \text{and} \quad B_{\mathrm{r}} = g(0)$$

which corresponds to the linear approximation of the nonlinear system (1).

The robust feedback linearization is accomplished by using a linearizing control law of the form

$$u(x,v) = \alpha(x) + \beta(x)v \tag{7}$$

where v is a linear control, and the change of coordinates (diffeomorphism)

$$x_{\mathrm{r}} = \phi(x) \tag{8}$$

The expressions for $\alpha(x)$, $\beta(x)$ and $\phi(x)$ are recalled in Appendix I-B. More details on how the robust feedback linearization is derived are given in [2].

III. H_∞ Robust Stabilization

In this section, it is presented the method used to calculate the linear controllers for linearized systems (5) and (6). These are the controllers applied, together with the corresponding feedback linearization, to the nonlinear system (1).

The Glover-McFarlane method [3] deals with the H_∞ robust stabilization problem of perturbed linear plants, given by a normalized left coprime factorization. The case of a normalized right coprime factorization (needed for the nonlinear analysis in Section IV) is considered below.

Lemma 1: Consider a strictly proper[1] linear system, controllable and observable, with transfer matrix $G(s)$, which has a normalized right coprime factorization given by $G(s) = N_{\mathrm{R}}(s)M_{\mathrm{R}}^{-1}(s)$ and a family of perturbed plants, also controllable and observable, with transfer matrices

$$G_{\mathrm{P}}(s) = (N_{\mathrm{R}}(s) + \Delta_{\mathrm{N_R}}(s))(M_{\mathrm{R}}(s) + \Delta_{\mathrm{M_R}}(s))^{-1}$$

where $\Delta_{\mathrm{M_R}}(s)$ and $\Delta_{\mathrm{N_R}}(s)$ are stable unknown transfer matrices which represent the uncertainty in the system. A controller K that guarantees

$$\left\| M_{\mathrm{R}}^{-1}(s)(I - K(s)G(s))^{-1}\begin{bmatrix} K(s) & I \end{bmatrix} \right\|_\infty \leq \gamma$$

[1]Only the strictly proper case is of interest in this paper, since it deals with systems whose output depends directly only on the state of the system, not on its input.

for a given $\gamma > \gamma_{\min}$, that is, a controller K that guarantees that the closed-loop system is stable for uncertainties such that

$$\left\| \begin{bmatrix} \Delta_{\mathrm{N_R}}(s) \\ \Delta_{\mathrm{M_R}}(s) \end{bmatrix} \right\|_\infty < \frac{1}{\gamma}$$

is given by

$$K = \left[\begin{array}{c|c} A - ZC^{\mathrm{T}}C + \gamma^2 BB^{\mathrm{T}}XL^{-1} & ZC^{\mathrm{T}} \\ \hline \gamma^2 B^{\mathrm{T}}XL^{-1} & 0 \end{array}\right] \tag{9}$$

with $L = (1-\gamma^2)I + ZX$, where (A,B,C) is a minimal state-space realization of $G(s)$ and Z and X are the unique positive definite solutions to the algebraic Riccati equations

$$AZ + ZA^{\mathrm{T}} - ZC^{\mathrm{T}}CZ + BB^{\mathrm{T}} = 0$$
$$A^{\mathrm{T}}X + XA - XBB^{\mathrm{T}}X + C^{\mathrm{T}}C = 0$$

Proof: The proof is obtained by “dualizing” the one given in [3] for the case of a left coprime factorization. □

Remark: As shown in [4, Corollary 18.8], K is also an H_∞ controller for the normalized left coprime factorization of $G(s)$, but its above state space representation (9) is dual to the one usually obtained in the latter case.

IV. Robustness Properties

In this section, it is proved that the robustness properties of the controller obtained by the method of Glover-McFarlane for the linearized system are kept when this controller is applied, together with the robust feedback linearization, to the nonlinear system.

This demonstration uses the concept of local W-stability [5], [6], which allows the analysis of the local input-output stability of a nonlinear system by the means of a local version of the Small Gain Theorem. The local W-gain of a nonlinear system $\mathbf{H}$ is denoted by $\gamma_{\mathrm{W_l}}(\mathbf{H})$. One of the main properties of this gain is the following one:

Property 1: Let $\mathbf{H}$ be a nonlinear system with state x, equilibrium $x_0 = 0$, and linear approximation $\mathbf{H_l}$ around $x_0 = 0$. Assuming that $\mathbf{H_l}$ is detectable and stabilizable and has a transfer matrix H, then the local W-gain of $\mathbf{H}$ around $x_0 = 0$ is such that $\gamma_{\mathrm{W_l}}(\mathbf{H}) = \|H\|_\infty$.

Consider the nonlinear system $\mathbf{P}$ described by the state-space representation $\dot{x} = f(x) + g(x)u$, $y = x$. Suppose that this system has a normalized right coprime factorization [7] and that it is subject to uncertainties $\mathbf{\Delta_{N_R}}$ and $\mathbf{\Delta_{M_R}}$ as shown in Fig. 1.

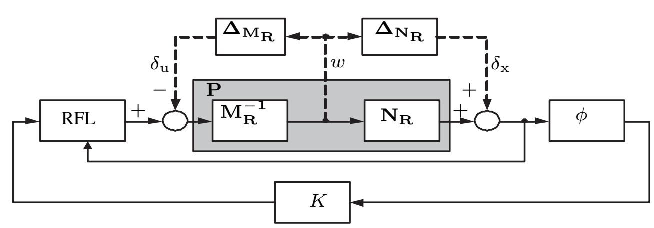

Fig. 1. Closed-loop system.

The perturbed nonlinear system has a normalized right coprime factorization given by [7]

$$\mathbf{N_R} : \; \begin{aligned} \dot{x} &= \tilde{f}(x) + g(x)w \\ y &= x + \delta_\mathrm{x} \end{aligned}$$

$$\mathbf{M_R^{-1}} : \begin{aligned} \dot{x} &= f(x) + g(x)(u - \delta_\mathrm{u}) \\ w &= (u - \delta_\mathrm{u}) - \tilde{h}(x) \end{aligned}$$

with

$$\tilde{f} = f - gg^\mathrm{T}\left(\frac{\partial V}{\partial x}\right)^\mathrm{T} \quad \text{and} \quad \tilde{h} = -g^\mathrm{T}\left(\frac{\partial V}{\partial x}\right)^\mathrm{T}$$

where $V(x)$ is a smooth proper positive definite solution of the Hamilton-Jacobi-Bellman equation

$$2\frac{\partial V(x)}{\partial x}f(x) - \frac{\partial V(x)}{\partial x}g(x)\left(g(x)\right)^\mathrm{T}\left(\frac{\partial V(x)}{\partial x}\right)^\mathrm{T} + x^\mathrm{T}x = 0$$

The nonlinear system $\mathbf{P}$ has a linear approximation with transfer matrix G and with state-space representation (A, B, C, D) where

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}, \; B = g(0)\;, \; C = I \text{ and } D = 0$$

Theorem 1: The linear controller K given by (9), combined with the robust feedback linearization and applied to the nonlinear system $\mathbf{P}$ (as shown in Fig. 1), ensures that the closed-loop is locally W-stable for nonlinear uncertainties $\mathbf{\Delta_{N_R}}$ and $\mathbf{\Delta_{M_R}}$ such that

$$\gamma_{\mathrm{W}_1}\left(\begin{bmatrix}\mathbf{\Delta_{N_R}} \\ \mathbf{\Delta_{M_R}}\end{bmatrix}\right) < \frac{1}{\gamma}$$

Proof: As seen in section III, by using the Glover-McFarlane method, it is possible to obtain, for the linearized system G, a controller K that guarantees

$$\left\| M_\mathrm{R}^{-1}(s)(I - K(s)G(s))^{-1}\begin{bmatrix}K(s) & I\end{bmatrix}\right\|_\infty \le \gamma \qquad (10)$$

where $M_\mathrm{R}^{-1}(s) = I + B^\mathrm{T}X(sI - A)^{-1}B$ and X is the unique positive definite solution of

$$A^\mathrm{T}X + XA - XBB^\mathrm{T}X + I = 0 \qquad (11)$$

For this controller K, the nonlinear system $\mathbf{T}$ with output w and inputs δ_x and δ_u is given by

$$\mathbf{T} \;: \quad \begin{aligned} \dot{x} &= f(x) + g(x)(u - \delta_\mathrm{u}) \\ w &= (u - \delta_\mathrm{u}) - \tilde{h}(x) \\ u &= \alpha(x + \delta_\mathrm{x}) + \beta(x + \delta_\mathrm{x})v \\ v &= K\phi(x + \delta_\mathrm{x}) \end{aligned}$$

This system, linearized around the origin, $(x = u = \delta_\mathrm{x} = \delta_\mathrm{u} = 0)$, using the results in (22), gives

$$T \;: \quad \begin{aligned} \dot{x} &= Ax + B(u - \delta_\mathrm{u}) \\ w &= (u - \delta_\mathrm{u}) - B^\mathrm{T}\tilde{X}x \\ u &= v \\ v &= K(x + \delta_\mathrm{x}) \end{aligned}$$

where $\tilde{X}$ is the positive definite solution of the Hamilton-Jacobi-Bellman equation for the linearized system

$$x^\mathrm{T}(A^\mathrm{T}\tilde{X} + \tilde{X}A - \tilde{X}BB^\mathrm{T}\tilde{X} + I)x = 0$$

which is equivalent to the Riccati equation (11). By uniqueness of this solution, $\tilde{X} = X$.

After some algebraic manipulation, and the use of the Laplace transform, the transfer matrix of the linearized system is obtained as

$$T(s) = M_\mathrm{R}^{-1}(s)(I - K(s)G(s))^{-1}\begin{bmatrix}K(s) & I\end{bmatrix}$$

From (10), it is known that $\|T(s)\|_\infty \le \gamma$. Therefore, by *Property 1*,

$$\gamma_{\mathrm{W}_1}(\mathbf{T}) = \|T(s)\|_\infty \le \gamma$$

Considering the closed-loop standard form in Fig. 2, the local version of the Small Gain Theorem [5], [6] implies that this closed-loop is locally-W-stable if

$$\gamma_{\mathrm{W}_1}\left(\begin{bmatrix}\mathbf{\Delta_{N_R}} \\ \mathbf{\Delta_{M_R}}\end{bmatrix}\right)\gamma_{\mathrm{W}_1}(\mathbf{T}) < 1$$

that is, the closed-loop system is locally-W-stable for uncertainties $\mathbf{\Delta_{N_R}}$ and $\mathbf{\Delta_{M_R}}$ such that

$$\gamma_{\mathrm{W}_1}\left(\begin{bmatrix}\mathbf{\Delta_{N_R}} \\ \mathbf{\Delta_{M_R}}\end{bmatrix}\right) < \frac{1}{\gamma}$$

□

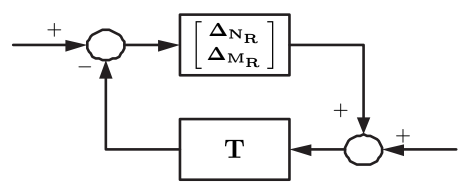

Fig. 2. Standard form for closed-loop system.

The statement of the above theorem is not valid with the classical feedback linearization. Thus, there is no guarantee that the robustness obtained by a controller K for the linearized system in the Brunovsky form is kept when the same controller is applied, together with the classical feedback linearization, to the nonlinear system. This is illustrated through the application example in the next section.

V. Application to a Magnetic Bearing

Consider the magnetic bearing system [8] depicted in Fig. 3, which is composed by a planar rotor disk and two sets of stator electromagnets, the first acting on the y-direction and the second on the x-direction. This system may be decoupled into two subsystems, one for each direction, with similar equations. Here only the subsystem in the y-direction is given.

The rotor is positioned by the magnetic forces F_1 and F_2 generated by the stator electromagnetic circuits. These forces are produced by the currents i_1 and i_2 in each stator coil and these currents depend on the voltages e_1 and e_2 applied to each stator. The inputs to the magnetic bearing system are the voltages e_1 and e_2. The measurable signals are the rotor position y, the rotor velocity $\dot{y}$ and the currents i_1 and i_2.

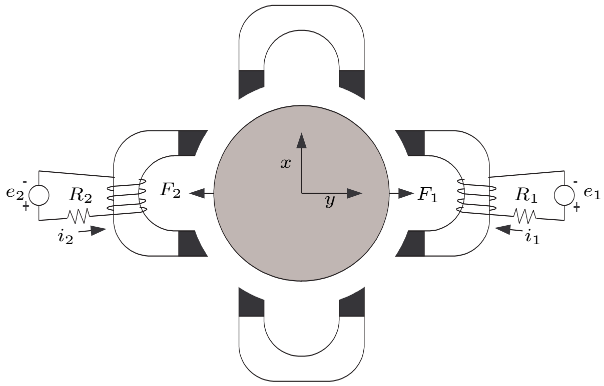

Fig. 3. Top view of a planar rotor disk magnetic bearing system [8].

A. System Model

Along the y-direction, the system is denoted by

$$\begin{cases} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{L_0}{m}\left(\left(\frac{i_1}{k-2y}\right)^2 - \left(\frac{i_2}{k+2y}\right)^2\right) \\ \frac{\mathrm{d}i_1}{\mathrm{d}t} = \left(\frac{k-2y}{L_0}\right)(e_1 - R_1 i_1) - 2\left(\frac{i_1}{k-2y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}i_2}{\mathrm{d}t} = \left(\frac{k+2y}{L_0}\right)(e_2 - R_2 i_2) + 2\left(\frac{i_2}{k+2y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} \end{cases}$$

with $k = 2g_0 + a$, where g_0 is the air gap when the rotor is in the position $y = 0$, a is a positive constant introduced to model the fact that the permeability of electromagnets is finite, L_0 is a positive constant depending on the system construction, m represents the mass of the rotor and R_1 and R_2 are the resistances in the first set of stator electromagnets.

Choosing the state variables $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i_1 - I_0$ and $x_4 = i_2 - I_0$ and the control inputs $u_1 = e_1 - R_1 I_0$ and $u_2 = e_2 - R_2 I_0$, where I_0 is a pre-magnetization current, the system may be written in the state-space form

$$\dot{x} = f(x) + g(x)u \tag{12}$$

with

$$f(x) = \begin{bmatrix} x_2 \\ \frac{L_0}{m}\left(\frac{(x_3+I_0)^2}{(k-2x_1)^2} - \frac{(x_4+I_0)^2}{(k+2x_1)^2}\right) \\ -\frac{R_1(k-2x_1)x_3}{L_0} - \frac{2x_2(x_3+I_0)}{k-2x_1} \\ -\frac{R_2(k+2x_1)x_4}{L_0} + \frac{2x_2(x_4+I_0)}{k+2x_1} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k-2x_1}{L_0} & 0 \\ 0 & \frac{k+2x_1}{L_0} \end{bmatrix}$$

whose only equilibrium point is $x_0 = 0$.

The nominal values of the system parameters are $m = 2$ kg, $g_0 = 1e{-3}$ m, $a = 1.25e{-5}$ m, $L_0 = 3e{-4}$ Hm, $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$ and $I_0 = 6e{-2}$ A.

B. Controller Design

Since the well-known conditions for the feedback linearization [1] are satisfied, the nonlinear system (12) may be linearized by feedback around its equilibrium point $x_0 = 0$. The output (2) is chosen as $\lambda(x) = \begin{bmatrix} x_1 & x_3 \end{bmatrix}^{\mathrm{T}}$. Two different controllers are designed: one that associates the classical feedback linearization with a linear H_∞ controller, and one that associates the robust feedback linearization with a linear H_∞ controller.

The linear H_∞ controllers are obtained using the Glover-McFarlane method with loop-shaping [9], which consists in applying the method to an augmented system $G_{\mathrm{a}} = WG$, where W is the weighting matrix that "shapes" the frequency response of G to provide a better performance.

1) Classical Feedback Linearization + H_∞: The classical linearization of the nonlinear model (12) is obtained by using the linearizing feedback control law (3) and the change of coordinates (4) with

$$\alpha_{\mathrm{c}}(x) = \begin{bmatrix} R_1 x_3 + \frac{2L_0 x_2(x_3+I_0)}{(k-2x_1)^2} \\ R_2 x_4 + \frac{2x_2 L_0(k+2x_1)(x_3+I_0)^2}{(k-2x_1)^3(x_4+I_0)} \end{bmatrix} \tag{13}$$

$$\beta_{\mathrm{c}}(x) = \begin{bmatrix} 0 & \frac{L_0}{k-2x_1} \\ \frac{-m(k+2x_1)}{2(x_4+I_0)} & \frac{L_0(k+2x_1)(x_3+I_0)}{(k-2x_1)^2(x_4+I_0)} \end{bmatrix} \tag{14}$$

$$\phi_{\mathrm{c}}(x) = \begin{bmatrix} x_1 \\ x_2 \\ \frac{L_0}{m}\left(\frac{(x_3+I_0)^2}{(k-2x_1)^2} - \frac{(x_4+I_0)^2}{(k+2x_1)^2}\right) \\ x_3 \end{bmatrix} \tag{15}$$

The linearized system is then

$$\dot{x}_{\mathrm{c}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{\mathrm{c}}} x_{\mathrm{c}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_{\mathrm{c}}} w$$

For this system, the linear control law is $w = K_{\mathrm{c}} x_{\mathrm{c}}$. In order to calculate the linear controller K_{c}, the first step is to determine the transfer matrix $G_{\mathrm{c}}(s)$, given by $G_{\mathrm{c}}(s) = (sI - A_{\mathrm{c}})^{-1} B_{\mathrm{c}}$. Then, it is possible to perform the loop-shaping, in the form $G_{\mathrm{ac}} = W_{\mathrm{c}} G_{\mathrm{c}}$, where W_{c} is chosen as

$$W_{\mathrm{c}} = \begin{bmatrix} \frac{10000(s+1)(s+0.5)}{s(s+10)} & 0 & 0 & \frac{50(s+1.2)}{s(s+10)} \\ 0 & 125 & 0 & 2 \\ 0 & 0 & 125 & 2 \\ 0 & 0 & 2 & 12 \end{bmatrix}$$

In Fig. 4, the singular value plots of G_{c} and $W_{\mathrm{c}} G_{\mathrm{c}}$ are shown. The weighting matrix W_{c} adds an integrator to the first line of transfer matrix G_{ac}, which is related to the rotor position x_1, to avoid steady-state errors, and zeros and poles to better shape the position response. To the other lines of transfer matrix G_{ac}, related to the velocity x_2 and the currents x_3 and x_4, only gains are added. W_{c} is not diagonal to avoid obtaining a decoupled controller that would result in a poor performance.

The next step is to calculate the controller K_{ac} for the augmented system G_{ac} using (9). For this a $\gamma_{\mathrm{c}} = 2.3$ is used, which gives a robustness index of 43%. The controller K_{c} is given by $K_{\mathrm{c}} = K_{\mathrm{ac}} W_{\mathrm{c}}$. The singular value plot of $K_{\mathrm{c}} G_{\mathrm{c}}$ is also shown in Fig. 4.

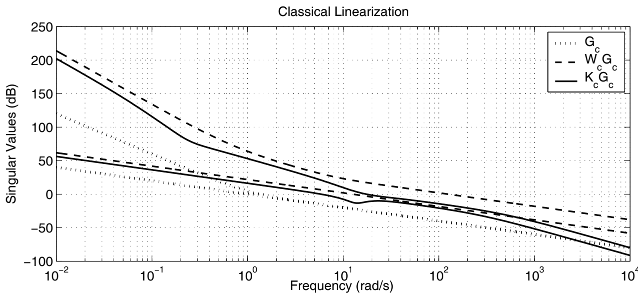

Fig. 4. Singular value plots of G_{c}, $W_{\mathrm{c}}G_{\mathrm{c}}$ and $K_{\mathrm{c}}G_{\mathrm{c}}$.

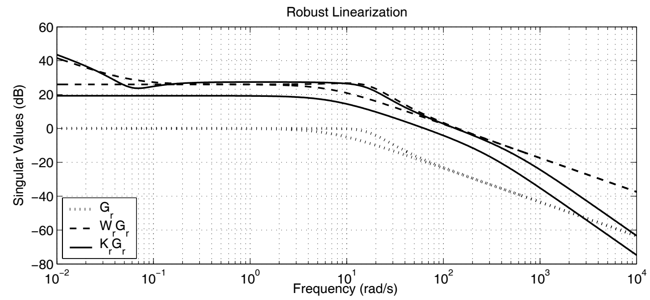

Fig. 5. Singular value plots of G_{r}, $W_{\mathrm{r}}G_{\mathrm{r}}$ and $K_{\mathrm{r}}G_{\mathrm{r}}$.

2) Robust Feedback Linearization + H_∞: The matrices L, T and R for nonlinear system (12) are

$$L = \begin{bmatrix} 0 & 0 & -\frac{2I_0R_1}{mk} & \frac{2I_0R_2}{mk} \\ 0 & -\frac{2I_0}{k} & -\frac{kR_1}{L_0} & 0 \end{bmatrix} \tag{16}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{8L_0I_0^2}{mk^3} & 0 & \frac{2L_0I_0}{mk^2} & -\frac{2L_0I_0}{mk^2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{17}$$

$$R = \begin{bmatrix} 0 & \frac{L_0}{k} \\ -\frac{mk}{2I_0} & \frac{L_0}{k} \end{bmatrix} \tag{18}$$

The results (13), (14), (15), (16), (17) and (18) substituted in expressions (19), (20) and (21) allow to determine, respectively, $\alpha(x)$, $\beta(x)$ and $\phi(x)$. The robust linearization of the nonlinear model (12) is obtained by using the linearizing feedback control law (7) and the change of coordinates (8). The linearized system is then

$$\dot{x}_{\mathrm{r}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{8L_0I_0^2}{mk^3} & 0 & \frac{2L_0I_0}{mk^2} & -\frac{2L_0I_0}{mk^2} \\ 0 & -\frac{2I_0}{k} & -\frac{kR_1}{L_0} & 0 \\ 0 & \frac{2I_0}{k} & 0 & -\frac{kR_2}{L_0} \end{bmatrix}}_{A_{\mathrm{r}}} x_{\mathrm{r}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k}{L_0} & 0 \\ 0 & \frac{k}{L_0} \end{bmatrix}}_{B_{\mathrm{r}}} v$$

with a linear control law $v = K_{\mathrm{r}}x_{\mathrm{r}}$.

The transfer matrix for this system is $G_{\mathrm{r}}(s) = (sI - A_{\mathrm{r}})^{-1}B_{\mathrm{r}}$. The loop-shaping $G_{\mathrm{ar}} = W_{\mathrm{r}}G_{\mathrm{r}}$ is done with the weighting matrix W_{r} chosen as

$$W_{\mathrm{r}} = \mathrm{diag}\left(\frac{1000}{s(s+10)}, 100, 20, 20\right)$$

The singular value plots of G_{r} and $W_{\mathrm{r}}G_{\mathrm{r}}$ are shown in Fig. 5. As in the classical case, the weighting matrix W_{r} adds an integrator to the first line of G_{ar}, to avoid steady-state errors in the position, and gains to the other lines of G_{ar} related to the velocity and the currents. A $\gamma_{\mathrm{r}} = 4.34$ is used to calculate the controller K_{ar} for the augmented system G_{ar}. This gives a robustness index of 23%. The controller K_{r} is given by $K_{\mathrm{r}} = K_{\mathrm{ar}}W_{\mathrm{r}}$, where K_{ar} is obtained from (9). The singular value plot of $K_{\mathrm{r}}G_{\mathrm{r}}$ is also shown in Fig. 5.

C. Controllers Analysis

For this analysis, the closed-loop formed when the controller K_{c} is applied to the linearized system G_{c} (respectively, to the nonlinear system together with the classical linearizing control) is called F_{c} (respectively, $\mathcal{F}_{\mathrm{c}}$). In the same form, the closed-loop formed when the controller K_{r} is applied to the linearized system G_{r} (respectively, to the nonlinear system together with the robust linearizing control) is called F_{r} (respectively, $\mathcal{F}_{\mathrm{r}}$).

A similar loop-shaping is used in the design of the two linear controllers, in order to provide a close nominal performance for systems $\mathcal{F}_{\mathrm{c}}$ and $\mathcal{F}_{\mathrm{r}}$, as shown in Fig. 6.

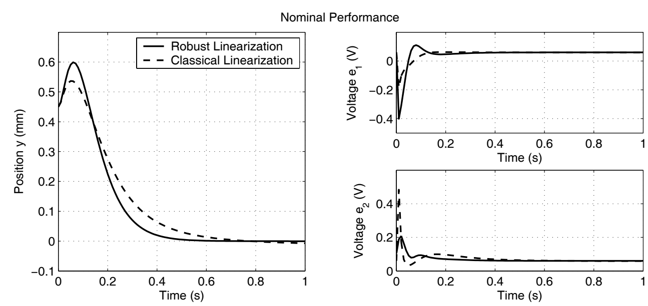

Fig. 6. Comparison of nominal performance.

The robustness guaranteed by the Glover-McFarlane method is of 23% for F_{r} and of 43% for F_{c}. Therefore, the robustness of the closed-loop system F_{c} is better than that of F_{r}. However, the only important point is to compare the robustness of $\mathcal{F}_{\mathrm{c}}$ and that of $\mathcal{F}_{\mathrm{r}}$. This comparison is made in the next subsection.

D. Simulation with Parameter Variations

The simulations are carried out with Simulink/Matlab, using the variable-step algorithm ode45 (Dormand-Prince) with a max step size of 1 ms. For these simulations, it is supposed that the parameters m, k, L_0, R_1 and R_2 may present variations of $\pm 10\%$, which results in 32 different combinations of their extreme values. All these combinations are tested. The results for the classical linearization are given in Fig. 7 and the results for the robust linearization are given in Fig. 8.

These results show that with all the considered parameter variations the robustly linearized system $\mathcal{F}_{\mathrm{r}}$ behaves

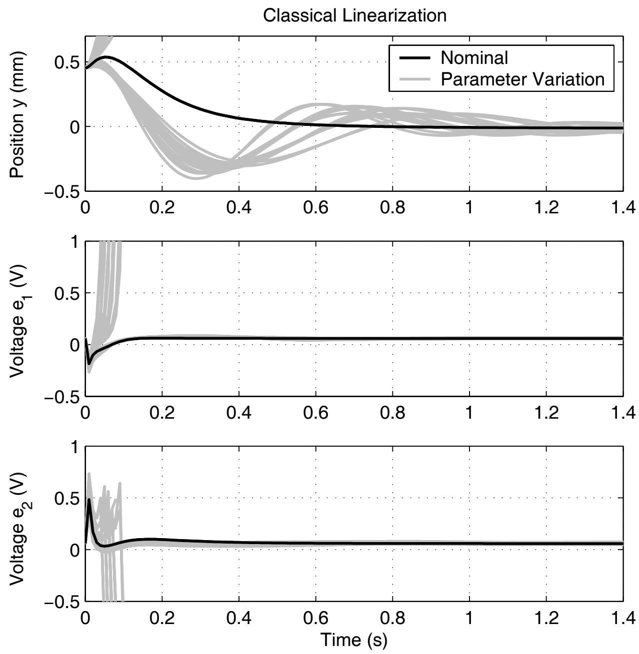

Fig. 7. Position y and voltages e_1 and e_2 for the classical linearization.

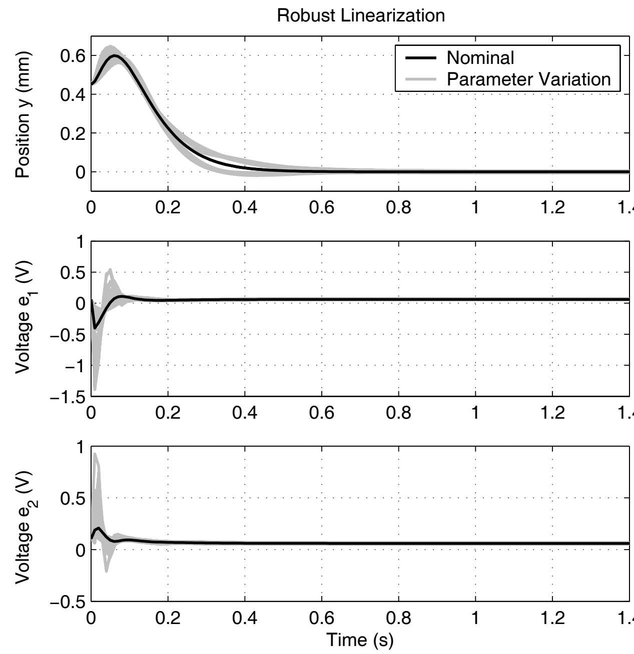

Fig. 8. Position y and voltages e_1 and e_2 for the robust linearization.

as desired, with performance close to the nominal one. Meanwhile, the classically linearized system $\mathcal{F}_{\rm c}$ is unstable for some combinations of parameters and presents a poor performance for the other ones.

VI. Concluding Remarks

As shown by the theory in Section IV and illustrated by the simulations in Section V-D, the use of the robust feedback linearization combined with a Glover-McFarlane H_∞ controller yields a **robust controller** for nonlinear systems. This is not true when the classical feedback linearization is used. In addition, the choice of the weighting matrix of the loop-shaping is much easier and intuitive when using the robust linearization (see Section V-B).

In the near future, this method will be applied to other control systems and experiments are planned.

Appendix I
Expressions for the Feedback Linearization

A. Classical Feedback Linearization

The expressions used in this linearization are:

$$\alpha_{\rm c}(x) = -M^{-1}(x) \begin{bmatrix} L_f^{r_1}\lambda_1(x) \\ \vdots \\ L_f^{r_m}\lambda_m(x) \end{bmatrix} \;,\; \beta_{\rm c}(x) = M^{-1}(x)$$

$$M(x) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}\lambda_1(x) & \cdots & L_{g_m}L_f^{r_1-1}\lambda_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_m-1}\lambda_m(x) & \cdots & L_{g_m}L_f^{r_m-1}\lambda_m(x) \end{bmatrix}$$

$$\phi_{\rm c}(x) = \begin{bmatrix} \phi_{{\rm c}_1}(x) & \cdots & \phi_{{\rm c}_m}(x) \end{bmatrix}^{\rm T}$$

$$\phi_{{\rm c}_i}(x) = \begin{bmatrix} \lambda_i(x) & L_f\lambda_i(x) & \cdots & L_f^{r_i-1}\lambda_i(x) \end{bmatrix}^{\rm T}$$

B. Robust Feedback Linearization

The expressions used in this linearization are:

$$\alpha(x) = \alpha_{\rm c}(x) + \beta_{\rm c}(x)LT^{-1}\phi_{\rm c}(x) \tag{19}$$

$$\beta(x) = \beta_{\rm c}(x)R^{-1} \tag{20}$$

$$\phi(x) = T^{-1}\phi_{\rm c}(x) \tag{21}$$

$$L = -M(0)\left.\frac{\partial\alpha_{\rm c}}{\partial x}\right|_{x=0} ,\; T = \left.\frac{\partial\phi_{\rm c}}{\partial x}\right|_{x=0} \text{ and } R = M^{-1}(0)$$

$$\left.\frac{\partial\alpha(x)}{\partial x}\right|_{x=0} = 0 \;,\; \left.\frac{\partial\phi(x)}{\partial x}\right|_{x=0} = I \text{ and } \beta(0) = I \tag{22}$$

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