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Receding Horizon H_∞ Control for Time-varying Sampled-data Systems

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***Abstract*— In this paper we consider the receding horizon H_∞ control problems for sampled-data systems. We transfer sampled-data systems into equivalent jump systems. We first consider the receding horizon H_∞ control problems for time-varying jump systems to obtain state feedback and output feedback receding horizon H_∞ controllers. We then apply the obtained results to sampled-data systems and give design methods of state feedback and output feedback receding horizon H_∞ controllers. We give a simple example to illustrate the theory.**

I. INTRODUCTION

The receding horizon control is widely used for real applications and its mathematical fundations have been established by various resarchers (for details see [15] and references therein). The receding horizon control was initially considered for stabilization of linear time varying systems ([12], [13]). Then later receding horizon H_∞ control was introduced. The state feedback case is studied in [9], [10], [14] and [16], while the output feedback case can be found in [7], [14] and [16]. Recently, receding horizon control with constrained input and/or output has been considered for time-varying discrete-time systems using linear matrix inequalities (LMIs) ([8], [11]).

Practical and modern control systems usually use digital computers as discrete-time controllers to control continuous-time systems ([3], [5]). Control systems using digital computer with AD/DA converters involve both continuous-time and discrete-time signals in the continuous-time framework and are called sampled-data systems. The H_∞ control theory for linear sampled-data systems is available and is usually established by reducing the H_∞ control problem, via so-called lifting, to an equivalent discrete-time H_∞ contol problem ([3] and the references therein). The H_∞ control theory for linear sampled-data systems can be also established via jump systems [5]. The receding horizon stabilization and robust stabilization problems for sampled-data systems are considered ([1], [2], [4]). However, to the best of our knowledge, receding horizon control with output feedback is not considered in the literature.

In this paper we consider the receding horizon H_∞ control problems for the sampled-data system

$$\begin{aligned}
\dot{x} &= A(t)x(t) + B_1(t)w(t) + B_2(t)\tilde{u}(t), \\
z(t) &= \begin{bmatrix} C_1(t)x(t) \\ D_{12}(t)\tilde{u}(t) \end{bmatrix}, \\
y(k) &= C_2(k)x(k\tau)
\end{aligned} \tag{1}$$

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where $\tau > 0$ is the sampling period, $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $\tilde{u} \in \mathbf{R}^{m_2}$ is the control input realized through a zero-order hold, i.e., $\tilde{u}(t) = u(k)$, $k\tau < t \le (k+1)\tau$, $z \in \mathbf{R}^{p_1+m_2}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the sampled measurement, $C_1 \in \mathbf{R}^{p_1\times n}$, $D_{12} \in \mathbf{R}^{m_2\times m_2}$ are bounded matrices and other matrices are bounded and of compatible dimensions. Here we assume $D_{12}(t) = D_{12}(k)$, $k\tau < t \le (k+1)\tau$ since $\tilde{u}(t) = u(k)$, $k\tau < t \le (k+1)\tau$. Since the system (1) is time-varying, the lifting technique is not suitable and we adopt the jump system approach to solve the receding horizon H_∞ control problems. Initial time t_0 is nonnegative, but for simplicity we assume $t_0 = k_0\tau$, $k_0 \ge 0$.

Since the control $\tilde{u}(t)$ is constant between two sampling instants, i.e., $k\tau < t \le (k+1)\tau$, we can introduce the following state space representation $\dot{\bar{x}} = 0$, $\bar{x}(k\tau^+) = u(k)$, $k\tau < t \le (k+1)\tau$. Then clearly $\tilde{u}(t) = \bar{x}(t)$. Let $x_e = [\,x' \quad \bar{x}'\,]'$ be the new state variable. Then the sampled-data system (1) is equivalent to the following jump system [5]:

$$\begin{aligned}
\dot{x}_e &= A_e(t)x_e(t) + B_{1e}(t)w(t), \ t \ne k\tau, \\
x_e(k\tau^+) &= A_{de}x_e(k\tau) + B_{2e}u(k), \\
z_c(t) &= C_{1e}(t)x_e(t), \\
z_d(k) &= \sqrt{\tau}D_{12}(k)u(k), \\
y(k) &= C_{2e}(k)x_e(k\tau)
\end{aligned} \tag{2}$$

where $t \ne k\tau$ implies $k\tau < t < (k+1)\tau$, $A_e(t) = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(t)$, $B_{1e}(t) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}(t)$, $A_{de} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $B_{2e} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $C_{1e}(t) = [\,C_1 \quad 0\,](t)$ and $C_{2e}(k) = [\,C_2 \quad 0\,](k)$.

This paper is organized as follows. In Section 2 we consider the receding horizon H_∞ control problems for jump systems. In Section 3, we consider the receding horizon H_∞ control problems for sampled-data systems. In Section 4 we give a simple example to illustarate the theory. In Section 5 we give the conclusion. In Appendix we give the proofs of all lemmas and theorems.

II. RECEDING HORIZON H_∞ CONTROL FOR JUMP SYSTEMS

A. Bounded Real Lemma

Consider

$$\begin{aligned}
\dot{x} &= A(t)x(t) + B(t)w(t), \ t \ne k\tau, \\
x(k\tau^+) &= A_d(k)x(k\tau) + B_d(k)w_d(k), \\
z(t) &= C(t)x(t), \\
z_d(k) &= C_d(k)x(k\tau) + D_d(k)w_d(k)
\end{aligned} \tag{3}$$

where $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_d}$ is the disturbance, $(z, z_d) \in \mathbf{R}^{p_1} \times \mathbf{R}^{p_d}$ is the controlled output and all matrices are bounded and of compatible dimensions. We can define the exponential stability, stabilizability and detectability and we can derive the Lyapunov stability lemma for (3) in the usual manner. For details see [5].

If the system (3) is exponentially stable, then for each $(w, w_d) \in L^2(k_0\tau, \infty; \mathbf{R}^{m_1}) \times l^2(k_0, \infty; \mathbf{R}^{m_d})$, the output $(z, z_d) \in L^2(k_0\tau, \infty; \mathbf{R}^{p_1}) \times l^2(k_0, \infty; \mathbf{R}^{p_d})$. If

$$\| z \|_{L^2}^2 + \| z_d \|_{l^2}^2 \leq \gamma^2(\| w \|_{L^2}^2 + \| w_d \|_{l^2}^2),$$

then the system (3) is said to fulfill the γ-disturbance attenuation. Here $\| z \|_{L^2}$ and $\| z_d \|_{l^2}$ are L^2- and l^2-norms on $[k_0\tau, \infty)$ and $[k_0, \infty)$, respectively.

For the ease of notation we often omit t and k in system matrices of (3). For the system (3) we consider the following Riccati equation

$$\begin{array}{rcl} -\dot{X}(t) & = & A'X(t) + X(t)A + C'C \\ & & \quad + \frac{1}{\gamma^2}X(t)BB'X(t), \ t \neq j\tau, \\ T_d(j) & > & aI \text{ for some } a > 0, \\ X(j\tau^-) & = & A_d'X(j\tau)A_d + C_d'C_d + (R_d'T_d^{-1}R_d)(j) \end{array} \tag{4}$$

with boundary condition $X(T) = Q$, $T \geq k_0\tau$ where

$$\begin{array}{rcl} T_d(j) & = & \gamma^2 I - D_d'D_d - B_d'X(j\tau)B_d, \\ R_d(j) & = & B_d'X(j\tau)A_d + D_d'C_d. \end{array} \tag{5}$$

We shall write the solution $X(t)$ of (4) as $X(t; T, Q)$.

Lemma 2.1: Let N be a nonnegative integer, $0 \leq \sigma < \tau$ and $T_k = (k + N)\tau + \sigma$. Suppose that $([C, C_d], [A, A_d])$ is detectable and for each $t \geq k_0\tau$ there exist bounded nonnegative matrices $Q(t)$ and $X(t)$ satisfying (4) with $X(T_k) = Q(T_k)$. Let $\hat{X}(t) = X(t; T_k, Q(T_k))$, $k\tau \leq t < (k+1)\tau$. If

$$\hat{X}(k\tau^-) \geq X(k\tau^-; T_k, Q(T_k)) \tag{6}$$

holds, then the system (3) is exponentially stable and fulfills the γ-disturbance attenuation.

Remark 2.1: If the following linear matrix inequality

$$\mathrm{diag}\{\hat{X}(k\tau^-), \gamma^2 I\} \geq M'(k)\mathrm{diag}\{\hat{X}(k\tau), I\}M(k) \tag{7}$$

holds, then by Schur complement formula, we obtain (6) where $M(k) = \begin{bmatrix} A_d & -B_d \\ C_d & -D_d \end{bmatrix}$.

B. State feedback control

Consider

$$\begin{array}{rcl} \dot{x} & = & A(t)x(t) + B_1(t)w(t), \ t \neq k\tau, \\ x(k\tau^+) & = & A_d(k)x(k\tau) + B_2(k)u(k), \\ z(t) & = & C_1(t)x(t), \\ z_d(k) & = & D_{12}(k)u(k) \end{array} \tag{8}$$

where $u \in \mathbf{R}^{m_2}$ is the control input, $C_1 \in \mathbf{R}^{p_1 \times n}$, $D_{12} \in \mathbf{R}^{m_2 \times m_2}$ are bounded matrices and other matrices are bounded and of compatible dimensions. Here we assume **A1:** $D_{12}'(k)D_{12}(k) = I$, $^\forall k \geq 0$, **A2:** $([C_1, 0], [A, A_d])$ is detectable. The idea of the receding horizon control is to design a controller by solving a finite horizon optimization problem over the interval $[k\tau, T_k]$ at each time $k\tau$. Thus the resulting controller depends on the system parameters up to $N\tau + \sigma$ time ahead. The optimization problem we consdier is the game defined by

$$\begin{array}{c} J(u, w; k\tau, x(k\tau), T_k) = \int_{k\tau}^{T_k} [|z(t)|^2 - \gamma^2|w(t)|^2]dt \\ + \sum_{j=k}^{k+N} |z_d(j)|^2 + x'(T_k)Q(T_k)x(T_k). \end{array}$$

The terminal state penalty Q is often incorporated into finite horizon control problems to allow for compromises between the norm of (z, z_d) and the size of the final state $x(T_k)$.

In order to find $u(k)$, we must integrate

$$\begin{array}{rcl} -\dot{X}(t) & = & \mathcal{R}(X(t)), \ t \neq j\tau, \\ X(j\tau^-) & = & \mathcal{R}_d(X(j\tau)) \end{array} \tag{9}$$

with boundary condition $X(T_k) = Q(T_k)$ backward from T_k to time $k\tau$. Here

$$\begin{array}{l} \mathcal{R}(X) = A'X(t) + X(t)A + C_1'C_1 + \frac{1}{\gamma^2}X(t)B_1B_1'X(t), \\ \mathcal{R}_d(X) = A_d'X(j\tau)A_d - (R_2'T_2^{-1}R_2)(j) \end{array}$$

where

$$T_2(j) = I + B_2'X(j\tau)B_2, \ R_2(j) = B_2'X(j\tau)A_d. \tag{10}$$

We shall denote by $X(t; t_2, Q)$, $t \in [t_1, t_2]$ the solution of (9) with boundary condition $X(t_2) = Q$. The following lemmas are useful for the receding horizon H_∞ control problem.

Lemma 2.2: Let X be a nonnegative solution of (9). Then for all $0 \leq t_1 \leq t_2 \leq t_3$ and for all $Q \geq 0$ we have the following.
(a) $X(t_1; t_2, X(t_2; t_3, Q)) = X(t_1; t_3, Q)$.
(b) If $Q_1 \geq Q_2 \geq 0$, then $X(t_1; t_2, Q_1) \geq X(t_1; t_2, Q_2)$.

Lemma 2.3: Suppose Q is a bounded nonnegative solution of

$$\begin{array}{rcl} -\dot{Q}(t) & \geq & \mathcal{R}(Q(t)), \ t \neq j\tau, \\ Q(j\tau^-) & \geq & \mathcal{R}_d(Q(j\tau)) \end{array} \tag{11}$$

on the interval $[t_0, t_1]$ and X is a bounded nonnegative solution of (9) on the interval $[t_0, t_1]$ with $X(t_1) = Q(t_1)$ where T_{2Q} and R_{2Q} are defined by (10) with X replaced by Q. Then $X(t) \leq Q(t)$ for any $t \in [t_0, t_1]$.

Lemma 2.4: Suppose Q is a bounded nonnegative solution of (11). Suppose also that for each $t \geq 0$, there exists a bounded nonnegative solution $X(t)$ of (9) on the interval $[t, t + N\tau + \sigma]$ with boundary condition $X(t + N\tau + \sigma) = Q(t + N\tau + \sigma)$. Then $X(t_1; t_2, Q(t_2)) \geq X(t_1; t_3, Q(t_3))$ for all $0 \leq t_1 \leq t_2 \leq t_3$

Remark 2.2: Let $\hat{X}(t) = X(t; T_k, Q(T_k))$ for $k\tau \leq t < (k+1)\tau$. Then Lemma 2.4 implies that

$$\hat{X}(k\tau^-) \geq X(k\tau^-; T_k, Q(T_k)). \tag{12}$$

Hence we obtain the following

$$\begin{array}{rcl} -\dot{\hat{X}}(t) & = & \mathcal{R}(\hat{X}(t)), \ t \neq k\tau, \\ \hat{X}(k\tau^-) & \geq & \mathcal{R}_d(\hat{X}(k\tau)). \end{array}$$

Theorem 2.1: Assume **A1** and **A2**. Suppose Q is a bounded nonnegative solution of (11) and X is a bounded nonnegative solution of (9) with $X(T_k) = Q(T_k)$. Then the closed-loop system (8) with

$$u(k) = -\hat{T}_2^{-1}(k)\hat{R}_2(k)x(k\tau) \tag{13}$$

is exponentially stable and fulfills the γ-disturbance attenuation where $\hat{T}_2(k)$ and $\hat{R}_2(k)$ are defined by (10) with $X(k\tau)$ replaced by $\hat{X}(k\tau)$.

C. Output feedback control

Consider

$$\begin{aligned} \dot{x} &= A(t)x(t) + B_1(t)w(t),\ t \neq k\tau, \\ x(k\tau^+) &= A_d(k)x(k\tau) + B_2(k)u(k), \\ z(t) &= C_1(t)x(t), \\ z_d(k) &= D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k\tau) \end{aligned} \tag{14}$$

where $y \in \mathbf{R}^{p_2}$ is the measurement and all matrices are bounded and of compatible dimensions. We assume **A1**, **A2** and **A3:** $([A, A_d], [B_1, 0])$ is stabilizable. Using the receding horizon control method, we shall design an output feedback controller such that the closed-loop system is exponentially stable and fulfills the γ-disturbance attenuation. Let X be the solution of (9) with $X(T_k) = Q(T_k)$. We consider the following Riccati equation [5]

$$\begin{aligned} \dot{Z}(t) &= \tilde{\mathcal{R}}(Z(t)),\ t \neq j\tau, \\ V_Z(j) &> aI \text{ for some } a > 0, \\ Z(j\tau^+) &= \tilde{\mathcal{R}}_d(Z(j\tau)). \end{aligned} \tag{15}$$

Here

$$\begin{aligned} &\tilde{\mathcal{R}}(Z) = A_X Z(t) + Z(t)A_X' + B_1B_1', \\ &\tilde{\mathcal{R}}_d(Z) = A_d Z(j\tau)A_d' - (R_{2Z}'T_{2Z}^{-1}R_{2Z})(j) \\ &\qquad +(F_{1Z}'V_Z F_{1Z})(j). \end{aligned}$$

where $A_X(t) = (A + \frac{1}{\gamma^2}B_1B_1'X)(t)$ and

$$\begin{aligned} T_{1Z}(j) &= \gamma^2 I - T_2^{-\frac{1}{2}}R_2 Z(j\tau)R_2'T_2^{-\frac{1}{2}}, \\ T_{2Z}(j) &= I + C_2 Z(j\tau)C_2', \\ R_{1Z}(j) &= T_2^{-\frac{1}{2}}R_2 Z(j\tau)A_d', \\ R_{2Z}(j) &= C_2 Z(j\tau)A_d', \\ S_Z(j) &= C_2 Z(j\tau)R_2'T_2^{-\frac{1}{2}}, \\ V_Z(j) &= (T_{1Z} + S_Z'T_{2Z}^{-1}S_Z)(j), \\ F_{1Z}(j) &= [V_Z^{-1}(R_{1Z} - S_Z'T_{2Z}^{-1}R_{2Z})](j). \end{aligned} \tag{16}$$

We shall denote by $Z(t; t_0, G)$ the solution of (15) with boundary condition $Z(t_0) = G$.

Lemma 2.5: Suppose G is a bounded nonnegative solution of

$$\begin{aligned} \dot{G}(t) &\geq \tilde{\mathcal{R}}(G(t)),\ t \neq j\tau, \\ V_G(j) &> aI \text{ for some } a > 0, \\ G(j\tau^+) &\geq \tilde{\mathcal{R}}_d(G(j\tau)) \end{aligned} \tag{17}$$

on the interval $[t_0, t_3]$ and Z is a bounded nonnegative solution of (15) on the interval $[t_0, t_3]$ with $Z(t_0) = G(t_0)$. Here V_G is defined by (16) with Z replaced by G. Then we have

(a) $Z(t) \leq G(t)$, for any $t \in [t_0, t_3]$.

(b) $Z(t_3; t_2, G(t_2)) \geq Z(t_3; t_1, G(t_1))$ for all $t_1 \leq t_2 \leq t_3$.

Remark 2.3: Suppose that there exists a nonnegative solution $Z(t)$ of (15) with boundary condition $Z(\tilde{t}_0) = G(\tilde{t}_0)$ for each $\tilde{t}_0$. Let $\hat{Z}(t) = Z(t; L_k, G(L_k))$ for $(k-1)\tau < t \leq k\tau$ where $L_k = (k - N_z)\tau - \sigma_z$, $0 \leq \sigma_z < \tau$ and N_z is a nonnegative integer. Then Lemma 2.5 implies

$$\hat{Z}(k\tau^+) \geq Z(k\tau^+; L_k, G(L_k)). \tag{18}$$

Hence from (15) we obtain

$$\begin{aligned} \dot{\hat{Z}}(t) &= \tilde{\mathcal{R}}(\hat{Z}(t)),\ (k-1)\tau < t < k\tau, \\ \hat{V}_Z(k) &> aI \text{ for some } a > 0, \\ \hat{Z}(k\tau^+) &\geq \tilde{\mathcal{R}}_d(\hat{Z}(k\tau)) \end{aligned} \tag{19–21}$$

where $\hat{T}_{1Z}$, $\hat{T}_{2Z}$ and etc are defined by (16) with $Z(k\tau)$ replaced by $\hat{Z}(k\tau)$.

If the inequality (21) is strict and $\hat{Z}$ is positive definite, then by direct calculation and using Schur complement formula (for details see [7]), we have $0 < \text{diag}\{\Pi_1(\hat{Z}), I\}$ where

$$\begin{aligned} \Pi_1(\hat{Z}) &= \hat{Z}^{-1}(k\tau) + C_2'C_2 - A_d'\hat{Z}^{-1}(k\tau^+)A_d \\ &\quad -\tfrac{1}{\gamma^2}(R_2'T_2^{-1}R_2)(k). \end{aligned}$$

Theorem 2.2: Assume **A1**-**A3** and $k_0\tau \geq N_z\tau + \sigma_z$. Suppose that there exists a bounded nonnegative solution $Q(t)$ to (11) and that for each $t \geq k_0\tau$ there exists a bounded nonnegative solution $X(t)$ satisfying (9) on the interval $[L_k, T_k]$ with boundary condition $X(T_k) = Q(T_k)$, denoted by $X(t; T_k, Q(T_k))$. Suppose further that there exists a bounded nonnegative solution $G(t)$ to

$$\begin{aligned} \dot{G}(t) &\geq \tilde{\mathcal{R}}(G(t)),\ t \neq j\tau, \\ V_G(j) &> aI \text{ for some } a > 0, \\ G(j\tau^+) &\geq \tilde{\mathcal{R}}_d(G(j\tau)) + \epsilon I \end{aligned} \tag{22}$$

for some $\epsilon > 0$ and for each $t \geq k_0\tau$, there exists a bounded nonnegative solution denoted by $Z(t; L_k, G(L_k))$ to

$$\begin{aligned} \dot{Z}(t) &= \tilde{\mathcal{R}}(Z(t)),\ t \neq j\tau, \\ V_Z(j) &> aI \text{ for some } a > 0, \\ Z(j\tau^+) &= \tilde{\mathcal{R}}_d(Z(j\tau)) + \epsilon I \end{aligned} \tag{23}$$

on the interval $[L_k, k\tau]$ with boundary condition $Z(L_k) = G(L_k)$. Then the output feedback controller

$$\begin{aligned} \dot{p} &= A_{\hat{X}}(t)p(t),\ t \neq k\tau, \\ p(k\tau^+) &= \hat{A}_d(k)[p(k\tau) + \hat{Z}(k\tau)C_2'y(k)], \\ u(k) &= \hat{C}_d(k)[p(k\tau) + \hat{Z}(k\tau)C_2'y(k)] \end{aligned} \tag{24}$$

exponentially stabilizes the system (14) and fulfills the γ-disturbance attenuation where $\hat{A}_d(k) = A_{dcl}(k)\Phi(k)$, $\hat{C}_d(k) = -(\hat{T}_2^{-1}\hat{R}_2\Phi)(k)$, $A_{dcl}(k) = (A - B_2\hat{T}_2^{-1}\hat{R}_2)(k)$ and $\Phi(k) = [I + \hat{Z}(k\tau)C_2'(k)C_2(k)]^{-1}$.

III. Receding horizon H_∞ control for sampled-data systems

Recall the sampled-data system (1) and the equivalent jump system (2). We assume **S1:** $D_{12}'(t)D_{12}(t) = I$, $^\forall t \geq 0$, **S2:** (C_1, A) is detectable and **S3:** (A, B_1) is stabilizable. Note that the assumptions **S2** and **S3** imply the conditions **A2** and **A3** for the equivalent jump system (2) [5].

A. State feedback control

Consider the sampled-data system (1) with $C_2 = I$ and the equivalent jump system (2). Let $Q(t) = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}' & Q_2 \end{bmatrix}(t)$ be a bounded nonnegative solution of (11) and $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t)$ be a bounded nonnegative solution of (9) with $X(T_k) = Q(T_k)$ where $X_1,\ Q_1 \in \mathbf{R}^{n\times n}$, $X_{12},\ Q_{12} \in \mathbf{R}^{n\times m_2}$ and $X_2,\ Q_2 \in \mathbf{R}^{m_2\times m_2}$. Then from (11) we obtain

$$\begin{array}{l} -\dot{Q}(t) \geq A_e'Q(t) + Q(t)A_e + C_{1e}'C_{1e} \\ \qquad\qquad + \frac{1}{\gamma^2}Q(t)B_{1e}B_{1e}'Q(t),\ t \neq j\tau, \\ \begin{bmatrix} \mathcal{Q}_1(j\tau^-) & Q_{12}(j\tau^-) \\ Q_{12}'(j\tau^-) & Q_2(j\tau^-) \end{bmatrix} \geq 0 \end{array} \tag{25}$$

and from (9) we have

$$\begin{array}{l} -\dot{X}(t) = A_e'X(t) + X(t)A_e + C_{1e}'C_{1e} \\ \qquad\qquad + \frac{1}{\gamma^2}X(t)B_{1e}B_{1e}'X(t),\ t \neq j\tau, \\ \begin{bmatrix} \mathcal{X}_1(j\tau^-) & X_{12}(j\tau^-) \\ X_{12}'(j\tau^-) & X_2(j\tau^-) \end{bmatrix} = 0 \end{array} \tag{26}$$

where

$$\begin{array}{rcl} \mathcal{Q}_1(j\tau^-) & = & Q_1(j\tau^-) - Q_1(j\tau) \\ & & + Q_{12}(j\tau)[\tau I + Q_2(j\tau)]^{-1}Q_{12}'(j\tau), \\ \mathcal{X}_1(j\tau^-) & = & X_1(j\tau^-) - X_1(j\tau) \\ & & + X_{12}(j\tau)[\tau I + X_2(j\tau)]^{-1}X_{12}'(j\tau). \end{array}$$

In this case the state feedback law

$$u(k) = -[\tau I + \hat{X}_2(k\tau)]^{-1}\hat{X}_{12}'(k\tau)x(k\tau) \tag{27}$$

stabilizes the system (1) and the closed-loop system (1) with (27) fulfills the γ-disturbance attenuation where $\hat{X}(t) = X(t; T_k, Q(T_k))$ for $k\tau \leq t < (k+1)\tau$.

Theorem 3.1: Assume **S1** and **S2**. Suppose $Q(t)$ is a bounded nonnegative solution to (25) and $X(t)$ is a bounded nonnegative solution to (26) with $X(T_k) = Q(T_k)$. Then the closed-loop system (1) with the state feedback controller (27) is exponentially stable and fulfills the γ-disturbance attenuation.

B. Output feedback control

Consider the sampled-data system (1) and the equivalent jump system (2). Suppose that bounded nonnegative solutions $Q(t)$ and $X(t)$ satisfy (25) and (26), respectively with $X(T_k) = Q(T_k)$. Let

$$\begin{array}{rcl} A_X(t) & = & (A_e + \frac{1}{\gamma^2}B_{1e}B_{1e}X)(t) \\ & = & \begin{bmatrix} A + \frac{1}{\gamma^2}B_1B_1'X_1 & B_2 + \frac{1}{\gamma^2}B_1B_1'X_{12} \\ 0 & 0 \end{bmatrix}(t). \end{array}$$

Let $G(t) = \begin{bmatrix} G_1 & G_{12} \\ G_{12}' & G_2 \end{bmatrix}(t)$ be a bounded nonnegative solution of (22) and $Z(t) = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{12}' & Z_2 \end{bmatrix}(t)$ a bounded nonnegative solution of (23) with $Z(L_k) = G(L_k)$ where Z_1, $G_1 \in \mathbf{R}^{n\times n}$, $Z_{12},\ G_{12} \in \mathbf{R}^{n\times m_2}$ and $Z_2,\ G_2 \in \mathbf{R}^{m_2\times m_2}$. Then from (22) we obtain

$$\begin{array}{l} \dot{G}(t) \geq A_XG(t) + G(t)A_X' + B_{1e}B_{1e}',\ t \neq j\tau, \\ V_{Ge}(j) > aI \text{ for some } a > 0, \\ \begin{bmatrix} \Theta_G(j) - \epsilon I & G_{12}(j\tau^+) \\ G_{12}'(j\tau^+) & G_2(j\tau^+) - \epsilon I \end{bmatrix} \geq 0 \end{array} \tag{28}$$

and from (23) we have

$$\begin{array}{l} \dot{Z}(t) = A_XZ(t) + Z(t)A_X' + B_{1e}B_{1e}',\ t \neq j\tau, \\ V_{Ze}(j) > aI \text{ for some } a > 0, \\ \begin{bmatrix} \Theta_Z(j) - \epsilon I & Z_{12}(j\tau^+) \\ Z_{12}'(j\tau^+) & Z_2(j\tau^+) - \epsilon I \end{bmatrix} = 0 \end{array} \tag{29}$$

where $W(j) = [\tau I + X_2(j\tau)]^{-\frac{1}{2}}$,

$$\begin{array}{rcl} \mathcal{Z}_1(j) & = & Z_1(j\tau)[I + C_2'C_2Z_1(j\tau)]^{-1}, \\ V_{Ze}(j) & = & \gamma^2 I - WX_{12}'(j\tau)\mathcal{Z}_1(j)X_{12}(j\tau)W, \\ \Theta_Z(j) & = & Z_1(j\tau^+) - [\mathcal{Z}_1(j) + \mathcal{Z}_1(j)X_{12}(j\tau)W \\ & & \times V_{Ze}^{-1}(j)WX_{12}'(j\tau)\mathcal{Z}_1(j)]. \end{array}$$

and $\mathcal{G}_1$, V_{G_e}, Θ_G are defined by above equations with Z replaced by G. From (24) we obtain the following controller

$$\begin{array}{rcl} \dot{p} & = & (A + \frac{1}{\gamma^2}B_1B_1'\hat{X}_1)(t)p(t) \\ & & + (B_2 + \frac{1}{\gamma^2}B_1B_1'\hat{X}_{12})(t)\tilde{v}(t),\ t \neq k\tau, \\ p(k\tau^+) & = & \hat{A}_d[p(k\tau) + \hat{Z}_1(k\tau)C_2'y(k)], \\ u(k) & = & \hat{C}_d[p(k\tau) + \hat{Z}_1(k\tau)C_2'y(k)] \end{array} \tag{30}$$

where $\tilde{v}(t)$ is given by $\tilde{v}(t) = u(k)$, $k\tau \leq t < (k+1)\tau$, $\hat{A}_d(k) = [I + \hat{Z}_1(k\tau)C_2'(k)C_2(k)]^{-1}$, $\hat{C}_d(k) = -[\tau I + \hat{X}_2(k\tau)]^{-1}\hat{X}_{12}'(k\tau)\hat{A}_d(k)$.

Theorem 3.2: Assume **S1-S3** and $k_0\tau \geq N_z\tau + \sigma_z$. Suppose that there exists a bounded nonnegative solution $Q(t)$ to (25) and that for each $t \geq k_0\tau$, there exists a bounded nonnegative solution $X(t)$ satisfying (26) on the interval $[L_k, T_k]$ with boundary condition $X(T_k) = Q(T_k)$ denoted by $X(t; T_k, Q(T_k))$. Suppose further that there exists a bounded nonnegative solution $G(t)$ to (28) for some $\epsilon > 0$ and for each $t \geq k_0\tau$, there exists a bounded nonnegative solution $Z(t)$ to (29) on the interval $[L_k, k\tau]$ with boundary condition $Z(L_k) = G(L_k)$ denoted by $Z(t; L_k, G(L_k))$. Then the closed-loop system (1) with the controller (30) is exponentially stable and fulfills the γ-disturbance attenuation.

IV. Example

Consider the periodic system

$$\begin{array}{rcl} \dot{x} & = & \frac{1}{4}(1 + \sin 2\pi t)x(t) + w(t) + \tilde{u}(t), \\ z(t) & = & \begin{bmatrix} x'(t) & \tilde{u}'(t) \end{bmatrix}', \\ y(k) & = & x(0.5k). \end{array}$$

Here the period of this system is 1 [sec] and the sampling period $\tau = 0.5$ [sec]. Let $N = N_z = 1$, $\sigma = \sigma_z = 0$ and $\gamma = 10$. Then there exists a τ-periodic nonnegative solutions of (25) and (28) respectively and we use

$$\begin{array}{rcl} Q((k+1)\tau) & = & Q(k\tau) = \begin{bmatrix} 40.8697 & 17.5464 \\ 17.5464 & 12.1652 \end{bmatrix}, \\ G((k-1)\tau) & = & G(k\tau) = \begin{bmatrix} 13.7280 & 2.4132 \\ 2.4132 & 4.5000 \end{bmatrix} \end{array}$$

to obtain $X(t;(k+1)\tau, Q((k+1)\tau))$ and $Z(k\tau;(k-1)\tau, G((k-1)\tau)$, respectively. In this case

$$Z(k\tau;(k-1)\tau, G(k\tau)) = \begin{cases} \begin{bmatrix} 7.572 & 1.949 \\ 1.949 & 4.500 \end{bmatrix}, & k = 0, 2, . \\ \begin{bmatrix} 9.813 & 2.101 \\ 2.101 & 4.500 \end{bmatrix}, & k = 1, 3, . \end{cases}$$

and the output feedback receding horizon H_∞ controller is given by

$$\begin{aligned} \dot{p} &= a_1(t)p(t) + a_2(t)\tilde{v}(t), \ t \neq k\tau, \\ p(k\tau^+) &= 0.1167p(k\tau) + 0.8833y(k), \\ u(k) &= -0.2387p(k\tau) - 1.8074y(k) \end{aligned}$$

for $k = 2m$ and

$$\begin{aligned} \dot{p} &= a_1(t)p(t) + a_2(t)\tilde{v}(t), \ t \neq k\tau, \\ p(k\tau^+) &= 0.0925p(k\tau) + 0.9075y(k), \\ u(k) &= -0.1725p(k\tau) - 1.6932y(k) \end{aligned}$$

for $k = 2m+1$, $m = 0, 1,..$ where $a_1(t) = \frac{1}{4}(1 + \sin 2\pi t) + \frac{1}{\gamma^2}\hat{X}_1(t)$, $a_2(t) = 1 + \frac{1}{\gamma^2}\hat{X}'_{12}(t)$ and

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_{12} \\ \hat{X}'_{12} & \hat{X}_2 \end{bmatrix}(t) = \hat{X}(t) = X(t;(k+1)\tau, Q((k+1)\tau)), \quad k\tau \le t < (k+1)\tau.$$

The simulation result of the system with the obtained output feedback controller is given in Figure 1 with $x(0) = 0$ and $w(t) = e^{-2t}\cos 2t$.

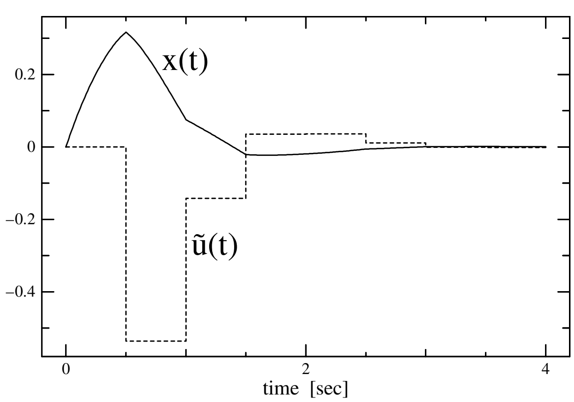

Fig. 1. A simulation result

V. Conclusion

We have considered the receding horizon H_∞ control problems for sampled-data systems. We have given sufficient conditions for the existence of state feedback controllers and output feedback controllers, in terms of the H_∞ Riccati inequalities (equations) with jumps.

Appendix

Proof of Lemma 2.1. Using (4) and (6) we have

$$-\dot{\hat{X}}(t) = A'\hat{X}(t) + \hat{X}(t)A + \tilde{C}'\tilde{C}$$

for $k\tau \le t < (k+1)\tau$ and

$$\hat{X}(k\tau^-) \ge A'_d\hat{X}(k\tau)A_d + \tilde{C}'_d\tilde{C}_d \qquad (31)$$

where $\tilde{C} = \begin{bmatrix} C \\ \frac{1}{\gamma}B'\hat{X} \end{bmatrix}$, $\tilde{C}_d = \begin{bmatrix} C_d \\ \hat{T}_d^{-\frac{1}{2}}\hat{R}_d \end{bmatrix}$ and $\hat{T}_d$, $\hat{R}_d$ are defined by (5) with $X(k\tau)$ replaced by $\hat{X}(k\tau)$. Since $([C, C_d], [A, A_d])$ is detectable, so is $([\tilde{C}, \tilde{C}_d], [A, A_d])$. Hence by Lyapunov lemma for jump systems, the system (3) is exponentially stable. Differentiating $x'(t)\hat{X}(t)x(t)$ and integrating it from $k_0\tau$ to ∞, we obtain the rest of the proof (for details see [5]).

Proof of Lemma 2.2. The proof is obvious.

Proof of Lemma 2.3. A brief outline of the proof is as follows. We first establish $X(s) \le Q(s)$, $s \in [j_1\tau, t_1]$, $j_1\tau \le t_1 < (j_1+1)\tau$. Then using the jump equation, we show $X(j_1\tau^-) \le Q(j_1\tau^-)$. Next we show $X(s) \le Q(s)$ on the interval $[(j_1-1)\tau, j_1\tau)$. Repeating these arguments, $X(t) \le Q(t)$, $t \in [t_0, t_1]$ will be established.
Step 1: Consider the functional

$$\begin{aligned} J(u, w; s, x(s), t_1) = \textstyle\int_s^{t_1}[|z(t)|^2 - \gamma^2|w(t)|^2]dt \\ +x'(t_1)Q(t_1)x(t_1) \end{aligned}$$

subject to $\dot{x} = A(t)x(t) + B_1(t)w(t)$ and $z(t) = C_1(t)x(t)$ where $j_1\tau < s \le t_1$. Then differentiating $x'(t)X(t)x(t)$ and $x'(t)Q(t)x(t)$ and integrating them from s to t_1 we have

$$\begin{aligned} &J(w; s, x(s), t_1) \\ &= x'(s)X(s)x(s) - \gamma^2 \textstyle\int_s^{t_1} |w(t) - \bar{w}(t)|^2 dt \\ &\le x'(s)Q(s)x(s) - \gamma^2 \textstyle\int_s^{t_1} |w(t) - \frac{1}{\gamma^2}B'_1(t)Q(t)x(t)|^2 dt. \end{aligned}$$

Hence $X(s) \le Q(s)$, $s \in (j_1\tau, t_1]$. Since $X(j_1\tau^+) = X(j_1\tau)$ and $Q(j_1\tau^+) = Q(j_1\tau)$, we have $X(j_1\tau) \le Q(j_1\tau)$.
Step 2: We introduce the functional

$$\begin{aligned} J(u, w; j_1\tau, x(j_1\tau), t_1) = |z_d(j_1)|^2 \\ + \textstyle\int_{j_1\tau}^{t_1}[|z(t)|^2 - \gamma^2|w(t)|^2]dt + x'(t_1)Q(t_1)x(t_1) \end{aligned}$$

subject to (8). Then considering $\max_w \min_u J(u, w; j_1\tau, x(j_1\tau), t_1)$ and using $X(j_1\tau) \le Q(j_1\tau)$, we obtain $X(j_1\tau^-) \le Q(j_1\tau^-)$.
Step 3: Now we assume that $X(t) \le Q(t)$, $t \in [j_1\tau^-, t_1]$ and introduce the functional

$$\begin{aligned} J(u, w; s, x(s), t_1) = |z_d(j_1)|^2 \\ + \textstyle\int_s^{t_1}[|z(t)|^2 - \gamma^2|w(t)|^2]dt + x'(t_1)Q(t_1)x(t_1) \end{aligned}$$

subject to (8) where $(j_1-1)\tau < s \le j_1\tau$. Then using $X(j_1\tau^-) \le Q(j_1\tau^-)$ and as in the proof of Steps 1 and 2, we obtain $X(s) \le Q(s)$.
Continuing in this way, we have the assertion.

Proof of Lemma 2.4. Similar to [6], by combining Lemmas 2.2 and 2.3 we have the assertion.

Proof of Theorem 2.1. The closed-loop system (8) with (13) is given by

$$\begin{aligned} \dot{x} &= A(t)x(t) + B_1(t)w(t), \ t \neq k\tau, \\ x(k\tau^+) &= A_{dcl}(k)x(k\tau), \\ z(t) &= C_1(t)x(t), \\ z_d(k) &= -(D_{12}\hat{T}_2^{-1}\hat{R}_2)(k)x(k\tau) \end{aligned}$$

where $A_{dcl} = A_d - B_2\hat{T}_2^{-1}\hat{R}_2$. Then using **A2** we can show that $([C_1, -D_{12}\hat{T}_2^{-1}\hat{R}_2], [A, A_{dcl}])$ and $([\tilde{C}_1, -D_{12}\hat{T}_2^{-1}\hat{R}_2], [A, A_{dcl}])$ are detectable where $\tilde{C}_1 = \begin{bmatrix} C_1 \\ \frac{1}{\gamma}B_1'\hat{X} \end{bmatrix}$. By Lemma 2.4 and Remark 2.2 we have (12) and by direct calculation we obtain

$$\begin{array}{rcl} -\dot{\hat{X}}(t) & = & A'\hat{X}(t) + \hat{X}(t)A + \tilde{C}_1'\tilde{C}_1, \ t \neq k\tau, \\ \hat{X}(k\tau^-) & \geq & A_{dcl}'\hat{X}(k\tau)A_{dcl} \\ & & +(-D_{12}\hat{T}_2^{-1}\hat{R}_2)'(-D_{12}\hat{T}_2^{-1}\hat{R}_2). \end{array}$$

Hence by Lemma 2.1 we have the assertion.

Proof of Lemma 2.5. The proof is similar to those of Lemmas 2.3 and 2.4.

Proof of Theoem 2.2. We shall show that the closed-loop system (14) with the controller (24)

$$\begin{array}{rcl} \dot{\hat{x}} & = & A_c(t)\hat{x}(t) + B_c(t)w(t), \ t \neq k\tau, \\ \hat{x}(k\tau^+) & = & A_{dc}(k)\hat{x}(k\tau), \\ z(t) & = & C_c(t)\hat{x}(t), \\ z_d(k) & = & C_{dc}(k)\hat{x}(k\tau) \end{array} \tag{32}$$

is exponentially stable and fulfills the γ-disturbance attenuation. Here $\hat{x} = \begin{bmatrix} x \\ x-p \end{bmatrix}$, $A_{dc} = \bar{A}_{dc}L_c$, $C_{dc} = \bar{C}_{dc}L_c$, $L_c = \mathrm{diag}\{I, \Phi\}$, $B_c = [\,B_1' \quad B_1'\,]'$, $C_c = [\,C_1 \quad 0\,]$

$$\begin{array}{rcl} A_c & = & \begin{bmatrix} A & 0 \\ -\frac{1}{\gamma^2}B_1B_1'\hat{X} & A_{\hat{X}} \end{bmatrix}, \\ \bar{A}_{dc} & = & \begin{bmatrix} A_d - B_2\hat{T}_2^{-1}\hat{R}_2 & B_2\hat{T}_2^{-1}\hat{R}_2 \\ 0 & A_d \end{bmatrix}, \\ \bar{C}_{dc} & = & [\,-D_{12}\hat{T}_2^{-1}\hat{R}_2 \quad D_{12}\hat{T}_2^{-1}\hat{R}_2\,]. \end{array}$$

By Lemma 2.5, Remark 2.3 and using (23) we have

$$\begin{array}{rcl} \dot{\hat{Z}}(t) & = & \tilde{\mathcal{R}}(\hat{Z}(t)), \ (k-1)\tau < t < k\tau, \\ \hat{V}_Z(k) & > & aI \text{ for some } a > 0, \\ \hat{Z}(k\tau^+) & > & A_d\Phi\hat{Z}(k\tau)(A_d\Phi)' + \tilde{B}_d\tilde{B}_d' \end{array}$$

where $\tilde{B}_d(k) = [\,\hat{R}_{2Z}'\hat{T}_{2Z}^{-1} \quad \hat{F}_{1Z}'\hat{V}_Z^{\frac{1}{2}}\,](k)$. By **A3**, $([A_{\hat{X}}, A_d\Phi], [B_1, \tilde{B}_d])$ is stabilizabe and hence $(A_{\hat{X}}, A_d\Phi)$ is exponentially stable. By **A2** and a simple calculation we can show that $([C_c, C_{dc}], [A_c, A_{dc}])$ is detectable. Next we shall show that $P(t) = \mathrm{diag}\{X(t), \gamma^2\hat{Z}^{-1}(t)\}$ satisfies

$$\begin{array}{rcl} -\dot{P}(t) & = & A_c'P(t) + P(t)A_c + C_c'C_c \\ & & +\frac{1}{\gamma^2}P(t)B_cB_c'P(t), \ (k-1)\tau < t < k\tau, \\ P(k\tau^-) & \geq & A_{dc}'P(k\tau)A_{dc} + C_{dc}'C_{dc}. \end{array}$$

By direct calculation we have

$$\dot{P} + A_c'P + PA_c + C_c'C_c + \frac{1}{\gamma^2}PB_cB_c'P = \mathrm{diag}\{M_1, M_2\}$$

where

$$\begin{array}{rcl} M_1(t) & = & \dot{X} + \mathcal{R}(X), \\ M_2(t) & = & \gamma^2\hat{Z}^{-1}[-\dot{\hat{Z}} + \tilde{\mathcal{R}}(\hat{Z})]\hat{Z}^{-1}. \end{array}$$

By Remark 2.2, we have $M_1(t) = 0$, $M_2(t) = 0$. Since L_c is nonsingular, it is enough to show

$$\begin{array}{l} (L_c')^{-1}P(k\tau^-)L_c^{-1} - \bar{A}_{dc}'P(k\tau)\bar{A}_{dc} \\ \qquad -\bar{C}_{dc}'\bar{C}_{dc}(k) \equiv \Psi_c(k) \geq 0. \end{array}$$

Indeed by direct calculation we have

$$\Psi_c(k) = \mathrm{diag}\{\Psi_{c1}(k), \Psi_{c2}(k)\}$$

where

$$\begin{array}{rcl} \Psi_{c1}(k) & = & \hat{X}(k\tau^-) - \mathcal{R}_d(\hat{X}), \\ \Psi_{c2}(k) & = & \gamma^2\Pi_1(\hat{Z}) + \gamma^2C_2'\hat{T}_{2Z}(k)C_2. \end{array}$$

By Remarks 2.2 and 2.3 we have $\Psi_c \geq 0$. Hence similar to the proof of Lemma 2.1 the closed-loop system (32) is exponentially stable and fulfills the γ-disturbance attenuation.

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