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A Causal Discrete-time Estimator-Predictor for Unicycle Trajectory Tracking

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Abstract— **This paper proposes a nonlinear sampled-data (SD) control approach for the trajectory tracking of a class of nonlinear differentially flat systems that encompass the unicycle, which is widely used in the context of unmanned aerial vehicle (UAV). The nonlinear SD control method relies on the flatness property for the generation of appropriate trajectories, with the design of one-step predictive control laws, and on controller discretization by means of an averaging-like method. The causality problem that might arise in the implementation is avoided by using an estimator based on numerical integration techniques of sufficiently high order. Numerical simulations show that the proposed SD control law offers the best closed-loop performance when compared with nonlinear direct digital design for the trajectory tracking of a unicycle. Furthermore, the results show that the proposed scheme is less sensitive to quantization errors arising with finite word length and fixed point arithmetic microprocessors than nonlinear direct digital design. The SD control relies on closed-form integrability of the UAV.**

I. INTRODUCTION

Autonomous trajectory tracking of UAV requires real-time control that is implemented in discrete-time (DT). Digital laws generally have to meet the following requirements: (i) a performance level in the same order of magnitude as that of their continuous-time (CT) counterpart; (ii) causality, i.e. during $[kT,\,(k+1)T)$ the controller provides a control input u_k that is computed from measurements obtained at mT where $m < k$, m and k being integers; (iii) to account for the nonlinearities of the plant; (iv) to ensure that (i), (ii) and (iii) are not violated despite quantization errors that arise with finite wordlength implementation [1]. Requirements (i)-(iii) are explicitly taken into account in the control synthesis proposed in this paper by emulating as best as possible the behavior of the CT closed-loop system. Requirement (iv), which is related to the digital implementation, is verified by means of fixed point simulations with a finite wordlength constraint of 16 bits.

DT control law synthesis for linear systems has reached a certain level of maturity [1], [2]. In the case of nonlinear systems, however, synthesizing DT controllers providing closed-loop stability and satisfactory performance is still a challenging task, particularly when the implementation hardware constrains the choice of the sampling and control update rates. The approaches proposed so far to obtain a DT control law for a nonlinear CT plant are local digital redesign [1] and direct digital design [3], [4], [5]. In general, since the DT plant model is only an approximation of the true nonlinear DT plant, the stability of the resulting closed-loop SD system cannot be guaranteed. Recently, Nesic *et al.* [6] derived conditions, for small T, that warrants closed-loop stability for nonlinear SD systems whose DT controllers are obtained with a direct digital design approach. In [7], a SD control synthesis using the flatness property of a class of nonlinear systems is proposed. Recall that the flatness property of a nonlinear system allows [8] 1) linearizing a nonlinear system, and 2) inverting the dynamics without having to integrate the states. The latter feature is well suited to trajectory generation, for the synthesis of feedforward control laws, whereas the former characteristic enables asymptotic stabilization of the system around a desired trajectory [9]. In the first step of the method proposed in [7], a CT control law $u(t)$, where $t \in [kT,(k+1)T)$ and $k \in \mathcal{N}$, the set of natural numbers, is calculated over the time range $[kT,(k+1)T)$ based on the flatness property of the nonlinear CT plant. Such CT control steers the nonlinear system from state x_k at time $t = kT$ to the desired state x^d_{k+1} at time $t = kT + T$. In the second step, a SD controller is calculated by averaging the CT law over the time interval $[kT,(k+1)T)$. However the method proposed in [7] requires the nonlinear plant state x_k at the same time instant t_k at which the digital board is expected to provide the DT control u_k, thereby resulting in a causality problem.

The contributions of the paper are threefold. First, to circumvent the causality problem found with the SD control method in [7], we propose a SD control that uses an estimator based on classical numerical integration methods expanded to a sufficiently high order. Second, we present a conversion of the flatness-based CT controller to a SD controller by means of a nonlinear averaging-like technique. It is shown that the error induced by the estimator-based SD controller with respect to a CT control law is bounded in $\mathcal{O}\left(T^2\right)$, which is particularly useful for digital implementations performed with a relatively large sample period T. In the ideal case of a null estimation error, the responses of the SD system match exactly those of the CT system, at the sampling instants. Third, a nonlinear SD control law is proposed and applied for trajectory tracking of a rotorcraft-like UAV modelled as a unicycle [10]. The exact computation of the SD control law results from the closed-form integrability of the unicycle. The closed-loop SD

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system, which is characterized by output sampling and input zero-order hold, ensures closed-loop stability and tracking performance even for relatively large sampling periods, whereas this is not the case for the method using a local discretization technique. Furthermore, the simulations show that the proposed scheme results in satisfactory performance when implemented with a 16 bit wordlength and fixed-point arithmetic.

II. Assumptions

The following assumptions are assumed to hold throughout the paper unless stated otherwise.

Assumption 1 (Lifting of signals) DT and CT lifted signals [2] are represented as $x_k = x(kT)$ and $x_k(\tau) = x(kT+\tau)$, respectively. Note that $x_k(0) = x_k$. The same holds for input signal $u(t)$.⋈

Assumption 2 (Sampler and hold) Under SD control, the nonlinear plant is preceded by a zero-order hold $\mathcal{H}$ and followed by the ideal sampler $\mathcal{S}$. Hold and sampler are synchronized at the sampling period T. Therefore, the SD control input is piecewise continuous with period T.⋈

Assumption 3 (Nonlinear differentially flat systems, [8]) Consider the nonlinear affine-in-the-input CT system

$$\dot{x} = f(x) + G(x)u \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^p, 1 \leq p \leq n$ and $G(x) = \begin{bmatrix} g_1(x)^T & \mathbf{0}^T_{(n-p)\times p} \end{bmatrix}^T$ with $g_1(x) \in \mathcal{R}^{p\times p}$. Functions f and G are assumed Lipschitz with Lipschitz constants $L_k^f > 0$ and $L_k^g > 0$, respectively.⋈

A system is differentially flat if the states, inputs, and outputs can be expressed algebraically in terms of an output z and a finite number of its higher-order derivatives. Signal z, referred to as the flat output, has the same dimension as the control inputs. For system (1), let the flat output be $z = h(x)$, then. there exist functions α and β such that [8]

$$u = \alpha\left(z, \dot{z}, \ddot{z}, \dddot{z}, ...\right), x = \beta\left(z, \dot{z}, \ddot{z}, \dddot{z}, ...\right). \tag{2}$$

The following simplifying notation is used in the paper: $\overline{z}^{\omega} = (z, \dot{z}, \ddot{z}, \dddot{z}, ...)$, where $\omega \in \{\alpha, \beta\}$ specifies if the list $z, \dot{z}, \ddot{z}, \dddot{z}$ is that of $\alpha()$ or of $\beta()$.

Remarks Nonlinear differentially flat system is the possibility to inverse the nonlinear system without having to integrate the state. This allows generating an open-loop control based on trajectory generation. Suppose the steering trajectory $x^t(t)$ is expressed as a polynomial $p_k(t)$ for $t \in [0, nT)$, with $n \geq 1$, that satisfies relevant start- and end-point conditions in terms of position, speed and acceleration continuity. From $z_k(\tau) = h(p_k(\tau))$, the computation of the feedforward term $u_k(\tau) = \alpha\left(\overline{z}_k^{\alpha}(\tau)\right)$ is straightforward. As a matter of fact, [7] proposes to control the plant by means of a feedback control strategy that is expressed in function of $u_k(\tau)$. To achieve trajectory tracking, the starting and the ending points of $p_k(t)$ are selected such that $x^t(t) \to x^d(t)$. In the next section, an extension of the control strategy in [7] is proposed.⋈

III. Sampled-data Control of Nonlinear Differentially Flat Systems

A. Control Objectives

Integration (1) on $[0, \tau)$, $\tau \in [0, T)$, as

$$x_k(\tau) = x_k + \int_0^{\tau} f(x_k(\nu))d\nu + \int_0^{\tau} G(x_k(\nu))u_k(\nu)d\nu \tag{3}$$

where $x_k(\tau)$ denotes the system state resulting from the action of the CT control input lifted as $u_k(\tau)$, and $x_k = x_k(\tau)|_{\tau=0}$. The control objectives can then be stated as follows. Let $x^d(t)$ be the trajectory to be tracked, which is assumed known at least one time step before the digital board generates the control input signal. At time $t = kT$, given the knowledge of x_{k-1}, x_k^d and x_{k+1}^d , the DT control law $\mathcal{U}_k$ is devised by following a two-step procedure, which consists in 1) determining a CT control law $u_k(\tau)$ that steers the system trajectories $x_k(\tau)$ towards the reference trajectory $x_k^d(\tau)$, and 2) averaging $u_k(\tau)$ over $[0, T)$ in order to provide $\mathcal{U}_k$. These steps are formalized by fulfilling the following two control objectives.

Objective 1 (Convergence in the CT domain) For $\tau \in [0, T)$, control input $u(kT + \tau)$ is such that $\lim_{k\to\infty} \left\|x_k - x_k^d\right\| = 0$.

Objective 2 (SD control): Determine the actual DT control input $\mathcal{U}_k$, which is constant over $[kT, (k+1)T)$, such that

$$\lim_{k\to\infty} \left\|x_k^d - \overline{\overline{x}}_k\right\| = \mathcal{O}\left(T^2\right) \tag{4}$$

where $\overline{\overline{x}}_k(\tau)$ denotes the state of the plant under SD control, which is obtained by application of the sampled control $\mathcal{U}_k$ and the state estimator to be determined.

B. Proposed Approach

The proposed SD control law consists of 1) devising a CT scheme that determines the starting and ending points of polynomial $p_k(t)$ so as to track $x^d(t)$, 2) providing state estimation, and 3) computing the piecewise continuous control input $\mathcal{U}_k$.

STEP 1: Derivation of the CT steering control law $u_k(\tau)$.

Assume that x_k is available for the computation of $\mathcal{U}_k$. It is thus possible to determine $u_k(\tau)$ that steers (3) from x_k to a targeted state x_{k+1}^t defined as

$$x_{k+1}^t = x_{k+1}^d + \mu_k\left(x_k - x_k^d\right) \tag{5}$$

to ensure that $\left\|x_{k+1}^t - x_{k+1}^d\right\| \leq \mu_k\left\|x_k - x_k^d\right\|$, where μ_k is a function of k and satisfies $0 < \mu_k < 1$. From the selection of a suitable interpolation polynomial $p_k(\tau)$, which is such that $p_k(0) = x_k$ and $p_k(T) = x_{k+1}^t$, and by using the flatness argument, one can determine $u_k(\tau) = \alpha\left(\overline{z}_k^{\alpha}(\tau)\right)$ as mentioned in the previous section such that the trajectory of the closed-loop system goes from x_k to x_{k+1}^t. Let $\widetilde{x}_1 = x_1 - x_1^d$, with $\|\widetilde{x}_1\| \neq 0$. Then

$$\lim_{k\to\infty} \left\|x_k^t - x_k^d\right\| = \|\widetilde{x}_1\| \lim_{k\to\infty} \prod_{j=1}^{j=k-1} \mu_j = 0. \tag{6}$$

Note that μ_k may be selected as an $n \times n$ matrix whose eigenvalues $\lambda_k^i, i \in \{1, 2, ...n\}$, are all inside the unit circle. Let $\lambda_k^M = \sup_{i\in\{1,2,...n\}} \left|\lambda_k^i\right|$. Therefore, the following inequality holds for all k:

$$\left\|x_{k+1}^t - x_{k+1}^d\right\| \leq \lambda_k^M \left\|x_k - x_k^d\right\| \tag{7}$$

where $\lambda_k^M < 1$. This leads to (6), where μ_j is replaced by λ_k^M. Considering a matrix instead of a scalar may be advantageous if one can establish a linear error-like dynamics between two sampling instants. For instance, one can exploit the property that flat systems can be linearized by endogenous state feedback control laws [8], which in turn allows to impose an exponentially decreasing evolution of the error dynamics. The error dynamics, which results from the state feedback linearization, leads to equality (5) with a stable matrix μ_k. This approach is illustrated in Section IV with the control of a helicopter-like UAV.

STEP 2: Calculation of $\mathcal{U}_k$ by means of averaging.

From an implementation perspective, the CT scheme developed so far and which lies on the computation of x_{k+1}^t, $p_k(\tau)$ and $u_k(\tau)$ presents two drawbacks: 1) a causality problem which lies in the fact that x_k is not available for the computation of $\mathcal{U}_k$, and 2) the digital board provides a constant signal $\mathcal{U}_k$ rather than the CT control input $u_k(\tau)$. To solve these two problems, we propose to use a one-step estimation of x_k, denoted as $\widehat{x}_k$, so as to initialize $p_k(\tau)$ without the causality problem and a more accurate approximation $\mathcal{U}_k$ of $u_k(\nu)$ than in [7], which considered the time-averaged value given as

$$\mathcal{U}_k = \frac{1}{T}\int_0^T u(kT+\nu)d\nu. \tag{8}$$

With (8), $\mathcal{U}_k$ leads to an error in $\mathcal{O}\left(T^2\right)$ even when one neglects the causality constraint, i.e. $\widehat{x}_k = x_k$.

At $\tau = T$, (3) can be expressed as

$$\begin{aligned} x_{k+1} = x_k + \textstyle\int_0^T f(x_k(\nu))d\nu + \\ \textstyle\int_0^T G(x_k(\nu))u_k d\nu + \int_0^T G(x_k(\nu))\left(u_k(\nu) - \mathcal{U}_k\right)d\nu \end{aligned} \tag{9}$$

It is clear that if $\int_0^T g_1(x_k(\nu))d\nu \neq 0$,then $\int_0^T G(x_k(\nu))u_k(\nu)d\nu$ and $\int_0^T G(x_k(\nu))\mathcal{U}_k d\nu$ lead to the same state update of x_k , which equivalently leads to

$$\int_0^T G(x_k(\nu))\left(u_k(\nu) - \mathcal{U}_k\right)d\nu = 0 \tag{10}$$

provided

$$\mathcal{U}_k = \left(\int_0^T g_1(x_k(\nu))d\nu\right)^{-1} \int_0^T g_1(x_k(\nu))u_k(\nu)d\nu. \tag{11}$$

From the flatness property, $x_k(\nu) = \beta\left(\overline{z}_k^\beta(\nu)\right)$ can be computed on $[0, T)$ by considering $p_k(\nu)$ and (5). $x_k(\nu)$ is determined such that $\mathcal{U}_k$ (11) steers the state of the system from x_k ($=\widehat{x}_k$) to x_{k+1}^t, which implies that the limit in (6) is obtained and, thus, Objective 1 is satisfied.

STEP 3: Design of state estimator.

For the case $\widehat{x}_k \neq x_k, \mathcal{U}_k$ tries to steer system (1) from an estimated state $\widehat{x}_k$ of x_k to a neighborhood of the targeted state x_{k+1}^t. Estimation $\widehat{x}_k$ can be computed from a fixed-step numerical integration of (1) on $[kT, (k+1)T)$

$$\widehat{x}_k = R_q\left(x_{k-1}, \mathcal{U}_{k-1}\right). \tag{12}$$

Since $\widehat{x}_k$ rather than x_k is used, target state x_{k+1}^t obtained in the ideal case with (5) is now computed as

$$\widehat{x}_{k+1}^t = x_{k+1}^d + \mu_k\left(\widehat{x}_k - x_k^d\right). \tag{13}$$

The one-step estimator R_q is chosen to provide an estimation error in $\mathcal{O}\left(T^q\right)$, where $q \in \mathcal{N}^+$ depends on the numerical integration order. A fixed-step Runge-Kutta (RK) method [11] of sufficiently high order provides a good trade-off between computation complexity and integration accuracy for systems that are not too stiff. One now prove the convergence of the estimator-based SD control. Proposition 1 states how far from the desired trajectory x_k^d the actual state $\overline{\overline{x}}$ is. Note that $\overline{\overline{x}}_k(0) = \widehat{x}_k(0)$ The following requirement about (1) and the steering control law $u_k(\tau)$ described in *STEP 1* is needed.

Assumption 4 Desired trajectories x_k^d and interpolation function p_k are such that $u_k(\tau)$ is bounded and such $\int_0^T g_1(x_k(\nu))d\nu \neq 0$ is verified.

Proposition 1 Consider the discrete-time control law $\mathcal{U}_k$ given in (11) and shown in Figure 1. When Assumption 4 is met, Objective 2 is satisfied, that is

$$\lim_{k\to\infty} \left\|x_k^d - \overline{\overline{x}}_k\right\| = \mathcal{O}\left(T^2\right), \tag{14}$$

provided the control law applied to the CT system, (1), is calculated with the interpolation function $p_k(\tau)$, whose start- and end-points are determined from (13) and state estimator R_q ($q \geq 2$), (12), respectively. ⋈

Proof Since the estimator R_q is assumed to be of order q, the estimation error is such that $\|x_k - \widehat{x}_k\| = \mathcal{O}\left(T^q\right)$ [11]. Then integration of (1) on $[0, \tau)$ with $\tau \in [0, T)$

$$\overline{\overline{x}}_k(\tau) = \overline{\overline{x}}_k + \int_0^\tau f(\overline{\overline{x}}_k(\nu))d\nu + \int_0^\tau G(\overline{\overline{x}}_k(\nu))\mathcal{U}_k d\nu \tag{15}$$

compared to (3) leads to

$$
\begin{array}{l}
\left\|\overline{\widehat{x}}_k(\tau) - x_k(\tau)\right\| \leq \mathcal{O}\left(T^q\right) \\
+L_k^f \int_0^\tau \left\|\overline{\widehat{x}}_k(\tau) - x_k(\tau)\right\| d\nu \\
+L_k^g \int_0^\tau \left\|\overline{\widehat{x}}_k(\tau) - x_k(\tau)\right\| \mathcal{U}_k d\nu \\
+\int_0^\tau G(x_k(\nu))\left(\mathcal{U}_k - u_k(\tau)\right) d\nu
\end{array} \quad (16)
$$

when using the fact that $\overline{\widehat{x}}_k = \widehat{x}_k$ at the beginning of the sampling period. Furthermore, when $\tau = T$, the last term of (16) is zero according to property expressed in (10). Thus applying Gronwall-Bellman's lemma and taking $\tau = T$, the following bound can be obtained:

$$
\left\|\overline{\widehat{x}}_{k+1} - x_{k+1}\right\| \leq \mathcal{O}\left(T^q\right) + I_k \quad (17)
$$

where

$$
\begin{array}{l}
I_k = \int_0^T \lambda(s)\mu(s)e^{\int_s^T \mu(r)dr} ds. \\
\lambda(s) = \mathcal{O}\left(T^q\right) + \kappa(s) \\
\kappa(s) = \int_0^s G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right) \\
\mu(s) = L_k^f + L_k^g \left|\mathcal{U}_k\right|
\end{array} \quad (18)
$$

Given that

$$
\int_0^T \mathcal{O}\left(T^q\right)\mu(s)e^{\int_s^T \mu(r)dr} ds = \mathcal{O}\left(T^q\right), \quad (19)
$$

integration by parts of

$$
\begin{array}{l}
\int_0^T \kappa(s)\mu(s)e^{\int_s^T \mu(r)dr} ds = \left[e^{L(T-s)}\kappa(s)\right]_T^0 \\
+\int_0^T \frac{d\kappa(s)}{ds} e^{L(T-s)} ds
\end{array} \quad (20)
$$

along with the property expressed in (10) lead to I_k

$$
\begin{array}{l}
I_k \\
= \mathcal{O}\left(T^q\right) + \int_0^T G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right) e^{L_k(T-s)} ds \\
= \mathcal{O}\left(T^q\right) \\
+\int_0^T G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right)\left(1 + \sum_{i=1}^{+\infty} \frac{L_k(T-s)^i}{i!}\right) ds \\
= \mathcal{O}\left(T^q\right) \\
+\int_0^T G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right)\sum_{i=1}^{+\infty} \frac{L_k(T-s)^i}{i!} ds \\
= \mathcal{O}\left(T^q\right) \\
+\sup_{\tau\in[0,T)} \left\|G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right)\right\| \mathcal{O}\left(T^2\right)
\end{array} \quad (21)
$$

with $L_k = L_k^f + L_k^g \left|\mathcal{U}_k\right|$. Furthermore, since CT control law $u_k(\tau)$ is designed to achieve $x_{k+1}^d = x_{k+1}^t$, then from (5) and from noticing that $\widehat{x}_{k+1}^t - x_{k+1}^t = \mu_k\left(\widehat{x}_k - x_k\right)$,

$$
\begin{array}{l}
\left\|x_{k+1}^d - \overline{\widehat{x}}_{k+1}\right\| \leq \left\|x_{k+1}^d - \widehat{x}_{k+1}^t\right\| \\
+\left\|\widehat{x}_{k+1}^t - x_{k+1}^t\right\| + \left\|\overline{\widehat{x}}_{k+1} - x_{k+1}\right\| \\
\leq \mu_k \left\|x_k^d - \overline{\widehat{x}}_k\right\| + (1+\mu_k)\mathcal{O}\left(T^q\right) + \mathcal{H}_k\mathcal{O}\left(T^2\right) \\
\leq \mathcal{O}\left(T^q\right) + \prod_{i=1}^{i=k}\mu_i \left\|x_1^d - \overline{\widehat{x}}_1\right\| \\
+\left(\mathcal{H}_k + \sum_{i=1}^{i=k-1}\left(\mathcal{H}_i \prod_{j=i+1}^{j=k}\mu_j\right)\right)\mathcal{O}\left(T^2\right)
\end{array} \quad (22)
$$

where $\widehat{x}_{k+1}^t = \overline{\widehat{x}}_{k+1}$, $x_{k+1}^t = x_{k+1}, \mathcal{H}_k = H_k + \mu_k H_{k-1}, H_0 = 0$,and $H_k = \sup_{\tau\in[0,T)} \left\|G(x_k(\nu))\left(\mathcal{U}_k - u_k(\nu)\right)\right\|$, which exists and is finite from Assumption 4. Let $\mu_j \leq \overline{\mu} < 1$ for all j, then

$$
\begin{array}{l}
\mathcal{H}_k + \sum_{i=1}^{i=k-1}\left(\mathcal{H}_i \prod_{j=i+1}^{j=k}\mu_j\right) \\
\quad \leq \sup_{i\in\{1,\ldots,k\}} \mathcal{H}_i \sum_{j=0}^{j=k-1} \overline{\mu}^j \\
\quad \leq \sup_{i\in\{1,\ldots,k\}} \mathcal{H}_i \frac{1-\overline{\mu}^k}{1-\overline{\mu}}
\end{array} \quad (23)
$$

As $\overline{\mu} < 1$, the rightmost term in (22) is finite and the limit of Objective 2, (4), follows immediately if $q \geq 2$.□

Remarks 1) If $\int_0^T g_1(x_k(\nu))d\nu = 0$, $\mathcal{U}_k$ determined by (11) is no longer valid. The time-averaged value $U_k = \frac{1}{T}\int_0^T u(kT+\nu)d\nu$ proposed in [7] is an appropriate choice; 2) One can show that DT control law $\mathcal{U}_k$ of (11) converges to u_k as T decreases to zero. Expand U_k as

$$
\begin{array}{l}
\mathcal{U}_k(T) = \left(\int_0^T g_1(x_k(\nu))d\nu\right)^{-1} \int_0^T g_1(x_k(\nu))u_k(\nu)d\nu \\
= \left(\int_0^T \left[g_1(x_k(0) + \sum_{i=1}^{i=+\infty} \frac{g_1^{(i)}(x_k(\nu)\nu^i}{i!}\right] d\nu\right)^{-1} \\
\times \int_0^T \left[g_1(x_k(0)u_k(0) + \sum_{i=1}^{i=+\infty} \frac{[g_1(x_k(\nu)u_k(\nu)]^{(i)}\nu^i}{i!}\right] d\nu \\
= \left(g_1(x_k(0)T + \mathcal{O}\left(T^2\right))\right)^{-1} \\
\times \left(g_1(x_k(0)u_k(0)T + \mathcal{O}\left(T^2\right))\right)
\end{array} \quad (24)
$$

thus $\lim_{T\to 0} U_k(T) = u_k$ ⋈ .

IV. ROTORCRAFT-LIKE UAV TRAJECTORY TRACKING

It is desired to make a helicopter-like UAV track a sinusoidal trajectory. The closed-form integrability of the plant is used to exactly compute the DT control $\mathcal{U}_k$. When equipped with autopilots, the planar motion of a helicopter-like UAV [10] can be described as

$$
\begin{array}{l}
\dot{x}_1 = x_3\cos\left(x_4\right), \dot{x}_2 = x_3\sin\left(x_4\right), \\
\dot{x}_3 = u_1, \dot{x}_4 = u_2,
\end{array} \quad (25)
$$

where (x_1, x_2), (x_3, x_4) and (u_1, u_2) represent the planar location, the speed and heading angle rate and the actuation signals, respectively. Output variables (x_1, x_2) are flat for (25). Thus, one can obtain the following relationships

$$
\begin{array}{l}
x_3 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}, x_4 = \tan^{-1}\left(\frac{\dot{x}_2}{\dot{x}_1}\right) \\
u_1 = \frac{\dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}, u_2 = \frac{\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1}{\dot{x}_1^2 + \dot{x}_2^2}.
\end{array} \quad (26)
$$

Let $\widetilde{x} = x - x^d$ where $x^T = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right]$ and x^d is the trajectory to track. Since (25) is flat, the control

$$
\begin{array}{l}
u_1 = v_1\cos(x_4) + v_2\sin(x_4) \\
u_2 = \frac{v_2\cos(x_4) - v_1\sin(x_4)}{x_3} \\
[v_1\ v_2]^T = \frac{d^2x^d}{dt^2} - k_1\frac{d\widetilde{x}}{dt} - k_2\widetilde{x}
\end{array} \quad (27)
$$

in closed loop with (25) results in the linearized dynamics

$$
\left[\begin{array}{c} \frac{d\widetilde{x}}{dt} \\ \frac{d^2\widetilde{x}}{dt^2} \end{array}\right] = A_d \left[\begin{array}{c} \widetilde{x} \\ \frac{d\widetilde{x}}{dt} \end{array}\right], A_d = \left[\begin{array}{cc} 0 & I \\ -k_2 & -k_1 \end{array}\right]. \quad (28)
$$

As x_k and $[dx/dt]_k$ are not available at kT, its estimates $\widehat{x}_k$ and $\left[\widehat{dx/dt}\right]_k$ are used in the controller implementation and are computed from the 4^{th}-order RK method, yielding

$$\begin{aligned} \widehat{x}_k &= R_4\left(x_{k-1}, \mathcal{U}_{k-1}\right) \\ \left[\widehat{\frac{dx}{dt}}\right]_k &= f(\overline{\widehat{x}}_k) + g(\overline{\widehat{x}}_k)\mathcal{U}_{k-1} \end{aligned} \tag{29}$$

where $\mathcal{U}_{k-1}$ is determined in the sequel. Adopt

$$\xi = \begin{bmatrix} \overline{\widehat{x}}_k - x_k^d \\ \left[\widehat{\frac{dx}{dt}}\right]_k - \left[\frac{dx^d}{dt}\right]_k \end{bmatrix}. \tag{30}$$

From (28), the targeted state at $(k+1)T$ is computed as

$$\begin{bmatrix} \widehat{x}_{k+1}^t \\ \left[\frac{d\widehat{x}^t}{dt}\right]_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1}^d \\ \left[\frac{dx^d}{dt}\right]_{k+1} \end{bmatrix} + e^{A_d T}\xi \tag{31}$$

Finally, from (11) and (26), DT control $\mathcal{U}_k$ is computed as

$$\begin{aligned} \mathcal{U}_k &= \frac{1}{T}\int_0^T [\dot{x}_3 \ \dot{x}_4]^T dt \\ &= \frac{1}{T}\begin{bmatrix} \sqrt{t_1^2 + t_2^2} - \sqrt{e_1^2 + e_2^2} \\ \tan^{-1}\left(\frac{t_2}{t_1}\right) - \tan^{-1}\left(\frac{e_2}{e_1}\right) \end{bmatrix} \end{aligned} \tag{32}$$

with

$$\left[\frac{d\widehat{x}^t}{dt}\right]_{k+1} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \left[\widehat{\frac{dx}{dt}}\right]_k = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{33}$$

In short, the computation of $\mathcal{U}_k$ (29)-(32), which is the controller output at instant kT, requires the knowledge of x_{k-1}, u_{k-1}, x_k^d and $\left[dx^d/dt\right]_k$. The aforementioned DT control law is used to track $[x^d(t) \ y^d(t)] = [50\sin(2t) \ t]$ for $t \in [0, 4 \text{ s}]$. Controller gain matrices are $k_1 = 14I_2$ and $k_2 = 100I_2$ where I_2 is the 2×2 identity matrix. Initial error $\widetilde{x}(0) = [10 \ 2]$ is introduced to excite the plant. The sample period is $T = 30$ ms. Since nonlinear DT control is typically designed in the CT domain and then implemented in DT as is, without any conversion, supposing that T is small enough, the proposed approach is compared to the linearizing law (27) labeled as CT+Hold, which is discretized by simply introducing a sampler and a ZOH at its input and output, respectively. Furthermore, a time delay of one sampling period T is simulated to represent the causality constraint inherent in the digital board in which the law is executed. The proposed SD control law provides a time response with smoother transients and a smaller tracking error than those obtained with the linearizing controller CT+HOLD, as shown in Figures 2 and 3. Transients of input signals u_1 are smaller than for CT+HOLD, as shown in Figure 4. It should be noted that the sampling time modifies the speed and angle responses (V, α) such that responses of CT+HOLD with $T = 5$ ms are similar to those of the proposed SD controller with $T = 30$ ms. This fact may be significant in practice, especially when speed is limited to positive values, as is the case in [10]. Figures 5 and 6 show results obtained when the discrete-time control laws are implemented with a finite wordlength constraint of 16 bits and fixed-point arithmetic so as to reproduce as faithfully as possible a realistic digital implementation of the controller. As illustrated in Figure 5 and 6, the proposed control law provides satisfactory responses, which are less oscillatory than those obtained with CT+Hold.

V. CONCLUSION

This paper extended the sampled-data control approach for a class of flat systems as described in [7] by proposing solutions to two well-known issues: 1) to avoid causality problems at the implementation stage, an estimator is proposed to provide the latest available state to the digital controller, and 2) an averaging-based DT control law is presented to compensate for nonlinear-in-the-input terms of the system. The proposed approach is applied to the trajectory tracking control of a rotorcraft-like UAV. The DT control is derived from the closed-form integrability property of the UAV model. Numerical simulations show the effectiveness of the proposed approach in terms of response accuracy when relatively large sampling time and fixed-point, finite wordlength implementations are considered. A possible extension to this work consists of determining an appropriate choice of steering trajectories that meet boundary conditions, and path and control constraints. Furthermore, the approach presented in this paper could be extended to a multistep model predictive control synthesis.

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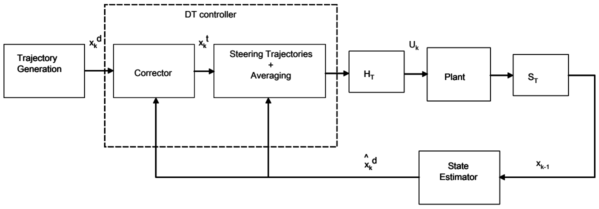

Figure 1: Structure of sampled-data control system

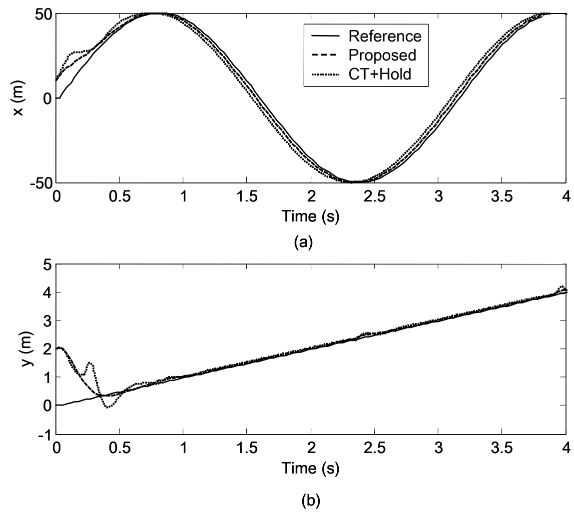

Figure 2: Position of the UAV stabilized with (a) the proposed and (b) the CT+Hold controllers at $T = 30$ ms

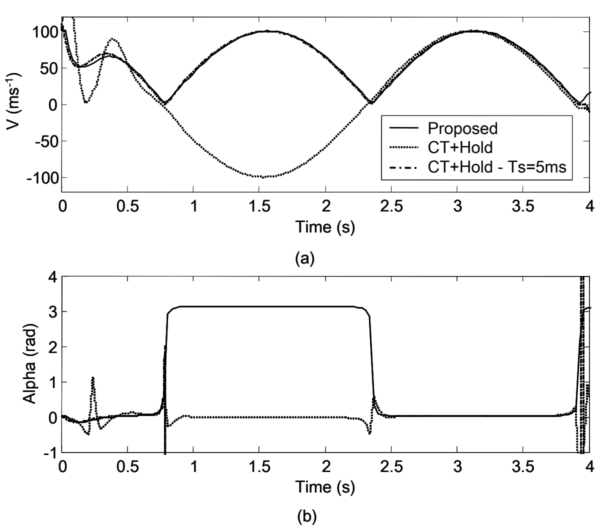

Figure 3: Speed and orientation of the UAV stabilized with (a) the proposed ($T = 30$ ms) and (b) the CT+Hold controllers at $T = 5$ ms and 30 ms

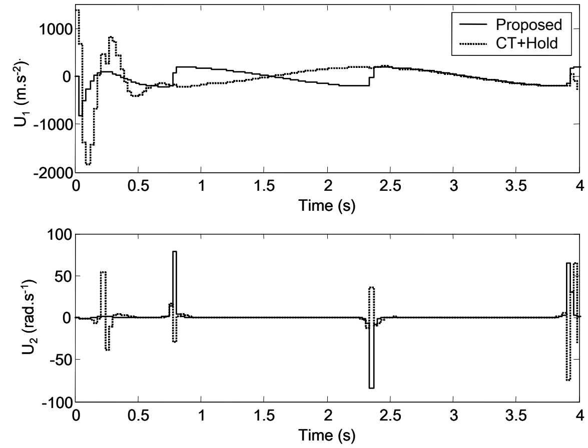

Figure 4: Discrete-time control signals - $T = 30$ ms

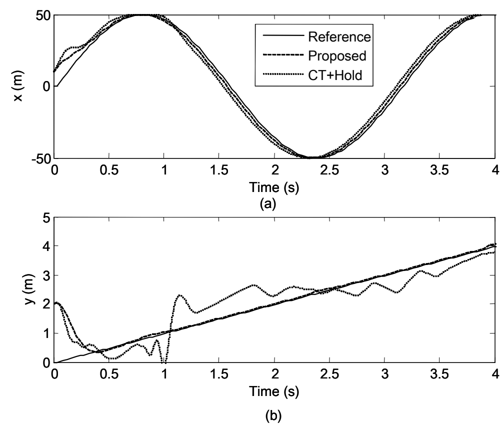

Figure 5: Position of the UAV stabilized with (a) the proposed and (b) the CT+Hold controllers at $T = 30$ ms with a fixed-point,16 bit word length implementation

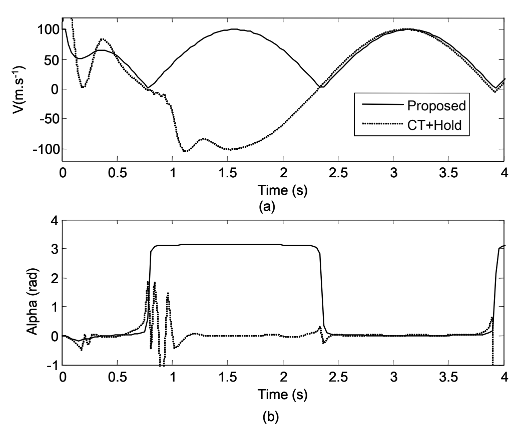

Figure 6: Speed and orientation of the UAV stabilized with (a) the proposed and (b) the CT+Hold controllers at $T = 30$ ms with a fixed-point, 16 bit word length implementation