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Nonlinear n-th Cost Cumulant Control and Hamilton-Jacobi-Bellman Equations for Markov Diffusion Process

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Abstract— **A general nonlinear stochastic system with non-quadratic cost function is considered for cost cumulant control of a Markov diffusion problem. The Hamilton-Jacobi-Bellman equation for the n-th cost moment case is derived as a necessary condition for optimality. The n-th cost cumulant Hamilton-Jacobi-Bellman equation derivation procedure is given. Second, third, and fourth cost cumulant Hamilton-Jacobi-Bellman equations are derived using the proposed procedure. The solutions of the nonlinear cost cumulant control problem is discussed using the state dependent Riccati equation method.**

I. Introduction

We propose to minimize a nonlinear Markov diffusion process using the cumulants of a non-quadratic cost function. The final objective of cost cumulant control, which is also known as statistical control or cost density control, is to manipulate the cost distribution by controlling each cumulant.

The cost cumulant control started with Sain. An open-loop minimum cost variance problem was solved by him in [8]. Sain and Liberty continued to study the characteristics of cost cumulant control in [10]. Liberty studied the quadratic nature of the minimal cost cumulant control in [5]. However their study was restricted to the linear system and quadratic cost case. The relationship between the cost cumulant control and risk-sensitive control was established in [11]. Sain *et al.* published results for the second cost cumulant control in the full-state-feedback case [12].

Why cumulants? Cumulants have more intuitive meanings than moments. The first cumulant is the mean, the second cumulant is the variance, and the third cumulant is the skewness and the fourth cumulant is the kurtosis. The second cumulant shows the variation of the distribution around the mean, the third cumulant implies the departure from symmetry, and the fourth cumulant shows the flatness of the distribution. Moreover, higher order cumulants have decreasing significance, thus approximating using first few cumulants are a good approximation to the general problem.

Section II discusses the necessary mathematical preliminaries, Section III presents the Hamilton-Jacobi-Bellman (HJB) equation for the n-th moment case, Section IV discusses the n-th cumulant HJB equation, and Section V discusses the solution procedure for the cost cumulant control problem. Finally, conclusions are given in the last section.

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II. Mathematical Preliminaries

Consider the nonlinear stochastic differential equation:

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dw(t), \qquad (1)$$

where $t \in [t_0, t_F], x(t_0) = x_0, x(t) \in \mathbb{R}^n$ is the state, $u(t) \in U$ is the control action, and $dw(t)$ is a Gaussian random process of dimension d with zero mean and covariance of $W(t)dt$. A memoryless feedback *control law* is introduced as

$$u(t) = k(t, x(t)), \qquad (2)$$

where k is a nonrandom function with random arguments. Also, consider a non-quadratic cost function

$$J(t, x(t); k) = \int_t^{t_F} \Big[L(s, x(s), k(s, x(s))) \Big] ds + \psi(x(t_F)). \qquad (3)$$

Let $Q_0 = [t_0, t_F) \times \mathbb{R}^n$ and $\bar{Q}_0$ denote the closure of Q_0. Assume that L is continuously differentiable and satisfies the polynomial growth condition. We assume that f, σ, and k satisfy the Lipschitz condition and the linear growth condition. The moments are defined as

$$M_i(t, x; k) = E\left\{ J^i(t, x; k) | x(t) = x \right\}.$$

$\mathcal{O}(k)$ is the backward evolution operator. given by

$$\begin{aligned} \mathcal{O}(k) \quad &= \quad \frac{\partial}{\partial t} + \left\langle f(t, x, k(t, x)), \frac{\partial}{\partial x} \right\rangle \\ &\quad + \frac{1}{2} \text{ tr } \left(\sigma(t, x) W(t) \sigma'(t, x) \frac{\partial^2}{\partial x^2} \right) \end{aligned} \qquad (4)$$

where (suppressing arguments)

$$\begin{aligned} \left\langle f, \frac{\partial}{\partial x} \right\rangle \quad &= \quad \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \triangleq \mathcal{O}^{(1)}(k) \\ \text{tr } \left(\sigma W \sigma' \frac{\partial^2}{\partial x^2} \right) \quad &= \quad \sum_{i,j=1}^{n} (\sigma W \sigma')_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \triangleq 2\mathcal{O}^{(2)}(k) \end{aligned} \qquad (5)$$

Now we define a moment around an arbitrary point, a, using Stieltjes integral.

$$M_i = \int_{-\infty}^{\infty} (x - a)^i dF$$

where $i = 0, 1, 2, \cdots$ and $M_0 = 1$ by definition. The moments are a set of descriptive constants of a distribution

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that are useful for measuring its properties. Formally, the cumulants $V_1, V_2, \cdots, V_i$ are defined by the identity in t.

$$\exp\left(\sum_{i=1}^{\infty} V_i \frac{t_i}{i!}\right) = \sum_{i=0}^{\infty} M_i \frac{t_i}{i!}$$

Note that V_0 is not defined. Furthermore, we have the following moment (characteristics function) and cumulant generating functions,

$$\phi(t) = \int_{-\infty}^{\infty} e^{jtx} dF \text{ and}$$

$$\psi(t) = \log \phi(t),$$

where j is the complex operator, $j^2 = -1$. Now, we present the moment-cumulant relationship from [4].

$$\begin{aligned} V_r &= r! \sum_{i=1}^{r} \sum_{\pi} \left(\frac{M_1}{1!}\right)^{\pi_1} \left(\frac{M_2}{2!}\right)^{\pi_2} \\ & \cdots \left(\frac{M_i}{i!}\right)^{\pi_i} \frac{(-1)^{\rho-1}(\rho-1)!}{\pi_1! \cdots \pi_i!} \end{aligned} \quad (6)$$

where $\pi_1 + \pi_2 + \cdots + \pi_i = \rho$, $\pi_1 + 2\pi_2 + \cdots + i\pi_i = r$, and π's are non-negative integers.

$$M_r = \sum_{i=1}^{r} \sum_{\pi} \left(\frac{V_1}{1!}\right)^{\pi_1} \left(\frac{V_2}{2!}\right)^{\pi_2} \cdots \left(\frac{V_i}{i!}\right)^{\pi_i} \frac{r!}{\pi_1! \cdots \pi_i!} \quad (7)$$

where $\pi_1 + 2\pi_2 + \cdots + i\pi_i = r$ and π's are non-negative integers.

III. N-TH MOMENT HAMILTON-JACOBI-BELLMAN EQUATION

Here we derive the Hamilton-Jacobi-Bellman (HJB) equation for n-th moment of the cost function, which is a necessary condition for the optimality. We will assume the existence of an optimal controller. The following algebraic identity will be used in the sequel. Also, for brevity we have removed all the proofs. Proofs are available by contacting the author.

Algebraic Identity: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Theorem 3.1: Let M_1 be an admissible mean cost function and K_{M_1} be the corresponding class of control laws. Assume $M_i^*(t,x) \in C_p^{1,2}(\bar{Q}_0)$ and the existence of an optimal controller $k^*_{M_i|M_1}$, where $i = 2, 3, \cdots$. Then $k^*_{M_i|M_1}$ and $M_i^*(t,x)$ satisfy the partial differential equation

$$\mathcal{O}(k^*_{M_i|M_1})[M_i^*(t,x)] + iM^*_{i-1}(t,x)L(t,x,k^*_{M_i|M_1}) = 0 \quad (8)$$

for $t \in T$, $x \in \mathbb{R}^n$, where

$$\begin{aligned} &\mathcal{O}(k^*_{M_i|M_1})[M_i^*(t,x)] + iM^*_{i-1}(t,x)L(t,x,k^*_{M_i|M_1}) \\ &= \min_{k \in K_{M_1}} \left\{ \mathcal{O}(k)[M_i^*(t,x)] + iM^*_{i-1}(t,x)L(t,x,k) \right\}, \end{aligned} \quad (9)$$

along with the boundary condition

$$M_i^*(t_F, x) = M^i(t_F, x) = \psi^i(x(t_F)), \quad i = 1, 2, 3, \cdots. \quad (10)$$

Theorem 3.1 gives the HJB Equation for any n-th moment of the cost function. We will utilize this theorem to find n-th cumulant HJB equation in the next section.

IV. N-TH CUMULANT HAMILTON-JACOBI-BELLMAN EQUATION

In order to find n-th cumulant HJB equation, we will utilize the moment-cumulant relationship and n-th moment HJB equation.

Lemma 4.1: The i-th cost moment, $M_i(t,x;k)$ is related to $i-1$-th cost moment, $M_{i-1}(t,x;k)$ by the following partial differential equation.

$$\mathcal{O}(k)[M_i(t,x;k)] + iM_{i-1}(t,x;k)L(t,x,k) = 0 \quad (11)$$

with the boundary condition $M_i(t_f, x; k) = \psi^i(x(t_F))$ where $i = 1, 2, \cdots$.

Note that first, second, and third moment HJB equations are given respectively as

$$\begin{aligned} 0 &= \mathcal{O}(k)M_1 + L \\ 0 &= \mathcal{O}(k)M_2 + 2M_1 L \\ 0 &= \mathcal{O}(k)M_3 + 3M_2 L. \end{aligned}$$

Lemma 4.2: The powers of cost moments M_i are related by the following partial differential equation.

$$\begin{aligned} &\mathcal{O}(k)[M_i^p(t,x;k)] = -piM_i^{p-1}(t,x;k)L(t,x,k) \\ &+\frac{p(p-1)}{2} M_i^{p-2}(t,x;k) \left\| \frac{\partial M_i}{\partial x} \right\|^2_{\sigma W \sigma'}. \end{aligned}$$

Lemma 4.3: The powers of the cost moments $M_i M_j$ are related by the following partial differential equation.

$$\begin{aligned} &\mathcal{O}(k)[M_i^p M_j^q] = -piM_i^{p-1}M_j^q M_{i-1}L \\ &-qjM_i^p M_j^{q-1} M_{j-1}L \\ &+\frac{p(p-1)}{2} M_i^{p-2} M_j^q \left\| \frac{\partial M_i}{\partial x} \right\|_{\sigma W \sigma'} \\ &+qpM_i^{p-1}M_j^{q-1} tr\left(\sigma W \sigma' \left(\frac{\partial M_j}{\partial x}\right)\left(\frac{\partial M_i}{\partial x}\right)'\right) \\ &+\frac{q(q-1)}{2} M_i^p M_j^{q-2} \left\| \frac{\partial M_j}{\partial x} \right\|^2_{\sigma W \sigma'} \end{aligned} \quad (12)$$

Now we propose a procedure to find the n-th order cumulant HJB equation:

- Use Equation (6) or (7) to find the relationship between n-th moment and n-th cumulant.
- Substitute M_n into Equation (9) and find the relationship between the lower order moments and cumulants.
- Prove that the above condition in step two is valid and determine the n-th cumulant HJB equation.

Using this procedure it is possible to determine any n-th cumulant HJB equation. As examples, we will find second, third and fourth cumulant HJB equations.

Theorem 4.1: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_2|M_1}$ and an optimum value function $V_2^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal second cumulant (variance) function V_2^* satisfies the following HJB equation.

$$\min_{k\in K_{M_1}} \mathcal{O}(k)[V_2^*(t,x)] + \left\| \frac{\partial V_1(t,x)}{\partial x} \right\|^2_{\sigma W \sigma'} = 0 \qquad (13)$$

for $(t, x) \in \bar{Q}_0$, together with the terminal condition, $V_2^*(t_F, x) = 0$.

Here, we present the third cumulant HJB equation.

Theorem 4.2: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_3|M_1}$ and an optimum value function $V_3^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal third cost cumulant (skewness) function V_3^* satisfies the following HJB equation.

$$\min_{k\in K_{M_1}} \left\{ \mathcal{O}(k)[V_3^*] + 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1}{\partial x}\right) \left(\frac{\partial V_2}{\partial x}\right)' \right) \right\} = 0 \qquad (14)$$

for $(t, x) \in \bar{Q}_0$, together with the terminal condition, $V_3^*(t_F, x) = 0$.

The minimization of second cumulant (variance) depends on the definition of the first cumulant (mean) value function V_1 as can be seen from Equation (13). The minimization of third cumulant (skewness) depends on the definition of both first and second cumulants value functions V_1 and V_2 as can be seen from Equation (14).

Here, we present the fourth cumulant (kurtosis) HJB equation.

Theorem 4.3: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_4|M_1}$ and an optimum value function $V_4^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal fourth cost cumulant (kurtosis) function V_4^* satisfies the following HJB equation.

$$\min_{k\in K_{M_1}} \mathcal{O}(k)[V_4^*] + tr\left(\sigma W \sigma' \left(\frac{\partial V_1}{\partial x}\right) \left(\frac{\partial V_3}{\partial x}\right)' \right) + 12 \left\| \frac{\partial V_1}{\partial x} \right\|^2_{(V_1 - V_2^2)\sigma W \sigma'} + 3 \left\| \frac{\partial V_2}{\partial x} \right\|^2_{\sigma W \sigma'} = 0 \qquad (15)$$

for $(t, x) \in \bar{Q}_0$, together with the terminal condition, $V_4^*(t_F, x) = 0$.

Using the presented method the first to fifty order cumulant HJB equations can be determined. To find the higher order ($10 > n \geq 6$), we require more partial differential equation lemmas such as the one's for $\mathcal{O}(k)[M_i^p M_j^q M_k^r]$. The necessary number of partial differential equations are summarized in the following table.

Table 1: Necessary Equations for n-th Cumulant Problem

No.	Necessary Theorem	Order of Cumulants
1	$\mathcal{O}(k)[M_i^p]$	$3 > n \geq 1$
2	$\mathcal{O}(k)[M_i^p M_j^q]$	$6 > n \geq 3$
3	$\mathcal{O}(k)[M_i^p M_j^q M_k^r]$	$10 > n \geq 6$
$\vdots$	$\vdots$	$\vdots$
m	$\mathcal{O}(k)[M_i^p M_j^q \cdots M_\eta^m]$	$\sum_{i=1}^{m+1} i > n \geq \sum_{i=1}^{m} i$

V. Solutions of Cost Cumulant Control

The general procedure for finding the solutions of cost cumulant control of a nonlinear system is discussed in this section. The solutions of first cumulant optimization (LQG)of a linear system is well known [3]. The second cumulant (minimal cost variance) case is given in [12]. Here, we derive the optimal controller for first two cumulants and first three cumulants of a general nonlinear system.

The procedure to find the optimal controller is given as follows.

- Decide on the number of cost cumulants to minimize.
- Derive the HJB partial differential equations for each of the cost cumulants.
- Create the partial differential equation to minimize using Lagrange multipliers.
- Solve for the optimal controller.
- Substitute the optimal controller back to the HJB equation and solve the HJB equation.

There are a number of approaches that one can take to obtain the solutions of the HJB equations: (1) Assume a linear system with quadratic cost and solve for the optimal controller, (2) find solutions of the HJB equation numerically [1], [6], or (3) utilize the state dependent Riccati equation (SDRE) approach [2]. In the sequel, first and third methods are investigated.

Cloutier and others have introduced the state-dependent Riccati equation (SDRE) approach to solve various nonlinear regulator problems [2]. Their approach is however for the time-invariant, infinite-time-horizon, nonlinear, and deterministic systems. We expand on their idea to the time-varying, nonlinear, finite-time-horizon, and stochastic systems. Then we will assume a linear system with quadratic cost and solve the cost cumulant problem and verify with the existing results.

A. Optimizing Solutions of the First Two Cumulants

Here we find the controller, k, that will minimize the value function,

$$V_1(t, x) + \gamma_2(t)V_2(t, x),$$

where $\gamma_2(t)$ is a time varying Lagrange multiplier. From the previous section, we have the following two partial differential equations as the necessary conditions for optimality.

$$\frac{\partial V_1}{\partial t} + g' \frac{\partial V_1}{\partial x} + k'B' \frac{\partial V_1}{\partial x} + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 V_1}{\partial x^2} \right) + h + k'Rk = 0 \qquad (16)$$

and

$$\frac{\partial V_2}{\partial t}+g'\frac{\partial V_2}{\partial x}+k'B'\frac{\partial V_2}{\partial x}+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_2}{\partial x^2}\right)$$
$$+tr\left(\sigma W\sigma'\frac{\partial V_1}{\partial x}\left(\frac{\partial V_1}{\partial x}\right)'\right)=0. \qquad (17)$$

Now we use the Lagrange multiplier method. Introduce a time varying Lagrange multiplier, $\gamma_2(t)$, and optimize the HJB equation for the first cumulant plus the Lagrange multiplier times the HJB equation of the second cumulant.

$$0=\min_k\left\{\frac{\partial V_1}{\partial t}+g'\frac{\partial V_1}{\partial x}+k'B'\frac{\partial V_1}{\partial x}\right.$$
$$+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_1}{\partial x^2}\right)+h+k'Rk+\gamma_2(t)$$
$$\times\left[\frac{\partial V_2}{\partial t}+g'\frac{\partial V_2}{\partial x}+k'B'\frac{\partial V_2}{\partial x}+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_2}{\partial x^2}\right)\right.$$
$$\left.\left.+tr\left(\sigma W\sigma'\frac{\partial V_1}{\partial x}\left(\frac{\partial V_1}{\partial x}\right)'\right)\right]\right\}.$$

The minimizing controller is obtained as

$$k^*=-\frac{1}{2}R^{-1}(t,x)B'(t,x)\left[\frac{\partial V_1}{\partial x}+\gamma_2(t)\frac{\partial V_2}{\partial x}\right]. \qquad (18)$$

Second order necessary condition, $R>0$, is satisfied also. Therefore the minimum is guaranteed, and the controller (18) is a candidate for an optimal first and second cost cumulant controller. Now we will find the solutions to the nonlinear first and second cost cumulant optimization problem using the SDRE method.

1) Nonlinear System First and Second Cumulant Minimization: Assume that

$$h(t,x)=x'Q(t,x)x$$

and

$$g(t,x)=A(t,x)x.$$

Now, we find the two partial differential equations from the necessary conditions of optimality, Equations (16) and (17). To find the first partial differential equation, we substitute the optimal controller, Equation (18), and the above assumptions into the Equation (16). Suppressing arguments for simplicity, we have

$$0=\frac{\partial V_1}{\partial t}+x'A'\frac{\partial V_1}{\partial x}-\left(\frac{\partial V_1}{\partial x}\right)'BR^{-1}\frac{1}{2}B'\frac{\partial V_1}{\partial x}+x'Qx$$
$$-\gamma_2\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}\frac{1}{2}B'\frac{\partial V_1}{\partial x}+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_1}{\partial x^2}\right)$$
$$+\frac{1}{4}\left(\frac{\partial V_1}{\partial x}\right)'BR^{-1}B'\frac{\partial V_1}{\partial x}+\frac{\gamma_2}{4}\left(\frac{\partial V_1}{\partial x}\right)'BR^{-1}B'\frac{\partial V_2}{\partial x}$$
$$+\frac{\gamma_2}{4}\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}B'\frac{\partial V_1}{\partial x}+\frac{\gamma_2^2}{4}\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}B'\frac{\partial V_2}{\partial x},$$

which can be simplified to

$$0=\frac{\partial V_1}{\partial t}+x'A'\frac{\partial V_1}{\partial x}-\frac{1}{4}\left(\frac{\partial V_1}{\partial x}\right)'BR^{-1}B'\frac{\partial V_1}{\partial x}$$
$$+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_1}{\partial x^2}\right)+x'Qx$$
$$+\frac{\gamma_2^2}{4}\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}B'\frac{\partial V_2}{\partial x}. \qquad (19)$$

This is one of the necessary partial differential equations for the first two cumulant problem. To find the other equation, we substitute the optimal controller (18) into the Equation (17), which simplifies to

$$0=\frac{\partial V_2}{\partial t}+x'A'\frac{\partial V_2}{\partial x}-\frac{1}{4}\left(\frac{\partial V_1}{\partial x}\right)'BR^{-1}B'\frac{\partial V_2}{\partial x}$$
$$-\frac{1}{4}\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}B'\frac{\partial V_1}{\partial x}+\frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_2}{\partial x^2}\right)$$
$$+\left(\frac{\partial V_1}{\partial x}\right)'\sigma W\sigma'\frac{\partial V_1}{\partial x}+\frac{\gamma_2}{2}\left(\frac{\partial V_2}{\partial x}\right)'BR^{-1}B'\frac{\partial V_2}{\partial x}. \qquad (20)$$

Assume that $V_1(t,x)$ and $V_2(t,x)$ are symmetric nonnegative definite matrices.

$$V_m(t,x)=x'\mathcal{V}_m(t,x)x+v_m(t)$$
$$\frac{\partial V_m}{\partial t}=x'\dot{\mathcal{V}}_m(t,x)x+\dot{v}_m(t)$$
$$\frac{\partial V_m}{\partial x}=2\mathcal{V}_m(t,x)x+vec\left\{x'\frac{\partial\mathcal{V}_m}{\partial x_i}x\right\}$$
$$\frac{\partial^2 V_m}{\partial x^2}=2\mathcal{V}_m(t,x)+2vec\left\{x'\frac{\partial\mathcal{V}_m}{\partial x_i}'\right\}$$
$$+2vec\left\{\frac{\partial\mathcal{V}_m}{\partial x_i}x\right\}+vec\left\{x'\frac{\partial\mathcal{V}_m^2}{\partial x_i^2}x\right\} \qquad (21)$$

where $m=1,2$ and $vec\{z_i\}=[z_1,z_2,\cdots,z_n]'$. The solution must satisfy the following symmetry conditions.

$$\sum_{k=1}^{n}\frac{\partial\mathcal{V}_{m_{ik}}(x)}{\partial x_j}x_k=\sum_{k=1}^{n}\frac{\partial\mathcal{V}_{m_{jk}}(x)}{\partial x_i}x_k$$

for $m=1,2, i=1,2,\cdots,n$ and $j=i+1,\cdots,n$. Now, the SDRE method is utilized. Substitute the above equations into Equation (19) and reduce it to

$$0=x'\dot{\mathcal{V}}_1x+x'Qx+\dot{v}_1-x'\mathcal{V}_1BR^{-1}B'\mathcal{V}_1x$$
$$-\frac{1}{2}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}'BR^{-1}B'\mathcal{V}_1x$$
$$-\frac{1}{2}x'\mathcal{V}_1BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}$$
$$-\frac{1}{4}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}'BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}$$
$$+\gamma_2^2x'\mathcal{V}_2BR^{-1}B'\mathcal{V}_2x$$
$$+\frac{\gamma_2^2}{2}x'\mathcal{V}_2BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}$$

$$
\begin{aligned}
&+\frac{\gamma_2^2}{2}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_2 x\\
&+\frac{\gamma_2^2}{4}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&+tr\left(\sigma W\sigma'\mathcal{V}_1\right)+tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_1{}'}{\partial x_i}\right\}\right)\\
&+tr\left(\sigma W\sigma' vec\left\{\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\right)\\
&+\frac{1}{2}tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_1^2}{\partial x_i^2}x\right\}\right)+x'A'2\mathcal{V}_1 x\\
&+x'A'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}
\end{aligned}
$$

From the above equation, we obtain the following necessary conditions for optimality.

$$
\begin{aligned}
\dot{v}_1 &= -tr(\sigma W\sigma'\mathcal{V}_1),\\
0 &= \dot{\mathcal{V}}_1+Q-\mathcal{V}_1BR^{-1}B'\mathcal{V}_1+\gamma_2^2\mathcal{V}_2BR^{-1}B'\mathcal{V}_2\\
&\quad +A'\mathcal{V}_1+\mathcal{V}_1A, \qquad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
&0=-\frac{1}{2}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_1 x\\
&-\frac{1}{2}x'\mathcal{V}_1BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\\
&-\frac{1}{4}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\\
&+\frac{\gamma_2^2}{2}x'\mathcal{V}_2BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&+\frac{\gamma_2^2}{2}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_2 x\\
&+\frac{\gamma_2^2}{4}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&+tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_1{}'}{\partial x_i}\right\}\right)\\
&+tr\left(\sigma W\sigma' vec\left\{\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\right)\\
&+\frac{1}{2}tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial^2\mathcal{V}_1}{\partial x_i^2}x\right\}\right)\\
&+x'A'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\} \qquad (23)
\end{aligned}
$$

The last equation is similar to Cloutier's necessary condition form optimality in deterministic LQR case [2]. To find the second Riccati-type equation, we let $\mathcal{V}_2(t,x)$ be a symmetric nonnegative definite matrix. Then we substitute Equations (21) with $m=2$ into Equation (20) to obtain the following conditions.

$$
\begin{aligned}
\dot{v}_2 &= -tr(\sigma W\sigma'\mathcal{V}_2)\\
0 &= \dot{\mathcal{V}}_2-\mathcal{V}_1BR^{-1}B'\mathcal{V}_2-\mathcal{V}_2BR^{-1}B'\mathcal{V}_1\\
&\quad -2\gamma_2\mathcal{V}_2BR^{-1}B'\mathcal{V}_2+4\mathcal{V}_1\sigma W\sigma'\mathcal{V}_1+A'\mathcal{V}_2+\mathcal{V}_2A \qquad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
0 \;=\; & x'A'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&-\frac{1}{2}x'\mathcal{V}_1BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&-\frac{1}{2}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_2 x\\
&-\frac{1}{4}vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&-\frac{1}{2}x'\mathcal{V}_2BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\\
&-\frac{1}{2}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_1 x\\
&-\frac{1}{4}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\\
&-\gamma_2 x'\mathcal{V}_2BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&-\gamma_2 vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{V}_2 x\\
&-\frac{\gamma_2}{2}vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' BR^{-1}B'vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\\
&+tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_2{}'}{\partial x_i}\right\}\right)\\
&+tr\left(\sigma W\sigma' vec\left\{\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\right)\\
&+\frac{1}{2}tr\left(\sigma W\sigma' vec\left\{x'\frac{\partial^2\mathcal{V}_2}{\partial x_i^2}x\right\}\right)\\
&+2x'\mathcal{V}_2\sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\\
&+2vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' \sigma W\sigma'\mathcal{V}_1 x\\
&+vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}' \sigma W\sigma' vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}. \qquad (25)
\end{aligned}
$$

Note that the suppressed arguments for $\mathcal{V}_1, \mathcal{V}_2, A, B, Q$, and R are (t,x). Thus, above equations are state dependent.

In summary, the nonlinear, non-quadratic first two cost cumulant optimization problem has the optimal controller given by

$$
\begin{aligned}
k^* \;=\; & -\frac{1}{2}R^{-1}(t,x)B'(t,x)\left[2\mathcal{V}_1x+vec\left\{x'\frac{\partial\mathcal{V}_1}{\partial x_i}x\right\}\right.\\
&\left.+\;\gamma_2(t)\left(2\mathcal{V}_2x+vec\left\{x'\frac{\partial\mathcal{V}_2}{\partial x_i}x\right\}\right)\right],
\end{aligned}
$$

where $\mathcal{V}$'s must satisfy two coupled Riccati-type Equations (22) and (24). Furthermore, two necessary conditions given by Equations (23) and (25) must be satisfied. In general these necessary conditions are difficult to satisfy for a given $A(t,x)$ in multivariable case. Thus, in the next section, we assume a linear system with a quadratic cost function to present the complete solution of the first two cost cumulant optimization problem.

2) First and Second Cost Cumulant Minimization: Linear System: Now we assume a linear system and a quadratic cost function. We verify that this solution is equivalent to the minimal cost variance solution of [12], [7]. Assume that $h(t,x) = x'Q(t)x, g(t,x) = A(t)x$, and $\sigma(t,x) = E(t)$, thus we have

$$\begin{aligned} L(t,x,k(t,x)) &= x'Q(t)x + k'R(t)k, \\ \psi(x(t_F)) &= x'(t_F)Q_F x(t_F), \text{ and} \\ f(t,x,k(t,x)) &= A(t)x + B(t)k(t,x). \end{aligned}$$

Furthermore, we assume quadratic form solutions

$$\begin{aligned} V_1 = x'\mathcal{V}_1 x + v_1 &\qquad V_2 = x'\mathcal{V}_2 x + v_2 \\ \frac{\partial \mathcal{V}_1}{\partial t} = x'\dot{\mathcal{V}}_1 x + \dot{v}_1 &\qquad \frac{\partial \mathcal{V}_2}{\partial t} = x'\dot{\mathcal{V}}_2 x + \dot{v}_2 \\ \frac{\partial \mathcal{V}_1}{\partial x} = 2\mathcal{V}_1 x &\qquad \frac{\partial \mathcal{V}_2}{\partial x} = 2\mathcal{V}_2 x \\ \frac{\partial^2 \mathcal{V}_1}{\partial x^2} = 2\mathcal{V}_1 &\qquad \frac{\partial^2 \mathcal{V}_2}{\partial x^2} = 2\mathcal{V}_2. \end{aligned} \tag{26}$$

Substitute the above equations into Equations (18) to obtain the optimal controller,

$$k = -R^{-1}B'(\mathcal{V}_1 + \gamma_2 \mathcal{V}_2)x.$$

Substituting Assumptions in (26) into Equations (16) and (17) we obtain the following coupled Riccati-type equations.

$$\begin{aligned} 0 &= \dot{\mathcal{V}}_1 + Q - \mathcal{V}_1 BR^{-1}B'\mathcal{V}_1 + \gamma_2^2 \mathcal{V}_2 BR^{-1}B'\mathcal{V}_2 \\ &\quad + A'\mathcal{V}_1 + \mathcal{V}_1 A \\ 0 &= \dot{\mathcal{V}}_2 - \mathcal{V}_1 BR^{-1}B'\mathcal{V}_2 - \mathcal{V}_2 BR^{-1}B'\mathcal{V}_1 \\ &\quad - 2\gamma_2 \mathcal{V}_2 BR^{-1}B'\mathcal{V}_2 + 4\mathcal{V}_1 \sigma W \sigma' \mathcal{V}_1 + A'\mathcal{V}_2 + \mathcal{V}_2 A \end{aligned}$$

with the boundary conditions $\mathcal{V}_1(t_F) = Q_F$ and $\mathcal{V}_2(t_F) = 0$. And the following equations:

$$\begin{aligned} \dot{v}_1 &= -tr(\sigma W \sigma' \mathcal{V}_1) \\ \dot{v}_2 &= -tr(\sigma W \sigma' \mathcal{V}_2). \end{aligned}$$

The suppressed arguments in the above equations are t. The above equations are equivalent to the minimal cost variance solution obtained in [12].

If we were to perform cost moment minimization, we will obtain nonlinear controller even for the linear, quadratic cost case. To see this, consider two partial differential equations for the first two moments from Equation (8):

$$\begin{aligned} 0 &= \frac{\partial M_1}{\partial t} + h + k'Rk + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 M_1}{\partial x^2}\right) \\ &\quad + g'\frac{\partial M_1}{\partial x} + k'B'\frac{\partial M_1}{\partial x} \\ 0 &= \frac{\partial M_2}{\partial t} + 2M_1(h + k'Rk) + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 M_2}{\partial x^2}\right) \\ &\quad + g'\frac{\partial M_2}{\partial x} + k'B'\frac{\partial M_2}{\partial x}. \end{aligned}$$

After multiplying a Lagrange multiplier and taking derivatives with respect to k, we find the optimal first two cumulant controller as

$$k^* = -\frac{1}{2 + 4\gamma M_1} R^{-1}B'\left(\frac{\partial M_1}{\partial x} + \gamma \frac{\partial M_2}{\partial x}\right).$$

If we let $M_i(t,x) = x'\mathcal{M}_i x + m_i$ for $i = 1, 2$, we obtain a nonlinear controller.

VI. Conclusions

This paper presented the necessary condition for n-th cost moment optimization problem in the form of HJB equation. The n-th cost cumulant optimization procedure is also discussed in detail. Second, third, and fourth cost cumulant HJB equations are derived utilizing the proposed procedure. Thus the necessary condition for any cost moment and cumulant can be derived. The solutions for any cost cumulant problem is also investigated in this paper. We proposed to use the Lagrange multiplier method with the HJB equations. Optimizing solutions for the first two cumulants are presented. For a nonlinear system with nonquadratic cost problem, we used the state dependent Riccati equation technique to find an optimal solution.

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