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On the properties of solutions to a differential inclusion associated with a nonsmooth constrained optimization problem

Paolo Nistri and Marc Quincampoix

Abstract— **In this paper we consider, under very general conditions, a constrained minimization problem and we associate to this problem a differential inclusion which has the property that all the trajectories converge to the set C of constrained critical points. The conditions on the functional to be minimized and on the function which defines the constraint are the minimal requirements on these data to use the tools of the nonsmooth analysis to show the convergence of the trajectories to C. Furthermore, if these functions are also subanalytic, then it is proved that any trajectory converges to a critical point and it has finite length. In fact, we show that these assumptions guarantee that the multivalued vector field defining the differential inclusion satisfies a Lojasiewicz-type inequality. The dependence of the rate of convergence on the values of the Lojasiewicz exponent is also shown.**

Notation:

$\mathbf{R}^n$: real $n-$space

$x = (x_1, \cdots, x_n) \in \mathbf{R}^n$: column vector

$\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$: scalar product of $x, y \in \mathbf{R}^n$

$\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$: Euclidean norm of $x \in \mathbf{R}^n$

$B = \{y \in \mathbf{R}^n : \|y\| < 1\}$

$\mathrm{int}(Q)$: interior of set $Q \subset \mathbf{R}^n$

$\mathrm{bd}(Q)$: boundary of Q

$\overline{\mathrm{co}}(Q)$: closure of the convex hull of Q

$\mathrm{d}(x, Q) = \inf_{y \in \mathbf{R}^n} \|x - y\|$: distance of $x \in \mathbf{R}^n$ from Q

$N_Q(x)$: normal cone to Q at x

$Q_1 \setminus Q_2$: difference of sets $Q_1, Q_2 \subset \mathbf{R}^n$

$Q + RB = \{x \in \mathbf{R}^n : x = y + Rz\,; y \in Q, z \in B\}$, $R \in \mathbf{R}$

$[0, \sigma]Q = \{x \in \mathbf{R}^n : x = \rho y\,; y \in Q, \rho \in [0, \sigma]\}$

$x : \mathbf{R} \to \mathbf{R}^n$: single-valued vector function

$\phi : \mathbf{R}^n \to \mathbf{R}$: single-valued scalar function

x' : (conventional) derivative of x

$\nabla \phi$: (conventional) gradient of ϕ

$\partial \phi$: generalized gradient of ϕ

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P. Nistri is with Dipartimento di Ingegneria dell'Informazione, University of Siena, Siena, Italy. pnistri@dii.unisi.it

M. Quincampoix is with Laboratoire de Mathématiques, Unité CNRS UMR 6205, Université de Bretagne Occidentale, 6 avenue Le Gorgeu, 29200 Brest, France. Marc.Quincampoix@univ-brest.fr

I. Introduction

In this paper, we consider the following minimization problem:

$$\min_S \phi, \quad (\mathbf{MP})$$

where $S = \{x \in \mathbf{R}^n : V(x) \leq 0\}$ is a compact set with nonempty interior and $V, \phi : \mathbf{R}^n \to \mathbf{R}$ are locally Lipschitz functions.

To solve this problem we associate to **(MP)** a differential inclusion and we look for conditions ensuring that all its trajectories starting from any point x_0 of a suitable neighborhood of S converge to the set of critical points of the functional ϕ constrained to the set S. Note that for convex data such differential inclusions known as variational inequalities has been considered in Section 3.5 of [1], and [5] (see also [15] for some extensions to nonconvex cases).

Our assumptions on the functions V, ϕ are quite general, indeed they represent the minimal requirements in order to apply our approach which is based on nonsmooth and set-valued analysis. In particular, no convexity assumptions are assumed neither on the cost functional ϕ nor on the constrained set S, that are the usual assumptions to treat constrained minimization problems. Nonsmooth constrained minimization problems with applications to linear and quadratic programming problems and their implementation via neural networks have been considered in [8], and in a more general setting like that of this paper in [13].

If we allow to consider trajectories outside the set S we need a condition which avoids the possibility of the convergence of one of them to a point which does not belong to S. To this aim, we assume throughout the paper the following crucial condition on the function V:

$$\begin{array}{c}\text{there exist } R > 0 \text{ and } m > 0 \text{ such that} \\ \inf_{x \in (S+RB)\setminus S}\ \min_{v \in \partial V(x)} \|v\| \geq m\end{array} \qquad \mathbf{(H1)}$$

where $\partial V(\cdot)$ denotes the generalized gradient of the locally Lipschitz function V. There are several equivalent definitions of the generalized gradient. One of them, which is useful for the practical computation of $\partial V(x)$, is based on the property that a locally Lipschitz function is differentiable almost everywhere in the sense of the Lebesgue measure. Indeed, it can be shown that ([14], Theorem 9.61)

$$\partial V(x) = \overline{\mathrm{co}}\{\lim_{n\to\infty} \nabla V(x_n) : x_n \to x, x_n \notin \mathbf{N}, x_n \notin \Omega_V\},$$

where Ω_V is the set of points in $x + \epsilon B$, $\epsilon > 0$, where V fails to be differentiable and N is an arbitrary set of measure zero.

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We refer the reader to the appendix of [13] which contains a full discussion concerning classes of sets satisfying the above assumptions (H1). In particular, any smooth set, any convex set, any set which is locally the epigraph of a Lipschitz function can be defined by a function V satisfying (H1).

Moreover, we assume the following condition on V:

$$\begin{array}{l}\text{there exist } \eta > 0 \text{ such that} \\ V(x) > \eta \text{ for any } x \notin S + RB\end{array} \qquad \textbf{(H2)}$$

Under these conditions, the next Theorem 1 will guarantee that the differential inclusion

$$x'(t) \in F(x(t)), \text{ for almost all (a.a.) } t \geq 0,$$

where $F : S + RB \to \mathbf{R}^n$ is given by

$$F(x) = \begin{cases} -\sigma\partial V(x) & \text{if } x \notin S, \\ -\partial\phi(x) & \text{if } x \in \text{int}\,(S), \\ -\partial\phi(x) - [0,\sigma]\partial V(x) & \text{if } x \in \text{bd}(S), \end{cases}$$

has, for $\sigma > 0$ sufficiently large, the property that any solution of the Cauchy problem

$$x'(t) \in F(x(t)), \text{ for a.a. } t \geq 0, \qquad (1)$$

$$x(0) = x_0, \qquad (2)$$

whenever $x_0 \in S + rB$, for some $0 < r < R$, converges to the set of constrained critical points of ϕ in S given by

$$C := \{x \in S : 0 \in \partial\phi(x) + N_S(x)\},$$

namely $\lim_{t\to+\infty} d(x(t), C) = 0$. Here $N_S(x)$ is the normal cone to S at x, see ([14], Proposition 6.5).

By a slightly different proof the same result has been obtained by the authors in [13] for a different differential inclusion associated with **(MP)**.

Furthermore, with respect to [13], in the present paper we address the problem of the convergence of any trajectory of (1)-(2) to a critical point $c \in C$ and that of the finite length of the trajectories. In Theorem 2 we provide conditions to solve these problems, such conditions are based on an extension to nonsmooth subanalytic functions ([6], Theorem 3.1) of the following result due to Lojasiewicz (see [9], [10] and [11]):

Let U be a nonempty open subset of $\mathbf{R}^n$ and $f : U \mapsto \mathbf{R}$ a real-analytic function. Let c be a critical point of f. Then there exists $\theta \in [0, 1)$ such that the quantity

$$\frac{|f(x) - f(c)|^\theta}{\|\nabla f(x)\|}$$

remains bounded in a neighborhood of c.

In fact, by using ([6], Theorem 3.1), we will show that if the locally Lipschitz functions ϕ and V are also subanalytic (see [4]) and regular ([14], Definition 7.25), then any trajectory $x(t)$, $t \geq 0$, of the differential inclusion does converge to a constrained critical point of C and it has finite length. Furthermore, Theorem 3 provides the rate of convergence of $x(t), t \geq 0$, to c according to the values of the Lojasiewicz exponent θ in $[0, 1]$.

These results has been also obtained in [6] for an unconstrained convex minimization problem.

II. Main Results

A. Convergence of trajectories for locally Lipschitz data

We can now prove the following

Theorem 1: Assume **(H1)-(H2)**, then there exist $\sigma > 0$ and $r > 0$ such that

(i) the set-valued map $x \mapsto F(x)$ in (1) is upper semicontinuous with nonempty, convex, compact values;

(ii) S is invariant under the dynamics of system (1),

(iii) for any $x_0 \in S + rB$, any solution $x(t)$, $t \geq 0$, of the Cauchy problem (1)-(2) reaches S in finite time;

(iv) $x(t)$, reaches C either in finite time or asymptotically i.e $\lim_{t\to+\infty} d(x(t), C) = 0$;
(v) $\liminf_{t\to+\infty} d(x(t), C) = 0$

Proof. Taking into account that the generalized gradient of a locally Lipschitz function is an upper semicontinuous set-valued map with convex, compact, nonempty values [14] property (i) follows immediately. Since, $x \mapsto \partial\phi(x)$ and $x \mapsto \partial V(x)$ are bounded on the boundary $\text{bd}(S)$ of the compact set S, then it is possible to choose $\sigma > 0$ large enough to have that the two sets $\{x \in \text{bd}(S) : 0 \in \partial\phi(x) + \mathbf{R}_+\partial V(x)\}$ and $\{x \in \text{bd}(S) : 0 \in \partial\phi(x) + [0,\sigma]\partial V(x)\}$ coincide. In fact, let $0 \in \partial\phi(\hat{x}) + \mathbf{R}_+\partial V(\hat{x}), \hat{x} \in \text{bd}(S)$, thus there exists $v \in \partial V(\hat{x})$ and $\rho \geq 0$ such that $\rho v \in -\partial\phi(\hat{x})$. Let

$$M_\phi := \max_{x\in\text{bd}(S)}\{\max_{u\in\partial\phi(x)} \|u\|\} < \infty$$

and

$$M_V := \max_{x\in\text{bd}(S)}\{\max_{v\in\partial V(x)} \|v\|\} < \infty.$$

It is easy to verify that for $\sigma > \dfrac{M_\phi}{M_V}, M_V > 0$, it turns out that $0 \in \partial\phi(\hat{x}) + [0,\sigma]\partial V(\hat{x})$. Observe that if $M_V = 0$ the assertion follows immediately. Moreover, by (H1) we have that $0 \notin F(x)$ for any $x \in (S + RB) \setminus S$, and by the definition of F it follows that, for any $x \in \text{int}(S)$, $0 \in F(x)$ is equivalent to $0 \in \partial\phi(x)$. Therefore, if $\sigma > \dfrac{M_\phi}{M_V}$ then $C = \{x \in S + RB : 0 \in F(x)\}$. In other words, the set of constrained critical points of ϕ in S coincides, when $\sigma > \dfrac{M_\phi}{M_V}$, with the set of unconstrained critical points of F in $S + RB$.

In virtue of (H2) it is easily seen that, for any $0 < \alpha < \eta$, we have

$$S \subset \{x \in \mathbf{R}^n : V(x) < \alpha\} \subset S + RB.$$

Let $r > 0$ be such that

$$S + rB \subset \{x \in \mathbf{R}^n : V(x) < \alpha\}.$$

Choose $x_0 \in (S + rB)\backslash S$ and a corresponding solution $x(t)$, $t \geq 0$, to (1)-(2). Let $V_\sigma(x) := \sigma V(x)$ and consider

$$\frac{d}{dt}\, V_\sigma(x(t)) \leq \max_{\zeta\in\partial V_\sigma(x(t))} \langle\zeta, x'(t)\rangle \leq -d^2(0, \partial V_\sigma(x(t))$$

for almost all $t \le \Theta_S(x(\cdot)) := \inf\{t > 0 : x(t) \in S\}$. By (H1), we obtain

$$\frac{d}{dt} V_\sigma(x(t)) \le -m^2\sigma^2, \tag{3}$$

for almost all $t \in [0, \Theta_S(x(\cdot))]$, thus by integrating (3) on $[0,t]$, $t \le \Theta_S(x(\cdot))$, we get

$$V_\sigma(x(t)) \le V_\sigma(x_0) - m^2\sigma^2 t \le \alpha\sigma - m^2\sigma^2 t.$$

Therefore $x(t)$ reaches S before $\dfrac{\alpha}{m^2\sigma}$ and so (iii) is also proved.

Moreover, for any $\epsilon \in (0, \alpha)$ by (H1) we have that

$$\max_{\zeta \in \partial V_\sigma(x)} \langle p, \zeta \rangle \ < -m^2\sigma^2,$$

for any $x \in \mathrm{bd}(S+\epsilon B)$ and for any $p \in -\partial V_\sigma(x)$. Thus, one can deduce that $S + \epsilon B$ is invariant under the dynamics (1), (cf. [12], [13]). Finally $S = \bigcap_{\epsilon>0}(S + \epsilon B)$ is also invariant by ([2], Theorem 5.4.6), and so (ii) is proved.

Consider now $x_0 \in S \backslash C$ and let $x(t), t \ge 0$, be a corresponding solution to (1)-(2). From (ii), we know that $x(t)$ remains in S for any $t \ge 0$. By our choice of $\sigma > 0$ in Theorem 1 we have that

$$C = \{x \in S : 0 \in \partial W(x)\},$$

where

$$W(x) := \phi(x) + \max\{0, V_\sigma(x)\},\ x \in \mathbf{R}^n. \tag{4}$$

That is C is also the set of critical points of W in S. Moreover, for almost all $t \le \Theta_C(x(\cdot)) := \inf\{t > 0 : x(t) \in C\}$, it results that

$$\frac{d}{dt} W(x(t)) \le -d^2(0, \partial W(x(t))) < 0. \tag{5}$$

Therefore $t \mapsto W(x(t))$ is strictly decreasing on the time interval $[0, \Theta_C(x(\cdot))]$.

We claim that for any $\varepsilon > 0$ the trajectory $x(t)$, $t \ge 0$, reaches the set

$$A_\varepsilon := \{x \in S : \min_{\zeta \in \partial W(x)} \|\zeta\| \le \varepsilon\}$$

in a finite time $\Theta_{A_\varepsilon}(x(\cdot))$. In fact, by (5) we obtain

$$\frac{d}{dt} W(x(t)) \le -\varepsilon^2, \ \text{ for almost all } \ t \le \Theta_{A_\varepsilon}(x(\cdot)).$$

Thus, integrating on $[0,t]$, $t \le \Theta_{A_\varepsilon}(x(\cdot))$, we get

$$W(x(t)) \le W(x_0) - \varepsilon^2 t.$$

From this we conclude that $\Theta_{A_\varepsilon}(x(\cdot))$ is finite; in fact if $\Theta_{A_\varepsilon}(x(\cdot)) = +\infty$ then $W(x_0) - \varepsilon^2 t \to -\infty$ as $t \to +\infty$ contradicting the fact that $x \mapsto W(x)$ assumes the minimum on the compact set S. Since $\varepsilon > 0$ is arbitrary we conclude that either there exists τ_C such that $x(\tau_C) \in C$ or

$$\lim_{t\to+\infty} \min_{\zeta \in \partial W(x(t))} \|\zeta\| = 0$$

namely $\lim_{t\to+\infty} d(x(t), C) = 0$. which is our claim (iv).

Finally, we have also that $\liminf_{t\to+\infty} d(x(t), C) = 0$. In fact, arguing by contradiction we have the existence of an $\varepsilon_0 > 0$ such that $x(t)$ does not reach the set A_{ε_0}. This contradiction proves (v). □

B. Convergence of trajectories for subanalytic and regular data

Let us recall that a function is said subanalytic if its graph is a subanalytic set, namely if it is represented by a projection of a finite number of intersections and unions of sets defined by finitely many analytic equalities or inequalities [4]. Moreover, for a locally Lipschitz function the notion of regularity can be found in ([14], Def. 7.25).

We are now in a position to prove our second main result.

Theorem 2: Assume **(H1)-(H2)** and let $\sigma, r > 0$ be the constants of Theorem 1. If V and ϕ are subanalytic and regular, then for any $x_0 \in S + rB$, every solution $x(t)$, $t \ge 0$, of the Cauchy problem (1)-(2) converges to a critical point of **(MP)** and it has finite length.

Proof. Observe that the function W defined in (4) is subanalytic, indeed the set of subanalytic functions from $\mathbf{R}^n \to \mathbf{R}$ is a ring with respect to the pointwise sum and product of functions and $x \to \max\{0, V_\sigma\}$ is a subanalytic function since $x \to V_\sigma(x)$ is subanalytic.

Let $x_0 \in S + rB$ be fixed and consider any solution $x(t)$, $t \ge 0$, to (1)-(2). From Theorem 1, we know that $x(t)$ reaches S in finite time and converges to C in finite or infinite time. Suppose that $\lim_{t\to+\infty} d(x(t), C) = 0$.

We now state the following Lemma, whose proof follows the lines of ([6], Theorem 4.5) where the involved functions are convex.

Lemma 1: Under the assumptions of Theorem 2, the trajectory $x(t)$, $t \ge 0$, converges to a critical point of W.

Proof. Because S is compact, by (iv) of Theorem 1, there exists a sequence $t_n \to +\infty$ and a critical point $c \in C$ with

$$\lim_{n\to\infty} x(t_n) = c.$$

Since W is subanalytic then W is constant on each connected component of C (see [7], Theorem 13). Observe that without loss of generality we can assume that $W(c) = 0$. We have that $W(x(t)) \to 0$ as $t \to +\infty$ since $d(x(t), C) \to 0$ as $t \to +\infty$ and W is constant on the connected component of C containing c. Therefore, since $t \to W(x(t))$ is a decreasing function, $W(x(t))$ is nonnegative for t sufficiently large. Applying the Lojasiewicz inequality for subanalytic functions ([6], Theorem 3.1), one concludes that there exists $\theta \in [0,1)$, $\lambda > 0$ and a neighborhood I of $c \in C$ of radius $\epsilon > 0$ given by $I := \{y \in \mathbf{R}^n : \|y - c\| < \epsilon\}$ such that

$$\frac{|W(x)|^\theta}{\min_{\zeta \in \partial W(x)} \|\zeta\|} < \lambda, \ \text{ for any } x \in I \backslash \{c\}. \tag{6}$$

Therefore, by Theorem 1, there exists $\tau > 0$ such that $x(t) \in S$ for any $t \ge \tau$ and

$$\begin{cases} 0 < \dfrac{(W(x(\tau)))^{1-\theta} - (W(x(t)))^{1-\theta}}{1-\theta} < \dfrac{\varepsilon}{3\lambda} \ \text{ for any } t > \tau \\ \|x(\tau) - c\| < \frac{1}{3}\varepsilon \end{cases} \tag{7}$$

For almost all $t \geq \tau$ we have

$$\frac{d}{dt}(W(x(t)))^{1-\theta} = (1-\theta)(W(x(t)))^{-\theta}\frac{d}{dt}(W(x(t)))$$
$$= -(1-\theta)(W(x(t)))^{-\theta}\|x'(t)\|^2 =$$
$$= -(1-\theta)\frac{\|x'(t)\|}{(W(x(t)))^{\theta}}\|x'(t)\|,$$

where the second equality is due to the fact that $W(x)$ is regular at any $x \in \mathbf{R}^n$ and $x(t)$ is absolutely continuous on any compact interval of $[0, +\infty)$, and thus, for almost all $t \geq 0$, the following chain rule holds true: $d/dt\, W(x(t)) = \langle \zeta, x'(t) \rangle$ for any $\zeta \in \partial W(x(t))$, see e.g. ([3], Lemma 1). In view of (6), this yields for almost all $t \in [\tau, \Theta_{\mathrm{bd}(I)}(x(\cdot))]$, where $\Theta_{\mathrm{bd}(I)}(x(\cdot)) := \inf\{t > \tau : x(t) \in \mathrm{bd}(I)\}$.

$$\frac{d}{dt}(W(x(t)))^{1-\theta} \leq -\frac{1-\theta}{\lambda}\|x'(t)\|. \tag{8}$$

By integrating (8) on the interval $[\tau, t]$ we obtain for any $t \in [\tau, \Theta_{\mathrm{bd}(I)}(x(\cdot))]$

$$\int_{\tau}^{t} \|x'(s)\|ds \leq -\frac{\lambda}{1-\theta}\int_{\tau}^{t}\frac{d}{dt}(W(x(s)))^{1-\theta}ds.$$

Hence

$$\int_{\tau}^{t} \|x'(s)\|ds \leq -\lambda\frac{(W(x(t)))^{1-\theta} - (W(x(\tau)))^{1-\theta}}{(1-\theta)}. \tag{9}$$

Thus using (7), one has

$$\int_{\tau}^{t} \|x'(s)\|ds < \frac{1}{3}\varepsilon, \tag{10}$$

and by (7) and (10), for any $t \in [\tau, \Theta_{\mathrm{bd}(I)}(x(\cdot))]$, we obtain

$$\|x(t) - c\| \leq \|x(\tau) - c\| + \int_{\tau}^{t}\|x'(s)\|ds \leq \frac{2}{3}\varepsilon.$$

Hence $x(t) \in I$ for any $t \geq \tau$, moreover we have that $\lim_{t\to+\infty}\|x(t) - c\| = 0$. □

To end the proof of Theorem 2 it is sufficient to observe that, in virtue of Lemma 1, inequality (10) holds for any $t \geq \tau$ and passing to the limit in (10) as $t \to \infty$ we obtain

$$\int_{\tau}^{\infty} \|x'(s)\|ds < \frac{1}{3}\varepsilon, \tag{11}$$

thus the trajectory $x(t), t \geq 0$, has finite length . □

A more precise description of the behavior of the solution to (1)-(2) can be provided in the case when the Lojasiewicz exponent θ has prescribed values in the interval $[0, 1)$. Specifically, we can prove the following result.

Theorem 3: Assume all the conditions of Theorem 2. Let $x(t), t \geq 0$, be a solution of (1)-(2) such that $\lim_{t\to+\infty} x(t) = c \in C$ and let θ be the Lojasiewicz exponent of W in a neighborhood of c. We have the following

(1) if $\theta \in (\frac{1}{2}, 1)$ then we have

$$\|x(t) - c\| \leq \frac{1}{(K_1 + K_2 t)^{\frac{1-\theta}{2\theta-1}}}, \quad t \geq 0;$$

for some $K_1, K_2 > 0$.

(2) if $\theta = \frac{1}{2}$ then we have

$$\|x(t) - c\| \leq K\, e^{-\alpha\, t}, \quad t \geq 0;$$

for some $K > 0$ and $\alpha > 0$.

(3) if $\theta \in (0, \frac{1}{2})$ then we have

$$x(t) = c, \quad \text{for any } t \geq t_c;$$

for some $t_c > 0$.

Proof. Letting $t \to +\infty$ in (9), using (6), we obtain

$$\int_{\tau}^{\infty} \|x'(s)\|ds \leq \frac{\lambda}{1-\theta}|W(x(\tau))|^{1-\theta}$$
$$\leq \frac{\lambda^{\frac{1}{\theta}}}{1-\theta}\|x'(\tau)\|^{\frac{1-\theta}{\theta}}.$$

Put $z(t) = \int_t^{\infty}\|x'(s)\|ds$, thus $z'(t) = -\|x'(t)\|$ and for $t \geq 0$ we have

$$0 < z(t) \leq \frac{\lambda^{\frac{1}{\theta}}}{1-\theta}\|x'(t)\|^{\frac{1-\theta}{\theta}}$$

or equivalently

$$z'(t) \leq -\alpha[z(t)]^{\frac{\theta}{1-\theta}}. \tag{12}$$

where $\alpha = \left[\frac{1-\theta}{\lambda^{\frac{1}{\theta}}}\right]^{\frac{\theta}{1-\theta}}$. Let $\theta \in (\frac{1}{2}, 1)$ and solve the differential equation

$$y'(t) = -\alpha[y(t)]^{\frac{\theta}{1-\theta}}, \quad t > 0 \tag{13}$$

with the initial condition $y(0) = z(0) = K > 0$. We obtain that

$$y^{\gamma}(t) = K_1 + K_2 t$$

where $\gamma = \frac{1-2\theta}{1-\theta} < 0, K_1 = K^{\gamma}, K_2 = -\alpha\gamma > 0$ and so

$$y(t) = \frac{1}{(K_1 + K_2 t)^{\frac{1-\theta}{2\theta-1}}}$$

For $\theta = \frac{1}{2}$ we obtain $y(t) = K\, e^{-\alpha\, t}, t \geq 0$. Finally, for $\theta \in (0, \frac{1}{2})$ we get

$$y^{\gamma}(t) = K^{\gamma} - \alpha\gamma t$$

where $\gamma = \frac{1-2\theta}{1-\theta} > 0$, and $-\alpha\gamma < 0$. Therefore $y^{\gamma}(t) = 0$ for $t \geq \frac{K^{\gamma}}{\alpha\gamma} := t_c$.

The conclusion follows from

$$\|x(t) - c\| \leq \int_{t}^{\infty}\|x'(s)\|ds = z(t), \quad t \geq 0,$$

and the fact that $z(t) \leq y(t)$. Indeed if $z(\hat{t}) = y(\hat{t})$ for some $\hat{t}$ then from (12)-(13) we have that $z'(\hat{t}) \leq y'(\hat{t})$. □

Remark 1: Since by Theorem 1 the set S is invariant with respect to the dynamics of (1) and $C = \{x \in S : 0 \in \partial W(x)\}$, one has only to compute the Lojasiewicz exponent on S, namely only for the function $W \cdot \psi_S$ (here ψ_S denotes the characteristic function of S).

Remark 2: Observe that in the case when the critical points of **(MP)** are isolated, Theorem 1 enables us to conclude that any trajectory of (1)-(2) converges to a critical point. Moreover, under this condition, a slight modification of the proof of Lemma 1, allows us to have the same conclusions of Theorem 2 without assuming that V and ϕ are regular functions.

III. EXAMPLES

The explicit computation of the Lojasiewicz exponent θ is not, in general, an easy problem (even for polynomial functions). However, as the following two dimensional simple examples will show, in some specific cases this is possible. These examples will also show that the Lojasiewicz exponent is strongly related to the constraint set S. Indeed, the same functional to be minimized under different constraints may have different exponents. Finally, it is easy to see that in the proposed examples all the assumptions of Theorem 2 are satisfied.

Example 1. We consider the constraint set $S = [-2, -1] \times [-1, 1]$, the function V given by $V(x, y) := d((x, y), S)$ and the cost functional $\phi(x, y) = x^2 + y^2$. As it is easily seen the set of critical point reduces to the single point $c = (-1, 0)$ (cf. Fig.1). The convergence of trajectory of system (1) to c is exponential. In fact one can easily check that the Lojasiewicz exponent of $W \cdot \psi_S$ is $\theta = \frac{1}{2}$ in a neighborhood of c relative to S.

Example 2. We consider the same $\phi(x, y) = x^2 + y^2$, while the constrained set is given by

$$S = \{(x, y) : |x + 2| + |y| \leq 1\},$$

and the function $V(x, y) := d((x, y), S)$. Once again the set of critical point reduces to the point $c = (-1, 0)$ (cf. Fig.2). However, the convergence of trajectories to (1) to c is in finite time. In fact one can easily check that the Lojasiewicz exponent of $W \cdot \psi_S$ in a neighborhood of c relative to S can be any $\theta < \frac{1}{2}$.

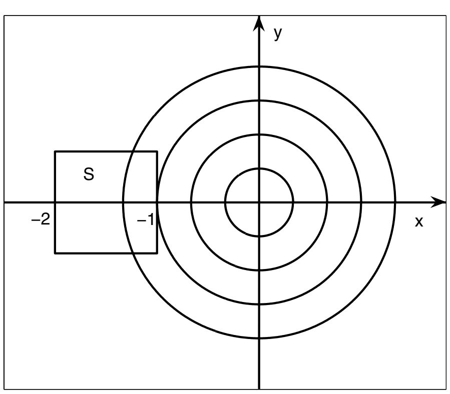

Fig. 1. Example 1: the set S and level sets of ϕ.

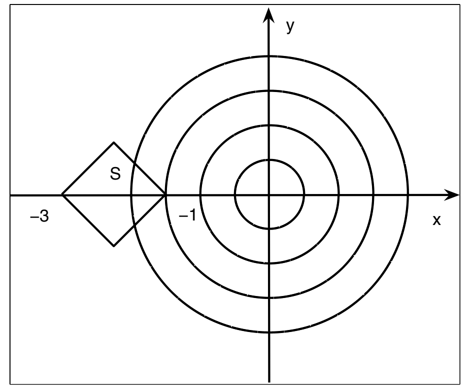

Fig. 2. Example 2: the set S and level sets of ϕ.

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