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Robustly stabilizing MPC for perturbed PWL systems

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***Abstract*— This paper deals with robustly stable model predictive control (MPC) of the class of piecewise linear systems. A piecewise linear feedback controller, that stabilizes the nominal system, is derived from linear matrix inequalities. Further, an algorithm is designed for constructing a polyhedral robustly positively invariant set for the system. First, a min-max feedback MPC scheme with known mode, based on a dual-mode approach that stabilizes the system, is presented. The second robustly stable MPC scheme is based on a semi-feedback controller, but this time the mode of the system is unknown.**

I. INTRODUCTION

A. Overview

In the area of hybrid systems, model predictive control (MPC) has recently attracted much interest due to its ability to handle systems with hard input-state constraints. Research has been focused on developing stabilizing MPC for hybrid systems and in particular for piecewise linear (PWL) and piecewise affine (PWA) systems: [1]–[5]. Since disturbances are always present, it is important that the MPC controller is robust. To guarantee constraint fulfillment for every possible disturbance realization within a certain set, the control action has to be chosen safe enough to cope with the effect of the worst disturbance realization. Because of this rigorous min-max approach, the control scheme for the class of perturbed PWA systems is computationally demanding (as it is a dynamic programming problem [1] or requires recasting the problem into a canonical form [6]).

In this paper we consider the class of PWL systems with additive disturbance. In Section II we derive a local controller for the nominal system in terms of linear matrix inequalities (LMI). In Section III we construct a convex robustly positively invariant set for the system. We propose two MPC algorithms for stabilizing a perturbed PWL system. Under the assumption that the mode is known, we derive a stable min-max feedback MPC scheme based on a dual-mode approach. The second MPC scheme assumes unknown mode, using a semi-feedback controller. From a computational point of view, the second scheme is less demanding (quadratic programming) than the first scheme (mixed-integer linear programming).

B. Notations and definitions

A PWA system with additive disturbance is defined as

$$x(k+1) = A_i x(k) + B_i u(k) + a_i + w(k), \text{ if } x(k) \in \mathcal{P}_i \quad (1)$$

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where x, u and w denote, respectively, the state, input and disturbance; $\{\mathcal{P}_i\}_{i\in\mathcal{I}}$ is a finite partition of $\mathbb{R}^n$. The closure $\mathrm{cl}(\mathcal{P}_i)$ is given by $\mathrm{cl}(\mathcal{P}_i) = \{x : E_i x \geq e_i\}$. When $a_i = 0$, $e_i = 0$, $\forall\, i \in \mathcal{I}$, we get a PWL system:

$$x(k+1) = A_i x(k) + B_i u(k) + w(k), \text{ if } x(k) \in \mathcal{P}_i \quad (2)$$

It is assumed that the disturbance belongs to a bounded polyhedron $w \in W$ and the control and state are required to satisfy the constraints $u \in U_\mathrm{c}$ and $x \in X_\mathrm{c}$; $X_\mathrm{c}, U_\mathrm{c}$ and W are all polytopes, with $0 \in U_\mathrm{c}, W$ and $0 \in \mathrm{int}(X_\mathrm{c})$.

Given two sets $Y, Z \subset \mathbb{R}^n$, the Minkowski sum of Y and Z is defined as: $Y \oplus Z = \{y + z : y \in Y, z \in Z\}$ and the Pontryagin difference as $Y \ominus Z = Y \oplus (-Z) = \{y \in \mathbb{R}^n : y \oplus Z \subseteq Y\}$. We denote with $M^\perp$ the orthogonal complement of a matrix M. We have then $M^T M^\perp = 0$ and $[M\ M^\perp]$ is nonsingular.

II. STABILIZING FEEDBACK CONTROLLER FOR THE NOMINAL PWL SYSTEM

In this section we design a local stabilizing feedback controller for the nominal PWL system. We discuss all the solutions of the matrix inequalities that appear by applying different levels of conservatism with the S-procedure. The nominal system associated to (2) is defined as:

$$x(k+1) = A_i x(k) + B_i u(k), \text{ if } x(k) \in \mathcal{P}_i \quad (3)$$

Now we determine a PWL state feedback controller $u(k) = F_i x(k)$, if $x(k) \in \mathcal{P}_i$ such that the nominal system (3) in closed-loop with this controller is stable. Such a controlleris derived from Lyapunov arguments. We search for a piecewise quadratic Lyapunov function [4] $V(x) = x^T P_i x$, if $x \in \mathcal{P}_i$, such that the following relations are satisfied:

$$\begin{cases} x^T(A_i + B_i F_i)^T P_j (A_i + B_i F_i)x - x^T P_i x < 0, \\ x^T P_i x > 0 \text{ for all } x \in \mathcal{P}_i \text{ and for all } (i,j) \in \mathcal{I}, \end{cases} \quad (4)$$

where we have considered[1] that $x \in \mathcal{P}_i$ and $(A_i + B_i F_i)x \in \mathcal{P}_j$. Since (4) has to be valid only for $x \in \mathcal{P}_i$, we can use the *S-procedure* [8] in order to reduce conservatism. One method to relax the matrix inequalities (4) is: find F_i, P_i, U_{ij}, V_i, for $(i,j) \in \mathcal{I}$, where U_{ij}, V_i have all entries non-negative that satisfy the following matrix inequalities:

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i)^T - P_i + E_i^T U_{ij} E_i < 0 \quad (5)$$

$$P_i > E_i^T V_i E_i \quad (6)$$

[1]For simplicity we assume that from a certain mode $i \in \mathcal{I}$ all the transitions to any other mode are possible. The case in which only some transitions are possible can be implemented straightforwardly [7].

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The symbol * is used to induce a symmetric structure in an LMI. We have the following solution for (5)–(6):

Theorem 2.1: The matrix inequalities (5)–(6) have a solution iff the following matrix inequalities have a solution

$$\begin{bmatrix} B_i^T P_j B_i & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i \end{bmatrix} < \begin{bmatrix} I & -F_i \\ * & F_i^T F_i \end{bmatrix} \quad (7)$$

$$P_i > E_i^T V_i E_i \quad (8)$$

where U_{ij}, V_i have all entries non-negative, $\forall (i,j) \in \mathcal{I}$.

Proof: It is easy to see that (5) can be written as

$$\begin{bmatrix} F_i \\ I \end{bmatrix}^T \begin{bmatrix} B_i^T P_j B_i & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i \end{bmatrix} \begin{bmatrix} F_i \\ I \end{bmatrix} < 0$$

We have $\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp = \begin{bmatrix} F_i \\ I \end{bmatrix}$ since $\begin{bmatrix} F_i \\ I \end{bmatrix}$ is a basis of ker($\begin{bmatrix} -I \\ F_i^T \end{bmatrix}$) (where ker($A$) denotes the kernel of the matrix A). Therefore, the previous formula can be written as

$$\begin{bmatrix} -I \\ F_i^T \end{bmatrix}^{\perp T} Q_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix}^\perp < 0 \quad (9)$$

where $Q_{ij} = \begin{bmatrix} B_i^T P_j B_i & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + E_i^T U_{ij} E_i \end{bmatrix}$.

Using now the Finsler's lemma [8], (9) is equivalent to

$$Q_{ij} < \sigma_{ij} \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \quad (10)$$

with $\sigma_{ij} \in \mathbb{R}$. Of course (10) has a solution if and only if

$$Q_{ij} < \sigma \begin{bmatrix} -I \\ F_i^T \end{bmatrix} \begin{bmatrix} -I & F_i \end{bmatrix} \quad (11)$$

has a solution, with $\sigma > 0$ (Take $\sigma > \max_{i,j}\{0, \sigma_{ij}\}$ for the implication "(10) $\Rightarrow$ (11)"; the other implication is obvious). Now if we divide (11) by $\sigma > 0$ and denote with $P_i \to 1/\sigma P_i$, $U_{ij} \to 1/\sigma U_{ij}$, $V_i \to 1/\sigma V_i$ we obtain that (11) is equivalent to (7). $\diamondsuit$

Now we discuss some possible relaxations for (5)–(6). The first relaxation is to replace (6) with $P_i > 0$.

Proposition 2.2: For $P_i > 0$, the matrix inequalities (5) are equivalent to

$$\begin{bmatrix} P_i - E_i^T U_{ij} E_i & * \\ A_i + B_i F_i & S_j \end{bmatrix} > 0 \quad (12)$$

$$0 < P_j \leq S_j^{-1}, \text{ for all } (i,j) \in \mathcal{I}. \quad (13)$$

Proof: With the relaxation $P_i > 0$, (5) is equivalent with

$$(A_i + B_i F_i)^T S_j^{-1} (A_i + B_i F_i) - P_i + E_i^T U_{ij} E_i < 0, \quad (14)$$

$$0 < P_j \leq S_j^{-1}, \text{ for all } (i,j) \in \mathcal{I}. \quad (15)$$

It is clear that if (5) has a solution, then there exists an $\epsilon > 0$ such that $(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i + E_i^T U_{ij} E_i < -\epsilon (A_i + B_i F_i)^T (A_i + B_i F_i)$. Then, we can take $S_j^{-1} = P_j + \epsilon I \geq P_j$ and thus we obtain (5)–(6). The other implication is obvious. Applying the well-known Schur complement to (12) we obtain the equivalent formulation (14). $\diamondsuit$

An algorithm for finding a solution for (12)–(13) is given in [7].

Now we discuss a second relaxation. If we do not apply the S-procedure condition "$x \in \mathcal{P}_i$", with the more conservative one "$x \in \mathbb{R}^n$", then (4) becomes:

$$(A_i + B_i F_i)^T P_j (A_i + B_i F_i) - P_i < 0, \; P_i > 0 \quad (16)$$

for all $(i,j) \in \mathcal{I}$. There are two methods to linearize (16). One is based on the well-known linearizing change of variable $S_i = P_i^{-1}, Y_i = F_i S_i$ (this type of linearization was used also in [2], [4]). Another linearization of (16) is $S_i = P_i^{-1}, Y_i = F_i G_i$ [9].

Proposition 2.3: The following LMIs in Y_i, S_i, G_i

$$\begin{bmatrix} G_i + G_i^T - S_i & * \\ A_i G_i + B_i Y_i & S_j \end{bmatrix} > 0 \quad (17)$$

for all $(i,j) \in \mathcal{I}$ have a solution if and only if $F_i = Y_i G_i^{-1}, P_i = S_i^{-1}$ are solutions of (16). $\diamondsuit$

The proof is straightforward, using the Schur complement (see [7] for details). If we can find P_i, F_i, using one of the approaches proposed before (Theorem 2.1 or Propositions 2.2 or 2.3), then the feedback controller $u(k) = F_i x(k)$ if $x(k) \in \mathcal{P}_i$ asymptotically stabilizes the origin of (3).

III. Convex robustly positively invariant set

In the sequel we assume that we have determined a state feedback controller $u(k) = F_i x(k)$ if $x(k) \in \mathcal{P}_i$ that stabilizes the nominal system (3) (cf. Section II). We define $A_{F_i} = A_i + B_i F_i$. Then the PWL system with additive disturbance (2) becomes:

$$x(k+1) = A_{F_i} x(k) + w(k), \text{ if } x(k) \in \mathcal{P}_i. \quad (18)$$

We define the following set:

$$X_F = \cup_{i \in \mathcal{I}} \{x \in \mathcal{P}_i : x \in X_c, F_i x \in U_c\}$$

Definition 3.1 ([10]): A set $\Omega \subseteq X_F$ is a *robustly positively invariant* (RPI) set for system (18) if for any $x \in \Omega \cap \mathcal{P}_i$ with $i \in \mathcal{I}$, we have $A_{F_i} x + w \in \Omega$ for all $w \in W$. The maximal (minimal) RPI set is defined as the largest (smallest) with respect to inclusion, RPI set for (18). $\diamondsuit$

It can be easily seen that both the minimal and the maximal RPI set associated to system (18) are in general non-convex sets. For system (18) the evolution of the mode $i = i(k)$ depends on the state $x(k)$. Nevertheless, for ease of computation of a convex (polyhedral) RPI set for (18), we will disregard this relation mode-state and we will consider that $i(k)$ evolves independently of $x(k)$ (i.e. $i(k+1) \in \mathcal{I}$ for all $k \geq 0$). This type of relaxation was used also in [2], [11] in order to obtain a convex invariant set for deterministic PWL systems. So, we replace the PWL system (18) with the following time-varying system

$$x_{k+1} = A_{F_{i(k)}} x_k + w_k, \; i(k+1) \in \mathcal{I} \quad (19)$$

where $i(\cdot)$ is a switching signal in $\mathcal{I}^{\mathbb{N}}$.

Definition 3.2: A set Ω is an RPI set for system (19) if for any $x \in \Omega$ we have that $A_{F_i}x + w \in \Omega$, for any possible switching $i \in \mathcal{I}$ and any admissible disturbance $w \in W$. $\diamondsuit$

In the sequel we construct an RPI set for system (19). We define the following set recursion:

$$\begin{aligned}
&\mathcal{O}_0^i = X_0^i = \{x : x \in X_c, F_i x \in U_c\},\\
&\mathcal{O}_t^i = \{x \in X_{F_i} : A_{F_i}x \oplus W \subseteq \cap_{j\in\mathcal{I}}\mathcal{O}_{t-1}^j\}
\end{aligned} \tag{20}$$

for any $i \in \mathcal{I}$ and $t = 1, 2, \ldots$

It is clear from (20) that $\mathcal{O}_{t+1}^i \subseteq \mathcal{O}_t^i$, and therefore $\mathcal{O}_t^i$ converges to $\mathcal{O}_\infty^i$. We define:

$$\mathcal{O}_\infty^i = \lim_{t\to\infty} \mathcal{O}_t^i = \cap_{t\geq 0}\mathcal{O}_t^i, \quad \mathcal{O}_\infty = \cap_{i\in\mathcal{I}}\mathcal{O}_\infty^i. \tag{21}$$

The properties of $\mathcal{O}_\infty$ are given in the following theorem:

Theorem 3.3: (i) The maximal RPI set included in $\cap_{i\in\mathcal{I}}X_{F_i}$ for the system (19) is the *convex* set $\mathcal{O}_\infty$.

(ii) The set $\mathcal{O}_\infty$ is an RPI set for the PWL system (18).

Proof: (i) It is easy to observe that since the sets X, U, and W are polytopes (described by a finite number of linear inequalities), all the sets $\mathcal{O}_t^i$ are also polytopes and thus convex. So, $\mathcal{O}_\infty^i$ is also convex. Since $\mathcal{O}_\infty$ is the intersection of convex sets, $\mathcal{O}_\infty$ is convex.

For any $x \in \mathcal{O}_\infty$ we have $x \in \mathcal{O}_{t+1}^i$ for all $i \in \mathcal{I}$ and $t \geq 0$. According to (20), $A_{F_i}x \oplus W \subseteq \cap_{j\in\mathcal{I}}\mathcal{O}_t^j$ for all $i \in \mathcal{I}$ and $t \geq 0$. Hence $A_{F_i}x \oplus W \subseteq \mathcal{O}_\infty$ for all $i \in \mathcal{I}$. Therefore, $\mathcal{O}_\infty$ is an RPI set for system (19).

Due to the recursion (20), $\mathcal{O}_\infty$ is the maximal RPI set for system (19) included in $\cap_{i\in\mathcal{I}}X_{F_i}$. Indeed, let $T \subseteq \cap_{i\in\mathcal{I}}X_{F_i}$ be an RPI set for the system (19) and let $x \in T$. Then from the definition of an RPI set for system (19), it follows that $A_{F_i}x \oplus W \subseteq T \subseteq \cap_{i\in\mathcal{I}}X_{F_i} \subseteq \cap_{j\in\mathcal{I}}\mathcal{O}_0^j$ for all $i \in \mathcal{I}$. This implies that $x \in \mathcal{O}_1^i$ for all $i \in \mathcal{I}$. Therefore, $T \subseteq \mathcal{O}_1^i$ for all $i \in \mathcal{I}$. By iterating this procedure we get $T \subseteq \mathcal{O}_t^i\ \forall t \geq 0$ and $i \in \mathcal{I}$. In conclusion $T \subseteq \mathcal{O}_\infty$, i.e. $\mathcal{O}_\infty$ is maximal.

(ii) It is clear that the set of trajectories of the PWL system (18) is a subset of the trajectories of the system (19). So, any RPI set of (19) is also an RPI set for (18). $\diamondsuit$

Because the sets $\mathcal{O}_t^i$ are described by a finite number of linear inequalities, it is important to know whether the set $\mathcal{O}_\infty$ can be *finitely determined*, i.e. whether there exists a finite t^* such that $\mathcal{O}_{t^*}^i = \mathcal{O}_{t^*+1}^i$ for all $i \in \mathcal{I}$ (therefore $\mathcal{O}_\infty = \cap_{i\in\mathcal{I}}\mathcal{O}_{t^*}^i$ is a polyhedral set). Using the recursion (20) and the commutativity property of intersection, we have: $\mathcal{O}_0 = \cap_{i\in\mathcal{I}}\mathcal{O}_0^i,\ \mathcal{O}_t = \cap_{i\in\mathcal{I}}\mathcal{O}_t^i$ for all $t \geq 1 \Rightarrow \mathcal{O}_{t+1} \subseteq \mathcal{O}_t$, and therefore, $\mathcal{O}_\infty = \cap_{t\geq 0}\mathcal{O}_t$. Now, $\mathcal{O}_t$ can be written in terms of Pontryagin differences:

$$\begin{aligned}
&Y_0 = \cap_{i\in\mathcal{I}}X_{F_i},\ \mathcal{O}_0 = Y_0;\\
&Y_1 = Y_0 \ominus W,\ \mathcal{O}_1 = \cap_{i\in\mathcal{I}}\{x \in \mathcal{O}_0 : A_{F_i}x \in Y_1\};\\
&Y_t = \cap_{(i_1,\ldots i_{t-1})\in\mathcal{I}\times\ldots\times\mathcal{I}}(Y_{t-1} \ominus A_{F_{i_1}}\ldots A_{F_{i_{t-1}}}W),\\
&\mathcal{O}_t = \cap_{(i_1,\ldots i_t)\in\mathcal{I}\times\ldots\times\mathcal{I}}\{x \in \mathcal{O}_{t-1} : A_{F_{i_1}}\ldots A_{F_{i_t}}x \in Y_t\}.
\end{aligned} \tag{22}$$

It is clear that $Y_{t+1} \subseteq Y_t$ (since $0 \in W$). Therefore, the limit of this sequence $Y_\infty = \cap_{t\geq 0}Y_t$ exists. We have the following stopping criterion for computing $\mathcal{O}_\infty$:

Theorem 3.4: If the free switching system $x(k+1) = A_{F_i}x(k)$ with $i \in \mathcal{I}$ is asymptotically stable and $\exists t_0 \geq 0$ such that $\mathcal{O}_{t_0}$ is bounded and $0 \in \text{int}(Y_\infty)$, then $\mathcal{O}_\infty$ is finitely determined and therefore also a polyhedral set.

Proof: From asymptotic stability we have:

$$\begin{cases}
A_{F_{i_1}}\ldots A_{F_{i_t}}x \to 0, \text{when } t \to \infty, \text{ for all } x \in \mathbb{R}^n\\
\mathcal{O}_{t_0} \text{ is bounded, } 0 \in \text{int}(Y_\infty)
\end{cases}$$

implies that there exists a $t^* \geq t_0$ such that for all $(i_1, \ldots, i_{t^*+1}) \in \mathcal{I} \times \ldots \times \mathcal{I}$, $A_{F_{i_1}}\ldots A_{F_{i_{t^*+1}}}x \in Y_\infty \subseteq Y_{t^*+1}$, for all $x \in \mathcal{O}_{t_0}$. Since $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t_0}$ we have : $A_{F_{i_1}}\ldots A_{F_{i_{t^*+1}}}x \in Y_{t^*+1}$, for all $x \in \mathcal{O}_{t^*}$. Therefore, according to the recursion (22), $\mathcal{O}_{t^*} \subseteq \mathcal{O}_{t^*+1}$. But $\mathcal{O}_{t^*+1} \subseteq \mathcal{O}_{t^*}$. In conclusion, we have $\mathcal{O}_{t^*} = \mathcal{O}_{t^*+1}$ and $\mathcal{O}_\infty = \mathcal{O}_{t^*}$. Since $\mathcal{O}_{t^*}$ is described by a finite number of linear inequalities, $\mathcal{O}_\infty$ is a polyhedral set. $\diamondsuit$

IV. ROBUST MPC WITH KNOWN MODE

In the sequel we propose two robustly stabilizing MPC schemes for PWL system (2). We consider two cases depending on whether the mode at each sample step is known or unknown. For each case we develop a robustly stable MPC scheme.

A. Feedback min-max MPC scheme

In this section we develop a stable MPC scheme for the PWL system (2), with known mode despite the presence of disturbances, based on a feedback min-max approach using a dual-mode MPC formulation. In order to determine a suitable control law, an optimal control problem $\mathcal{V}_N(.)$ with horizon N is solved. Let $\mathbf{w} = (w(0), \ldots, w(N-1))$ be a possible realization of the disturbance over the interval 0 to $N-1$. Efficient control in the presence of disturbances requires state feedback; so, the decision variable (for a given initial state x) in the optimal control problem is a control policy defined as $\pi = (u(x), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot))$ where $u(x) \in U_c$ and $\mu_k : X_c \to U_c$, $k = 1, \ldots, N-1$ is a state feedback control law. Let $x(k; x, \pi, \mathbf{w})$ denote the solution to (2) at step k. The feedback min-max optimization problem is defined as:

$$\mathcal{V}_N(x) : \min_{\pi} \max_{\mathbf{w}\in W^N} \sum_{k=0}^{N-1} l(x_k, u_k) \tag{23}$$

$$\begin{cases}
x_k = x(k; x, \pi, \mathbf{w}) \in X_c,\ \forall k = 1, \ldots, N-1\\
u_k = \mu_k(x(k; x, \pi, \mathbf{w})) \in U_c,\ \forall k = 0, \ldots, N-1\\
x_N = x(N; x, \pi, \mathbf{w}) \in \mathcal{O}_\infty,\ \forall \mathbf{w} \in W^N,
\end{cases}$$

where $l(x, u)$ is convex and such that $l(x, u) \geq \alpha(d(x, \mathcal{O}_\infty))$, if $x \notin \mathcal{O}_\infty$ and $l(x, u) = 0$, if $x \in \mathcal{O}_\infty$ with α a $K-$function [12]. The distance of a point x to the closed, convex set $\mathcal{O}_\infty$ is defined as $d(x, \mathcal{O}_\infty) = \min_{x^o\in\mathcal{O}_\infty} \|x - x^o\|$. In the sequel we consider $\|X\|$ as the $p-$norm ($\|X\|_p$, $p \geq 1$) for vectors and matrices.

For linear systems problem (23) can be solved using the extreme disturbance realizations [12]. In our setting, due to the nonlinearities of the system, this approach cannot be applied directly. To overcome this problem, we propose to restrict the admissible control policies π to only those that guarantee that, for every value of the disturbance, the mode $i(k)$ is unique at each sample step k but the state is not known (e.g. gear box with gear position being the mode):

$$x(k; x, \pi, \mathbf{w}) \in \mathcal{P}_{i(k)},\ \forall \mathbf{w} \in W^N. \tag{24}$$

It can be easily observed that imposing (24) to the system (2) the state set generated by the disturbance at each sample step k is a *convex* set:

$$x(k; x, \pi, W^k) = x(k; x, \pi, \mathbf{0}) + X(k; i(0)...i(k-1), W^k) \tag{25}$$

where the first term expresses the nominal trajectory corresponding to (3) and the second term represents a convex uncertainty set associated with the state, which depends on the switching mode sequence $i(0), \ldots, i(k-1)$ and on the set W^k. Since W is a bounded polyhedron with v vertices, let $\mathcal{L}_v^N$ denote the set of indexes ℓ such that $\mathbf{w}^\ell = (w(0)^\ell, \ldots, w(N-1)^\ell)$ takes values only on the vertices of W. Then, $\mathcal{L}_v^N$ is a finite set with the cardinality $V_N = v^N$. Further, let $\mathbf{u}^\ell = (u_0^\ell, \ldots, u_{N-1}^\ell)$ denote a control sequence associated with the ℓ-th disturbance realization $\mathbf{w}^\ell$ and let $x_k^\ell = x(k; x_0, \mathbf{u}^\ell, \mathbf{w}^\ell)$ be the solution of (2) with the additional constraint (24). Using (24), the optimization problem (23) becomes a finite-dimensional optimization problem

$$\mathcal{V}_N(x): \quad \min_{\mathbf{u}} \max_{\ell \in \mathcal{L}_v^N} \sum_{k=0}^{N-1} l(x_k^\ell, u_k^\ell) \tag{26}$$

$$\begin{cases} \text{constraint (24)}, x_0^\ell = x, \quad \forall \ell \in \mathcal{L}_v^N \\ x_k^\ell \in X_{\rm c},\ \ k = 1, ..., N-1,\ \forall \ell \in \mathcal{L}_v^N \\ u_k^\ell \in U_{\rm c},\ \ k = 0, ..., N-1,\ \forall \ell \in \mathcal{L}_v^N \\ x_N^\ell \in \mathcal{O}_\infty,\ \forall \ell \in \mathcal{L}_v^N \\ x_k^{\ell_1} = x_k^{\ell_2} \ \Rightarrow\ u_k^{\ell_1} = u_k^{\ell_2},\ \forall \ell_1, \ell_2 \in \mathcal{L}_v^N \end{cases}$$

The last constraint is the well-known *causality constraint* [12]. The optimization problem to be solved at step k is:

$$\mathcal{V}_{N-k}(x_k): \min_{\mathbf{u}} \max_{\ell \in \mathcal{L}_v^{N-k}} \sum_{j=0}^{N-k-1} l(x_{k+j|k}^\ell, u_{k+j|k}^\ell) \tag{27}$$

$$\begin{cases} \text{constraint (24)},\ x_{k|k}^\ell = x_k, \qquad \forall \ell \in \mathcal{L}_v^{N-k} \\ x_{k+j|k}^\ell \in X_{\rm c},\ \ j = 1, ..., N-k-1,\ \forall \ell \in \mathcal{L}_v^{N-k} \\ u_{k+j|k}^\ell \in U_{\rm c},\ \ j = 0, ..., N-k-1,\ \forall \ell \in \mathcal{L}_v^{N-k} \\ x_{N|k}^\ell \in \mathcal{O}_\infty,\ \forall \ell \in \mathcal{L}_v^{N-k} \\ x_{k+j|k}^{\ell_1} = x_{k+j|k}^{\ell_2} \Rightarrow u_{k+j|k}^{\ell_1} = u_{k+j|k}^{\ell_2}, \forall \ell_1, \ell_2 \in \mathcal{L}_v^{N-k} \end{cases}$$

where $x_{k+j|k}^\ell$ is the prediction of the state at step $k+j$ given by the model (2), corresponding to the ℓ-th disturbance realization $(w(0)^\ell, ..., w(N-k-1)^\ell)$ and applying the input sequence $u_{k|k}^\ell, ..., u_{N-1|k}^\ell$. The constraint (24) is imposed only to the states $x_{k+j|k}^\ell$ with $j = 1, ..., N-k-1$ and not to $x_{N|k}^\ell$. The only constraint on the state $x_{N|k}^\ell$ is the terminal constraint: $x_{N|k}^\ell \in \mathcal{O}_\infty$. We use a variable horizon scheme as in [12]. The feedback min-max MPC controller is based on a dual-mode approach. For any $k \geq 0$, given the current state x_k, the algorithm is formulated as follows:

Feedback min-max MPC algorithm (I)

- if $x_k \in \mathcal{O}_\infty \cap \mathcal{P}_i$ then take $u^{\rm RH}(x_k) = F_i x_k,\ \forall i \in \mathcal{I}$
- otherwise, solve (27) and set $u^{\rm RH}(x_k)$ to the first control in the optimal solution computed: $u_{k|k}^\ell$,

where $u^{\rm RH}(x)$ is the control input applied to the system according to the receding horizon strategy.

B. Stability

We give first some definitions [13]: a set $T_{\rm set}$ is *robustly stable* iff for all $\epsilon > 0$, there exists a $\gamma > 0$ such that $d(x_0, T_{\rm set}) < \gamma$ implies $d(x(k), T_{\rm set}) < \epsilon$ for all $k \geq 0$ and all admissible disturbance sequences. The set $T_{\rm set}$ is *robustly finite-time attractive* with domain of attraction X iff for all $x_0 \in X$ there exist a finite-time M such that $x(k) \in T_{\rm set}$ for all $k \geq M$. The set $T_{\rm set}$ is *robustly finite-time stable* with the domain of attraction X iff it is robustly stable and robustly finite-time attractive with domain of attraction X. We define also $\bar{X}_N = \{x \in \mathbb{R}^n :$ (26) has a solution for $x\}$

Theorem 4.1: If the optimization problem $\mathcal{V}_N(x_0)$ is feasible (hence has an optimum), then all subsequent optimization problems $\mathcal{V}_{N-k}(x_k)$ with $k = 1, ..., N-1$ are feasible. Moreover, at sample step N we have $x_N \in \mathcal{O}_\infty$.

Proof: At step $k = 0$, with the initial state $x_0 = x \in \mathcal{P}_{i_0}$, let $(u_{0|0}^{*\ell}, ..., u_{N-1|0}^{*\ell})$ be the optimal solution corresponding to the ℓ-th disturbance realization, satisfying the constraints (24), therefore producing the "certain" switching sequence $i_0, i_1, ..., i_{N-1}$. Let $x_{0|0}^\ell, ..., x_{N-1|0}^\ell$ be the corresponding state trajectories. From the causality constraints we have: $u_{0|0}^{*\ell_1} = u_{0|0}^{*\ell_2} = u_0^*$ for any $\ell_1 \neq \ell_2 \in \mathcal{L}_v^N$. Now the input u_0^* is applied and the disturbance takes a certain value $w_0 = \sum_{\ell \in \mathcal{L}_v^N} \mu_\ell w_0^\ell \in W$, where w_0^ℓ is a vertex of W and μ_ℓ are appropriate convex scalar weights. Therefore, $x_1 = A_{i_0} x + B_{i_0} u_0^* + w_0 = \sum_{\ell \in \mathcal{L}_v^N} \mu_\ell x_1^\ell$ with $x_1^\ell = A_{i_0} x + B_{i_0} u_0^* + w_0^\ell$, i.e. x_1 lies in the convex hull $\mathrm{co}\{x_1^\ell : \ell \in \mathcal{L}_v^N\}$. Define the following control sequence

$$(\sum_{\ell \in \mathcal{L}_v^{N-1}} \mu_\ell u_{1|0}^{*\ell}, ..., \sum_{\ell \in \mathcal{L}_v^{N-1}} \mu_\ell u_{N-1|0}^{*\ell}) \tag{28}$$

With this control sequence the state predictions at step $k = 1$ evolve in the convex hull of the predictions at step $k = 0$: $x_{1+j|1} \in \mathrm{co}\{x_{1+j|0}^\ell,\ \ell \in \mathcal{L}_v^{N-1}\}$, where $x_{1+j|1}$ with $j = 1, ..., N-1$ is the state prediction at step $k = 1$, for an arbitrary disturbance sequence. Similarly the input predictions evolves in the convex hull of the predictions made at time $k = 0$ (according to (28)). Moreover, the switching sequence is certain: $i_1, ..., i_{N-1}$ (we used that $X, U, \mathcal{O}_\infty$ are convex). Then, the problem $\mathcal{V}_{N-1}(x_1)$ is feasible and has an optimum. By induction, we can prove that all subsequent optimization problems $\mathcal{V}_{N-k}(x_k)$ are feasible. Furthermore, $\mathcal{V}_1(x_{N-1})$ is feasible. So, there exists an optimal input such that $x_N \in \mathcal{O}_\infty$. $\diamondsuit$

Theorem 4.2: The feedback min-max MPC law $u^{\mathrm{RH}}(.)$ given by the *algorithm (I)* makes $\mathcal{O}_\infty$ robustly finite-time stable for the system (2) in closed-loop with $u^{\mathrm{RH}}(x)$ with a region of attraction $\bar{X}_N$.

Proof: See [7]. $\diamondsuit$

The optimization problem (27) can be recast as a mixed-integer linear programming problem when the $p-$norm is either $\|\cdot\|_1$ or $\|\cdot\|_\infty$.

V. Robust MPC with unknown mode

A. Robust MPC using quadratic optimization problems

The maximal RPI set $\tilde{\mathcal{O}}_\infty$ included in X_F for (18) is (in general) not a convex set. The maximal RPI set $\tilde{\mathcal{O}}_\infty$, for which the nominal controller $u = F_i x$ is feasible, is in general small. Now we derive a robustly stable MPC scheme that uses prediction control trajectories which do not correspond to fixed state feedback control laws. Therefore, we enlarge the set of initial states that can be steered to a target set, close to the origin. We introduce a new control variable c_k such that the new input applied to the system is

$$u_k = F_i x_k + c_k, \text{ if } x_k \in \mathcal{P}_i. \tag{29}$$

Let N be the control horizon, then $c_k, ..., c_{k+N-1}$ represent degrees of design freedom and $c_{k+N+j} = 0, \ \forall j \geq 0$. In this case the PWL system (18) becomes

$$x_{k+1} = A_{F_i} x_k + B_i c_k + w_k, \text{ if } x_k \in \mathcal{P}_i. \tag{30}$$

Employing a reasoning similar to [14], the dynamics of (30) can be described by the autonomous PWL system

$$z_{k+1} = \mathcal{A}_i z_k + \mathcal{D} w_k, \text{ if } z_k \in \tilde{\mathcal{P}}_i \tag{31}$$

where $z \in \mathbb{R}^{n+Nm}$, $z = \begin{bmatrix} x \\ f \end{bmatrix}$, $f = [c_k^T, ..., c_{k+N-1}^T]^T$, $\mathcal{D} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\mathcal{A}_i = \begin{bmatrix} A_{F_i} & \begin{bmatrix} B_i & 0 & ... & 0 \end{bmatrix} \\ 0 & M \end{bmatrix}$, $M = \begin{bmatrix} 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$, $\mathrm{cl}(\tilde{\mathcal{P}}_i) = \{z : [E_i \ \ 0]z \geq 0\}$.

Clearly the stability properties of the matrices $\mathcal{A}_i$ depend only on the matrices A_{F_i}. We denote

$$\begin{aligned} X_{0,z}^i &= \{z = [x^T \ f^T]^T : x \in X_{\mathrm{c}} \cap \mathcal{P}_i, F_i x + c_k \in U_{\mathrm{c}}\} \\ &= \{z : [I \ \ 0]z \in X_{\mathrm{c}} \cap \mathcal{P}_i, [F_i \ \ I \ \ 0]z \in U_{\mathrm{c}}\}. \end{aligned}$$

Remark 5.1 If there exists an RPI set $\mathcal{O}$ for (18), then there must exist at least one RPI set $\mathcal{O}_z \subseteq \cup_{i \in \mathcal{I}} X_{0,z}^i$ for (31). Indeed, from definition (31), it is clear that $\mathcal{O}_z = \{z = [x^T \ 0]^T : x \in \mathcal{O}\}$ is an RPI set for this system. $\diamondsuit$

So, if the maximal RPI set $\tilde{\mathcal{O}}_\infty$ for (18) exists, then there exists also a maximal RPI set $\mathcal{O}_{\infty,z}$ for the augmented system (31) and the projection of $\mathcal{O}_{\infty,z}$ into the state space $\mathbb{R}^n$ (denoted with $\mathcal{O}_{\infty,zx}$) contains $\tilde{\mathcal{O}}_\infty$. Therefore, the benefits of using free control moves are clear now. The robust MPC algorithm is defined as follows:

Algorithm (II)

1) Off-line step: compute F_i and the maximal RPI set $\mathcal{O}_{\infty,z}$ for (31)
2) On-line step: each step k, given x_k solve

$$J_N^*(k) = \min_f f^T f, \text{ s.t. } z = [x_k^T \ f^T]^T \in \mathcal{O}_{\infty,z} \tag{32}$$

Implement the controller $u_k = F_i x_k + c_k$.

The maximal RPI set for (31) included in $\cup_{i \in \mathcal{I}} X_{0,z}^i$ is in general a union of polyhedral sets: $\mathcal{O}_{\infty,z} = \cup_{j=1}^q \mathcal{O}_z^j$, where $\mathcal{O}_z^j$ are polytopes. Therefore, at step 2 of Algorithm (II) we have to solve q quadratic programming (QP) problems, and then to choose f for which $f^T f$ is the smallest one.

Theorem 5.2: Given $x_0 \in \mathcal{O}_{\infty,zx}$, the receding horizon implementation of the Algorithm (II) asymptotically steers the trajectory of (30) to $\tilde{\mathcal{O}}_\infty$.

Proof: If $x_0 \in \mathcal{O}_{\infty,zx}$, then (32) has a solution at $k = 0$, $f_0^* = [c_0^{*T} ... c_{N-1}^{*T}]^T$. Moreover, there exists an $i_0 \in \mathcal{I}$ such that $x_0 \in \mathcal{P}_{i_0} \cap \mathcal{O}_{\infty,zx}$. Let us denote with $f_1^{\mathrm{feas}} = [c_1^{*T} ... c_{N-1}^{*T} \ \ 0]^T$. Applying the feedback input $u_0 = F_{i_0} x_0 + c_0^*$ to the system (30), and keeping in mind that $\mathcal{O}_{\infty,z}$ is an RPI set for (31), then we obtain $[x_1^T \ f_1^{\mathrm{feas}T}]^T \in \mathcal{O}_{\infty,z}$. Therefore, f_1^{feas} is feasible at $k = 1$. By induction, we can prove that given $x(k)$, for all $k \geq 1$ the optimization problem (32) has an optimal solution $f_k^* = [c_k^{*T}, ..., c_{k+N-1}^{*T}]^T$ and at sample step $k+1$ we have a feasible solution $f_{k+1}^{\mathrm{feas}} = [c_{k+1}^{*T} ... c_{k+N-1}^{*T} 0]^T$. In conclusion

$$J_N^*(k+1) - J_N^*(k) \leq -\|c_k^*\|^2. \tag{33}$$

Hence, the sequence $\{J_N^*(k)\}_{k \geq 0}$ is non-increasing and bounded below by 0. Therefore, it converges to $J_N^\infty < \infty$. Summing (33) from 0 to ∞ we obtain: $0 \leq \sum_{k \geq 0} \|c_k^*\|^2 \leq J_N^*(0) - J_N^\infty < \infty$. So, the series $\sum_{k \geq 0} \|c_k^*\|$ is convergent. We conclude that $c_k^* \to 0$ as $k \to \infty$. Therefore, $\lim_{k \to \infty} d(x_k, \tilde{\mathcal{O}}_\infty) = 0$, because $\tilde{\mathcal{O}}_\infty$ is the maximal set of states for which the controller $u = F_i x$ if $x \in \mathcal{P}_i$ is feasible. $\diamondsuit$

B. Robust MPC using a single QP problem

In this section we develop a new MPC scheme, such that we solve on-line a single quadratic optimization problem.

Off-line step

In this step we compute *off-line* the set of initial states and input correction sequences that steer these states to the RPI set $\mathcal{O}_\infty$ (cf. (21)) in N steps, using the controller (29). This set is obtained recursively as follows:

$$\mathcal{X}_0^i = \mathcal{O}_\infty^i, \ \forall i \in \mathcal{I}, \tag{34}$$

$$\mathcal{X}_{k+1}^i = \left\{ \begin{bmatrix} x \\ c \\ \tilde{c} \end{bmatrix} : \begin{matrix} \begin{bmatrix} A_{F_i} x + B_i c \oplus W \\ \tilde{c} \end{bmatrix} \in \bigcap_{j \in \mathcal{I}} \mathcal{X}_k^j \\ x \in X_{\mathrm{c}}, \ F_i x + c \in U_{\mathrm{c}} \end{matrix} \right\}$$

$k = 0, ..., N-1$ and $i \in \mathcal{I}$. Note that a similar recursion was proposed also in [11] in the context of gain scheduling for nonlinear systems. Clearly $\mathcal{X}_N^i \subseteq \mathbb{R}^{n+mN}$. We denote with X_k^i the projection of $\mathcal{X}_k^i$ into the state space $\mathbb{R}^n$. In conclusion the set of initial states that can be steered to

$\mathcal{O}_\infty$ in N steps, using the semi-feedback controller (29) is: $X_N = \cup_{i\in\mathcal{I}}(X_N^i \cap \mathcal{P}_i)$. Because X_c, U_c and W are polytopes and initially $\mathcal{X}_0^i = \mathcal{O}_\infty^i$, with $\mathcal{O}_\infty^i$ a polytope, we obtain that $\mathcal{X}_N^i$'s are polytopic sets. Hence, X_N^i is a polytope for any $i \in \mathcal{I}$. So, X_N is a union of polytopes.

Proposition 5.3: The set $\cup_{i\in\mathcal{I}}(\mathcal{X}_N^i \cap \tilde{P}_i)$ is an RPI set for the augmented system (31).

Proof: See [7]. $\diamondsuit$

Clearly, $\cup_{i\in\mathcal{I}}(\mathcal{X}_N^i \cap \tilde{P}_i) \subseteq \mathcal{O}_{\infty,z}$. The evolution of (30) under the input sequence (29), with the initial state x_0 is:

$$x_{k+1} = A_{F_{i(k)}} \ldots A_{F_{i(0)}} x_0 + \sum_{j=1}^{k+1} A_{F_{i(k+1)}} \ldots A_{F_{i(j)}} (B_{i(j-1)} c_{j-1} + w_{j-1}) \tag{35}$$

where $A_{F_{i(k+1)}} = I$ and $i(0), \ldots, i(k)$ is any feasible switching sequence corresponding to state sequence $x_0, \ldots, x_k$.

On-line step

Assume $x(k) \in \mathcal{P}_i$. At this stage, we solve *on-line*, at each step k, the following QP problem:

$$J_N^*(k) = \min_f f^T f, \text{ s.t. } [x_k^T \ f^T]^T \in \mathcal{X}_N^i \tag{36}$$

Then, according to the receding horizon strategy, the input applied to the system at step k is given by: $u_k = F_i x_k + c_k^*$. Once $x_k \in \mathcal{O}_\infty$, the MPC law is given by the local controller $u_k = F_i x_k$, which has the property that it keeps the state inside this RPI set for any disturbance in W.

Theorem 5.4: If the free switching system $x(k+1) = A_{F_i} x(k)$ with $i \in \mathcal{I}$ is asymptotically stable and the initial state $x_0 \in X_N$ then the feedback MPC law $u_k = F_i x_k + c_k^*$ drives the state x_k asymptotically to the RPI set $\mathcal{O}_\infty$.

Proof: Similarly as in Theorem 5.2 we conclude that

$$c_k^* \to 0 \text{ as } k \to \infty. \tag{37}$$

Let us now prove that $d(x_k, \mathcal{O}_\infty) \to 0$ as $k \to \infty$. Given $x_0 \in X_N$ there exists an $x_0^o \in \mathcal{O}_\infty$ such that $d(x_0, \mathcal{O}_\infty) = \|x_0 - x_0^o\|$ (since $\mathcal{O}_\infty$ is a closed, convex set). Now $x_1 = A_{F_{i(0)}} x_0 + B_{i(0)} c_0^* + w_0$. Let us define $x_1^o = A_{F_{i(0)}} x_0^o + w_0$. From the definition of $\mathcal{O}_\infty$ it is clear that $x_1^o \in \mathcal{O}_\infty$ and $d(x_1, \mathcal{O}_\infty) \leq \|x_1 - x_1^o\| \leq \|A_{F_{i(0)}}\| \ \|x_0 - x_0^o\| + \|B_{i(0)} c_0^*\|$ By induction, using (35), we can prove that

$$d(x_{k+1}, \mathcal{O}_\infty) \leq \|x_{k+1} - x_{k+1}^o\| \leq \|A_{F_{i(k)}} \ldots A_{F_{i(0)}} (x_0 - x_0^o)\| + \sum_{j=1}^{k+1} \|A_{F_{i(k+1)}} \ldots A_{F_{i(j)}} B_{i(j-1)} c_{j-1}^*\|, \tag{38}$$

for any feasible sequence of switches $i(0), \ldots, i(k)$, where $x_{k+1}^o = A_{F_{i(k)}} x_k^o + w_k \in \mathcal{O}_\infty$. Since the free switching system $x(k+1) = A_{F_i} x(k)$ with $i \in \mathcal{I}$ is asymptotically stable, then for all $x \in \mathbb{R}^n$ we have $\|A_{F_{i(k)}} \ldots A_{F_{i(j)}} x\| \to 0$ for j finite and $k \to \infty$. Using this and (37) in (38), we obtain $d(x_k, \mathcal{O}_\infty) \to 0$ as $k \to \infty$. $\diamondsuit$

For a worked example of the two MPC schemes proposed in this paper and an extension to PWA systems the reader is referred to [7].

VI. Conclusions

In this paper we have designed two stable MPC algorithms for perturbed PWL systems. First, a robustly stable feedback min-max MPC scheme is introduced, that uses the fact that the mode of the system is certain at each step. We incorporate feedback in the control, in order to increase the domain of the feasible control sequences. The second stable MPC scheme is based on unknown mode, using a semi-feedback controller. For this scheme we have to solve on-line quadratic optimization problems.

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References

[1] E. Kerrigan and D. Mayne, "Optimal control of constrained, piecewise affine systems with bounded disturbances," in *Proc. Conf. on Decision and Control*, Las Vegas, Nevada, Dec. 2003.

[2] M. Lazar, W. Heemels, S. Weiland, and A. Bemporad, "Stabilization conditions for model predictive control of constrained PWA systems," in *Proc. Conf. on Decision and Control*, Bahamas, Dec. 2004.

[3] ——, "On the stability of quadratic forms based model predictive control of constrained PWA systems," in *Proc. American Control Conference*, Oregon, 2005.

[4] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybride systems: An LMI approach," in *Proc. Conf. on Decision and Control*, Sydney, Dec. 2000.

[5] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, Mar. 1999.

[6] I. Necoara, B. D. Schutter, T. van den Boom, and J. Hellendoorn, "Model predictive control for perturbed continuous piecewise affine systems with bounded disturbances," in *Proc. Conf. on Decision and Control*, Bahamas, Dec. 2004.

[7] I. Necoara, B. De Schutter, T. van den Boom, and J. Hellendoorn, "Robustly stabilizing MPC for perturbed PWL systems: Extended report," Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands, Tech. Rep. 05-004a, Feb. 2005, URL: www.dcsc.tudelft.nl/~bdeschutter/pub/abstracts/05_004a_abstr.html.

[8] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishna, "Linear Matrix Inequalities in System and Control Theory," *SIAM, Studies in Applied Mathematics, Philadelphia*, 1994.

[9] J. Daafouz and J. Bernussou, "Parameter dependent lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems and Control Letters*, vol. 43, pp. 355–359, 2001.

[10] I. Kolmanovsky and E. Gilbert, "Theory and computation of disturbance invariance sets for discrete-time linear systems," *Mathematical Problems in Engineering: Theory, Methods and Applications*, vol. 4, pp. 317–367, 1998.

[11] L. Chisci, P. Falugi, and G. Zappa, "Gain-scheduling MPC of nonlinear systems," *Int. J. Robust and Nonlinear Control*, vol. 13, pp. 295–308, March-April 2003.

[12] P. Scokaert and D. Mayne, "Min-max feedback MPC for constrained linear system," *IEEE Transactions on Automatic Control*, vol. 43, no. 8, pp. 1136–1142, Aug. 1998.

[13] E. Kerrigan and J. Maciejowski, "Feedback min-max MPC using a single linear program: Robust stability and the explicit solution," *Int. J. Robust Nonlinear Control*, pp. 1–18, 2003.

[14] B. Kouvaritakis, J. Rossiter, and J. Schuurmans, "Efficient robust predictive control," *IEEE Transactions on Automatic Control*, vol. 45, no. 8, pp. 1545–1549, August 2000.