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Sampled-Data State Feedback Control of Piecewise-Affine Systems

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Abstract— **This paper addresses stability of sampled-data piecewise-affine (PWA) systems consisting of a continuous-time plant and a discrete-time emulation of a continuou-time state feedback controller. The paper presents conditions under which the trajectories of the sampled-data closed-loop system will exponentially converge to a neighborhood of the origin. Moreover, the size of this neighborhood will be related to bounds on perturbation parameters related to the sampling procedure, in particular, related to the sampling period. Finally, it will be shown that when the sampling period converges to zero the performance of the stabilizing continuous-time PWA state feedback controller can be recovered by the emulated controller.**

I. INTRODUCTION

PWA systems are multi-model systems that offer a good modeling framework for complex dynamical systems involving nonlinear phenomena. State and output feedback control of continuous-time PWA systems have received increasing interest over the last years. The research work has concentrated on Lyapunov-based controller synthesis methods for continuous-time PWA systems [3], [7], [8], [12], [13]. However, none of these approaches would be applicable directly to controller synthesis for computer-controlled or sampled-data PWA systems. This is the scenario mostly encountered in applications given the flexibility of control implementation in a microprocessor. References [3], [7], [8], [12], [13] consider continuous-time processes controlled by continuous-time controllers while the implementation in a microprocessor requires emulation of a continuous-time controller as a discrete-time controller. Although linear sampled-data systems are a well-studied matter [2], controller emulation for systems with possible discontinuities at the switching, such as sampled-data PWA systems, has not had many research contributions. In fact, only recently these systems have started to be addressed in the literature in references such as [4], [5], [6], [15], [16], [17]. The approach by Sun and Ge [15] established that, under certain conditions, the controllable subspaces of a continuous-time switched linear system and its discrete-time counterpart are the same. Canonical forms of switched linear systems based on controllability are presented in the more recent work of Sun [16]. The approach by Zhai *et. al* [17] considers stability analysis of switched systems that can switch between a set of continuous-time plants and a set of discrete-time plants but does not handle sampled-data systems involving a cascade of a discrete-time system between a sample-and-hold and a continuous-time system. Furthermore, it does not address controller design. The approach by Imura et. al. [4], [5], [6] was probably the first where the term "sampled-data PWA systems" is used, although the systems described in his work do not posses the typical structure of a continuous-time plant being controlled by a discrete-time controller. The problem addressed in [4], [5], [6] is one where the controller is continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, in these systems it is assumed that the designer has command over the switching times of the system, such as in the case of filling up two tanks and deciding when to switch the water between them. For this class of systems reference [6] presents a probabilistic analysis of controllability. This is not the case of interest in the current paper. The preliminary study of Imura [4], [5] is important as it highlights important limitations of current discrete-time PWA control methodologies when applied to the control of a physical continuous-time system. As mentioned in [4] unexpected phenomena such as chattering can occur, depending on the switching times. This increases the interest in studying computer implementations of controllers designed in continuous-time, as is suggested in this paper.

The problem that we propose to address is the one corresponding to the classical structure of a sampled-data system whereby the system is continuous and the controller is being implemented (emulated) in discrete-time inside a computer. Previous approaches to this classical structure of sampled-data control can be classified into two categories: i) discrete-time controller design to a discrete-time approximation of the continuous-time plant and ii) continuous-time controller design to a continuous-time plant followed by discrete-time emulation of the controller. To the best of the author's knowledge the only previous work in sampled-data PWA systems is the work of Imura *et. al.* [4], [5], [6] which, as already stated, does not address the classical structure of interest in this paper. For papers in sampled-data control for nonlinear systems (but not switched and therefore not applicable to PWA systems) that fall into category i) we refer the reader to [10] and references therein. For papers in sampled-data control for nonlinear systems (but not switched) that fall under category ii) we refer the reader to the recent paper by Khalil [9] and references therein. Note that these papers represent pioneering approaches in the field of nonlinear sampled-data control systems but always assume the plant dynamics to be locally Lipschitz. Therefore they do not include the possibility of having PWA dynamics that are switched with possible discontinuities in the plant dynamics at the switching. The interesting recent paper by Nesic and Teel [11] also falls under category i) described above but offers the advantage of treating the plant model as a differential inclusion, thus possibly enabling the treatment of discontinuous vector fields. In fact, one of the examples described in [11] deals with a hysteresis switched controller. Although potentially applicable to PWA systems, reference [11] does not address the problem of interest here, namely stability and performance recovery by emulation of a continuous-time PWA controller. Furthermore, to be able to embed PWA systems in the framework of [11], the plant dynamics would have to be embedded in a differential inclusion, which can potentially lead to conservative results instead of handling the PWA dynamics directly.

The paper starts by stating the problem assumptions. Then, the stability of the sampled-data system when a continuous-time controller is emulated in discrete-time is analyzed. Finally, the paper finishes with an example illustrating how the results can be used in practice, followed by the conclusions

II. Problem Assumptions

It is assumed that a PWA system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see [14] for generating such a partition). Following [7], [3], each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{z \,|\, H_i^T z - g_i < 0\}, \tag{1}$$

where $H_i = [h_{i1} \;\; h_{i2} \ldots h_{ip_i}]$, $g_i = [g_{i1} \;\; g_{i2} \ldots g_{ip_i}]^T$. Moreover the sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset I\!R^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{z}(t) = A_i z(t) + b_i + B_i u(t), \tag{2}$$

where $z(t) \in I\!R^n$ and $u(t) \in I\!R^m$. For system (2), we adopt the definition of trajectories or solutions presented in [7].

Definition 2.1: [7] Let $z(t) \in \mathcal{X}$ be an absolutely continuous function. Then $z(t)$ is a trajectory of the system (2) on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$ and Lebesgue measurable $u(t)$, the equation $\dot{z}(t) = A_i z(t) + b_i + B_i u(t)$ holds for all i such that $z(t) \in \overline{\mathcal{R}}_i$. □

Any two cells sharing a common facet will be called ***level-1*** neighboring cells. Let $\mathcal{N}_i$ = {***level*-1** neighboring cells of $\mathcal{R}_i$}. It is also assumed that vectors $c_{ij} \in I\!R^n$ and scalars d_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{z \in I\!R^n \,|\, c_{ij}^T z - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [3]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{z = l_{ij} + F_{ij} s \;|\; s \in I\!R^{n-1}\} \tag{3}$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in I\!R^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}^T and $l_{ij} \in I\!R^n$ is given by $l_{ij} = c_{ij}\left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. It is further assumed that matrices E_i and f_i exist such that $\mathcal{R}_i \subseteq \varepsilon_i$,

$$\varepsilon_i = \{z | \, \|E_i z + f_i\| \leq 1\}. \tag{4}$$

This ellipsoidal covering is especially useful in the case where $\mathcal{R}_i$ is a slab because in this case the matrices E_i and f_i are guaranteed to exist and the covering (having one degenerate ellipsoid ε_i) is exact, ***i.e.,*** $\varepsilon_i \subseteq \mathcal{R}_i$ and $\mathcal{R}_i \subseteq \varepsilon_i$. More precisely, if $\mathcal{R}_i = \{z \mid d_1 < c_i^T z < d_2\}$, then the degenerate ellipsoid is described by $E_i = 2c_i^T/(d_2 - d_1)$ and $f_i = -(d_2 + d_1)/(d_2 - d_1)$. Finally it is assumed, without loss of generality, that the control objective is to stabilize the system to the origin.

III. Stability of Sampled-Data PWA Systems

In this section a stability result is presented for the closed-loop sampled-data system that is obtained when a continuous-time state feedback controller is implemented on a digital computer. It is assumed that a continuous-time state feedback controller of the form

$$u = K_i z + m_i, \quad z \in \mathcal{R}_i \tag{5}$$

has already been designed such that the continuous-time closed-loop system is exponentially stable.[1] It is also assumed that the state z of the system is measured at a sampling rate $f_s = T^{-1}$, $T > 0$, and that the controller in the feedback loop appears between a sampler and a zero-order-hold. For the sampling instants, the plant state and the sampled state are the same and therefore the sampled-data system is described by the differential equation

$$\dot{z} = A_j z + b_j + B_j K_j z(kT) + B_j m_j, \tag{6}$$

for $z(t) \in \mathcal{R}_j$, $z(kT) \in \mathcal{R}_j$. However, for a given time t that is not a sampling instant, the general situation that should be considered is the one in which the state of the plant is in region $\mathcal{R}_i$ and the most recently sampled state is in region $\mathcal{R}_j$ with possibly $i \neq j$. The system is then described by the differential equation

$$\dot{z} = A_i z + b_i + B_i K_j z(kT) + B_i m_j, \tag{7}$$

for $z(t) \in \mathcal{R}_i$, $z(kT) \in \mathcal{R}_j$. This equation can be rewritten in the perturbed form

$$\dot{z} = \bar{A}_i z + \bar{b}_i + B_i \delta_{ij}, \tag{8}$$

for $z(t) \in \mathcal{R}_i$, $z(kT) \in \mathcal{R}_j$, where $\bar{A}_i = A_i + B_i K_i$, $\bar{b}_i = b_i + B_i m_i$ and

$$\delta_{ij} = K_j \left(z(kT) - z(t)\right) + \left(K_j - K_i\right) z(t) + \left(m_j - m_i\right). \tag{9}$$

Note that the first term in (9) represents the perturbation due to the error between the last available sample of the state and its current value. The second and third terms are associated with the perturbation due to the state and its most recent sample being possibly in different regions. The second term represents the perturbation due to a difference in the gain matrices in regions $\mathcal{R}_i$ and $\mathcal{R}_j$ and the third term represents the perturbation due to a difference in the affine control terms in the same regions. Given a continuous-time controller of the form (5), the first step in the procedure outlined in this paper is to search for a quadratic Lyapunov function of the form

$$V(z) = z^T P z \tag{10}$$

[1]For optimization programs whose solution (when it exists) yields exponentially stabilizing PWA controllers see [12], [13].

that proves stability of the continuous-time closed-loop system. This can be done by solving for fixed $\alpha \geq 0$ the following set of LMIs (see for example [13] for details on the derivation of these conditions):

$$P = P^T > 0,\ \lambda_i < 0,\ i = 1, \ldots, M,$$
$$\begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + \alpha P + \lambda_i E_i^T E_i & P\bar{b}_i + \lambda_i E_i^T f_i \\ (P\bar{b}_i + \lambda_i E_i^T f_i)^T & -\lambda_i \left(1 - f_i^T f_i\right) \end{bmatrix} < 0. \quad (11)$$

The results that follow assume that such a Lyapunov function can be found. Note however that not all continuous-time PWA systems that are stable admit a globally quadratic Lyapunov function (see [7] for counter-examples).

A. Conditions Independent of the Sampling Period

We now present the first result of this section. It gives conditions under which the trajectories of the sampled-data system (7) converge to a region around the closed-loop equilibrium point. Furthermore, it relates the size of this region to a measure of the perturbation term in the closed-loop system.

Theorem 3.1: Assume a Lyapunov function of the form (10) is found and is defined in $\mathcal{X} \subseteq I\!R^n$. Let the condition number of P be $\chi(P) = \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}$. Assume there are finite constants $N_{ij} > 0,\ \Delta_{K_{ij}} > 0$ such that $\|\delta_{ij}\| \leq N_{ij} + \Delta_{K_{ij}}\|z\|,\ i, j = 1, \ldots, M$. Let $N = \max_{i,j=1,\ldots,M}(N_{ij}),\ \Delta_K = \max_{i,j=1,\ldots,M}\left(\Delta_{K_{ij}}\right),\ B = \max_{i=1,\ldots,M}\|B_i\|$. Define

$$\mu_\theta = \frac{2\chi(P)B}{\alpha\theta - 2\chi(P)B\Delta_K}N$$

and the region

$$\mathcal{S}_\theta = \{z \in \mathcal{X} \mid \|z\| \leq \mu_\theta\}$$

for any positive constant $\theta < 1$ that verifies

$$\Delta_K < \frac{\alpha\theta}{2\chi(P)B}$$

Then, the trajectories of the closed-loop sampled-data system (8) converge exponentially to the set

$$\Omega = \left\{z \in \mathcal{X} \mid V(z) \leq \sigma_{max}(P)\mu_\theta^2\right\}$$

Proof: Using the dynamics (8), the derivative of the candidate Lyapunov function (10) for $z(t) \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$ along the trajectories of the system is

$$\dot{V}(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} + 2z^T P B_i \delta_{ij}$$

However, note that if a quadratic Lyapunov function is found by solving (11), using the $\mathcal{S}$-procedure (see [13] for details) it can be shown that for $z \in \mathcal{R}_i$

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < -\alpha z^T P z.$$

Therefore, for $z \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$ it follows that

$$\frac{d}{dt}V(z) < -\alpha z^T P z + 2z^T P B_i \delta_{ij}$$

Taking norms and using the bounds

$$\|\delta_{ij}\| \leq N_{ij} + \Delta_{K_{ij}}\|z\| \leq N + \Delta_K\|z\|$$

and $-z^T P z \leq -\sigma_{min}(P)\|z\|^2$ yields

$$\frac{d}{dt}V(z) < -\alpha\sigma_{min}(P)\|z\|^2 + 2\|z\|\sigma_{max}(P)B\left(N + \Delta_K\|z\|\right)$$

or, for any positive constant $\theta < 1$

$$\frac{d}{dt}V(z) < -(1-\theta)\alpha\sigma_{min}(P)\|z\|^2 - \theta\alpha\sigma_{min}(P)\|z\|^2 + 2\|z\|\sigma_{max}(P)B\left(N + \Delta_K\|z\|\right).$$

Therefore, for $0 < \theta < 1$, we have

$$\frac{d}{dt}V(z) < -(1-\theta)\alpha\sigma_{min}(P)\|z\|^2 \leq -(1-\theta)\alpha V(z) \quad (12)$$

for

$$\|z\| > \frac{2\chi(P)B}{\alpha\theta - 2\chi(P)B\Delta_K}N$$

provided

$$\Delta_K < \frac{\alpha\theta}{2\chi(P)B}. \quad (13)$$

As a result of (12), for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$V(z(t)) < V(z(t_0))e^{-(1-\theta)\alpha(t-t_0)}$$

Using the relation $\sigma_{min}(P)\|z\|^2 \leq V(z) \leq \sigma_{max}(P)\|z\|^2$ we can conclude that for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$\|z(t)\| \leq \|z(t_0)\|\chi^{\frac{1}{2}}(P)e^{-0.5(1-\theta)\alpha(t-t_0)}$$

Thus, there will be a positive and finite time t_1^θ such that $z(t_1^\theta) \in \mathcal{S}_\theta$ for any positive constant $\theta < 1$ that verifies (13). Note that $\mathcal{S}_\theta \subseteq \Omega$. This can be proved by contradiction. Assume that it is not true that $\mathcal{S}_\theta \subseteq \Omega$. Then, there exists at least one $z_0 \in \mathcal{S}_\theta$ for which $z_0^T P z_0 > \sigma_{max}(P)\mu_\theta^2$, a contradiction. Since $\dot{V} \leq 0$ at the boundary of Ω, Ω is an invariant set for system (8). Consequently, since $z(t_1^\theta) \in \mathcal{S}_\theta \subseteq \Omega$, $z(t) \in \Omega$ for all $t \geq t_1^\theta$ and for all $0 < \theta < 1$ that verifies (13). □

Remark 1: This result relates the size of the region to which the trajectories converge with the size of the perturbations. The smaller the size of the perturbations the smaller the size of the region, as expected. Note that for the case where $K_i = K_j,\ \Delta_K = 0$ and condition (13) is automatically verified. □

Remark 2: Bounds on δ_{ij} can be easily obtained in the case where all polytopic regions are bounded, by noticing that $\|z(kT) - z(t)\| \leq \max_{x \in \mathcal{R}_i, y \in \mathcal{R}_j}\|x - y\|$. These bounds are however potentially conservative and better ways of obtaining them should be investigated. In particular, the bound should depend on the sampling period T. □

The next section relates the bound on $\|z(kT) - z(t)\|$ to the sampling period T and offers a less conservative result that enables us to prove that if the sampling period converges to zero then the system is practically exponentially stable to the origin and the continuous-time behavior is recovered.

B. Conditions Dependent of the Sampling Period

Integrating equation (7) for $t \in [kT, (k+1)T]$ yields

$$z(t) - z(kT) = \int_{kT}^{t} A_{i(\tau)} z(\tau) d\tau + \int_{kT}^{t} b_{i(\tau)} d\tau + \int_{kT}^{t} B_{i(\tau)} d\tau \left(K_j z(kT) + m_j\right) \quad (14)$$

Therefore, letting $A = \max_{i=1,\ldots,M} \|A_i\|$, $b = \max_{i=1,\ldots,M} \|b_i\|$, $B = \max_{i=1,\ldots,M} \|B_i\|$ yields

$$\|z(t) - z(kT)\| \leq A \int_{kT}^{t} \|z(\tau)\| d\tau + (t - kT)\left(b + B\|K_j z(kT) + m_j\|\right) \quad (15)$$

Since all possible dynamics in a PWA system are affine, finite escape times cannot occur and therefore there will be a finite constant $Z(k,T) > 0$ such that

$$\|z(t)\|_{kT \leq t \leq kT+T} \leq Z(k,T) \quad (16)$$

Using the bound (16) in expression (15) leads to

$$\|z(t) - z(kT)\| \leq (t - kT)(AZ(k,T) + b + B\|K_j z(kT) + m_j\|) \quad (17)$$

Remark 3: Note that the smaller the sampling time T the smaller the bound $Z(k,T)$ and when $T \to 0$, $Z(k,T) \to \|z(kT)\|$. Note further that the Euler approximation for integration would lead to $Z(k,T) = \|z(kT)\|$ because $\int_{kT}^{t} \|z(\tau)\| d\tau \simeq \|z(kT)\|(t - kT)$. □

Letting $K = \max_{i=1,\ldots,M} \|K_i\|$, $m = \max_{i=1,\ldots,M} \|m_i\|$ yields

$$\|z(t) - z(kT)\| \leq (t - kT)(AZ(k,T) + b + BK\|z(kT)\| + Bm) \quad (18)$$

The worst possible (highest) bound is the one corresponding to $t = (k+1)T$, which leads to

$$\|z(t) - z(kT)\| \leq T\left(AZ(k,T) + b + BK\|z(kT)\| + Bm\right) \quad (19)$$

Recall that the expression for the perturbations developed in (9) was

$$\delta_{ij} = K_j\left(z(kT) - z(t)\right) + \left(K_j - K_i\right) z(t) + \left(m_j - m_i\right). \quad (20)$$

Let now $\Delta_{K_{ij}} = \|K_j - K_i\|$, $\Delta_{m_{ij}} = \|m_j - m_i\|$. Then we can write

$$\|\delta_{ij}\| \leq K\|z(t) - z(kT)\| + \Delta_{K_{ij}}\|z(t)\| + \Delta_{m_{ij}} \quad (21)$$

and therefore using (19) this finally yields

$$\|\delta_{ij}\| \leq N_{ij}(k,T) + \Delta_{K_{ij}}\|z\|, \; i,j = 1,\ldots,M \quad (22)$$

where

$$N_{ij}(k,T) = \Delta_{m_{ij}} + KT\left(AZ(k,T) + BK\|z(kT)\| + \bar{b}\right) \quad (23)$$

and $\bar{b} = b + Bm$. Using this bound and Theorem 3.1 the following result can now be stated.

Corollary 3.1: Assume a Lyapunov function of the form (10) is found and is defined in $\mathcal{X} \subseteq \mathbb{R}^n$. Let the condition number of P be $\chi(P) = \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}$. Let $N_{ij}(k,T) = \Delta_{m_{ij}} + KT\left(AZ(k,T) + b + BK\|z(kT)\| + Bm\right)$ where $\|z(t)\|_{kT \leq t \leq kT+T} \leq Z(k,T)$ and

$$A = \max_{i=1,\ldots,M} \|A_i\|, \; b = \max_{i=1,\ldots,M} \|b_i\|, B = \max_{i=1,\ldots,M} \|B_i\|,$$
$$\Delta_{K_{ij}} = \|K_j - K_i\|, \; \Delta_{m_{ij}} = \|m_j - m_i\|.$$

Furthermore, let $N(k,T) = \max_{i,j=1,\ldots,M}\left(N_{ij}\right)$, $\Delta_K = \max_{i,j=1,\ldots,M}\left(\Delta_{K_{ij}}\right)$, $\Delta_m = \max_{i,j=1,\ldots,M}\left(\Delta_{m_{ij}}\right)$. Define

$$\mu_\theta(k,T) = \frac{2\chi(P)B}{\alpha\theta - 2\chi(P)B\Delta_K} N(k,T)$$

and the region

$$\mathcal{S}_\theta(k,T) = \{z \in \mathcal{X} \mid \|z\| \leq \mu_\theta(k,T)\}$$

for any positive constant $\theta < 1$ that verifies

$$\Delta_K < \frac{\alpha\theta}{2\chi(P)B}$$

Then, in the absence of sliding modes, it follows that:

1) The trajectories of the closed-loop sampled-data system (8) converge exponentially to the set

$$\Omega(k,T) = \left\{z \in \mathcal{X} \mid V(z) \leq \sigma_{max}(P)\mu_\theta^2(k,T)\right\}.$$

2) When $T \to 0$, the trajectories of the closed-loop sampled-data system (8) are practically exponentially convergent to the origin. By this it is meant that $\mu_\theta(k,T) \to 0$ almost everywhere when $T \to 0$.

Proof: Result 1) follows directly from the proof of Theorem 3.1. Result 2) follows from the facts that:

- In the absence of sliding modes, chattering phenomena is ruled out in closed-loop. Therefore, $z(t)$ cannot stay at a region boundary for any time interval with positive Lebesgue measure.
- From the previous point, we conclude that for any $T > 0$, $k \geq 0$ the Lebesgue measure of the set $S_{k,T} = \{t \in [kT, (k+1)T] \mid \Delta_{K_{ij}}(t) > 0, \; \Delta_{m_{ij}}(t) > 0\}$ is zero since this set is composed of the time instants for which $z(t)$ crosses a region boundary in the time interval $[kT, (k+1)T]$.
- From the previous point, we conclude that $\Delta_{K_{ij}} \to 0$, $\Delta_{m_{ij}} \to 0$ as $T \to 0$ except on a set of time instants that has Lebesgue measure zero. Therefore, the measure of the set of times t for which $i(t), j(t)$ are different converges to zero as $T \to 0$ so $i = j \; a.e$ when $T \to 0$ and $i = j$ when $T = 0$.
- $Z(k,T) \to \|z(kT)\|$ when $T \to 0$ and, as seen before, $\Delta_{m_{ij}} \to 0 \; a.e$ as $T \to 0$. This together with the fact that $\|z(kT)\|$ is bounded for any $k \geq 0$ implies that $N(k,T) \to 0 \; a.e$ when $T \to 0$. Thus $\mu_\theta(k,T) \to 0 \; a.e$ when $T \to 0$, which finishes the proof. □

Remark 4: This result formally establishes the very important and desired property that a sampled-data PWA system converges to a closed-loop continuous-time PWA system when the sampling period converges to zero. As desired, all the stability guarantees for the closed-loop continuous-time

system can be recovered in a very specific sense, which in this case leads to practical exponential convergence to the origin. □

Remark 5: The result assumes the absence of sliding modes. Sliding modes can indeed be ruled out in feedback if the component of the vector fields perpendicular to the boundaries is continuous across the boundaries. This idea was first suggested for PWA systems in [12] to avoid the generation of sliding modes in closed-loop. If the feedback construction suggested in [12] is used, it can be shown following the reasoning explained in [12] that sliding modes are still ruled out in feedback for sampled-data PWA systems if the additional constraints $B_i = B_j = B,\ c_{ij}^T B\left(K_j - K_i + m_j - m_i\right) = 0,\ \forall i = 1, \ldots, M, \forall j \in \mathcal{N}_i$ ***are verified. Notice that these constraints are linear in the controller parameters and can easily be included in the optimization procedure suggested in [12] for systems with a constant input matrix*** B ***(such as the one presented in the example of the next section).*** □

Remark 6: Note that for the case of continuous PWA systems, the continuous vector field from the state equation (2) given by $f(z, u) = A_i z + b_i + B_i u$ ***and*** $f(z(kT), u(kT)) = A_j z(kT) + b_j + B_j u(kT)$ ***is locally Lipschitz in*** z ***with Lipschitz constant*** $L = \max_{i=1,\ldots,M} \|A_i\|$***. In this case, following the ideas presented in [9], the Gronwall-Bellman inequality applied to the integral of the dynamical equation (2) between*** kT ***and*** $t \leq KT + T$

$$
\begin{aligned}
z(t) = z(kT) + (t - KT) f(z(kT), u(kT)) + \\
\int_{kT}^{t} \left[f(z(\tau), u(kT)) - f(z(kT), u(kT)\right] d\tau
\end{aligned}
$$

enables us to show that

$$
\begin{aligned}
\|z(t) - z(kT)\| \leq \frac{1}{L}\left[e^{(t-KT)L} - 1\right] \cdot \\
\|A_j z(kT) + b_j + B_j u(kT)\|,\ kT \leq t \leq KT + T
\end{aligned}
$$

When the control input is replaced by its value $u(kT) = K_j z(kT) + m_j$***, it finally yields the bound***

$$
\|z(t) - z(kT)\| \leq \frac{1}{L}\left[e^{TL} - 1\right]\left[\|\bar{A}_j\| \|z(kT)\| + \|\bar{b}_j\|\right]. \tag{24}
$$

where we have used the fact that $t - KT \leq T$ ***for*** $kT \leq t \leq KT + T$ ***and*** $\bar{A}_j,\ \bar{b}_j$ ***are defined as before. Following the reasoning leading to (23) a new value for*** $N_{ij}(k, T)$ ***can be found as***

$$
N_{ij}(k, T) = \Delta_{m_{ij}} + \frac{K}{L}\left[e^{TL} - 1\right]\left[\|\bar{A}_j\| \|z(kT)\| + \|\bar{b}_j\|\right] \tag{25}
$$

Note that (23) and (25) become very similar ($\bar{A},\ \bar{b}$ ***are replaced by*** $\|\bar{A}_j\|,\ \|\bar{b}_j\|$***) for very small*** T ***if the Euler approximation is used in (23) to approximate*** $Z(k, T)$ ***by*** $\|z(kT)\|$***. Note however that (23) is more general and less restrictive than (25) because it is valid even for discontinuous PWA systems that are therefore not locally Lipschitz. The important point to make is that from expression (25) when*** $T \to 0,\ N_{ij} \to 0\ a.e.$ ***since*** $\|z(kT)\|$ ***is bounded and*** $z(kT) \to z(t)$ ***so that*** i, j ***become the same, except on a set of measure zero. This leads to the same result obtained in Corollary 3.1 when one replaces (23) by (25) for the special case of continuous PWA systems.*** □

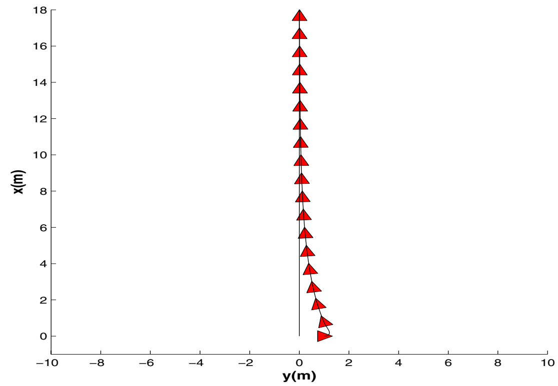

Fig. 1. $x - y$ trajectory with continuous-time controller, $\psi_0 = \frac{\pi}{2}$, $r_0 = 0$ rad/s, $y_0 = 1$ m

IV. EXAMPLE

The objective of this example is to design a controller that forces a cart on the $x - y$ **plane to follow the straight line** $y = 0$ **with a constant velocity** $U_0 = 1$ **m/s. It is assumed that a controller has already been designed to maintain a constant forward velocity. The cart's path is then controlled by the torque** u **about the** z**-axis according to the following dynamics:**

$$
\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{k}{I} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ U_0 \sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \\ 0 \end{bmatrix} u, \tag{26}
$$

where ψ **is the heading angle with time derivative** r, $I = 1$ **Kgm**^{2} **is the moment of inertia of the cart with respect to the center of mass and** $k = 0.01$ **Nms is the damping coefficient. Note that for this example** $B_i = B_j = B,\ c_{ij}^T = [1\ 0\ 0],\ c_{ij}^T B = 0$**. The state of the system is** $(z_1, z_2, z_3) = (\psi, r, y)$**. Assume the trajectories can start from any possible initial angle in the range** $\psi_0 \in [-\frac{3\pi}{5}, \frac{3\pi}{5}]$ **and any initial distance from the line. The function** $\sin(\psi)$ **is approximated by a PWA function (see [14]) yielding**

$$
\begin{aligned}
\mathcal{R}_1 &= \left\{ z \in I\!R^3 \mid z_1 \in \left(-\frac{3\pi}{5}, -\frac{\pi}{5}\right) \right\}, \\
\mathcal{R}_2 &= \left\{ z \in I\!R^3 \mid z_1 \in \left(-\frac{\pi}{5}, -\frac{\pi}{15}\right) \right\}, \\
\mathcal{R}_3 &= \left\{ z \in I\!R^3 \mid z_1 \in \left(-\frac{\pi}{15}, \frac{\pi}{15}\right) \right\},
\end{aligned}
$$

and $\mathcal{R}_4$ **is symmetric to** $\mathcal{R}_2$ **and** $\mathcal{R}_5$ **is symmetric to** $\mathcal{R}_1$**, all with respect to the origin. A controller was designed to stabilize the origin (inside region** $\mathcal{R}_3$**) yielding**

$$
\begin{aligned}
K_1 &= [\ -49.908 \quad -9.467 \quad -13.926\ ] \quad m_1 = 2.70 \times 10^{-6} \\
K_2 &= [\ -48.316 \quad -9.330 \quad -13.812\ ] \quad m_2 = 3.75 \times 10^{-7} \\
K_3 &= [\ -50.148 \quad -9.468 \quad -13.742\ ] \quad m_3 = 0.00 \times 10^{0} \\
K_4 &= [\ -48.316 \quad -9.330 \quad -13.812\ ] \quad m_4 = -m_2 \times 10^{0} \\
K_5 &= [\ -49.908 \quad -9.468 \quad -13.926\ ] \quad m_5 = -m_1 \times 10^{0}
\end{aligned}
$$

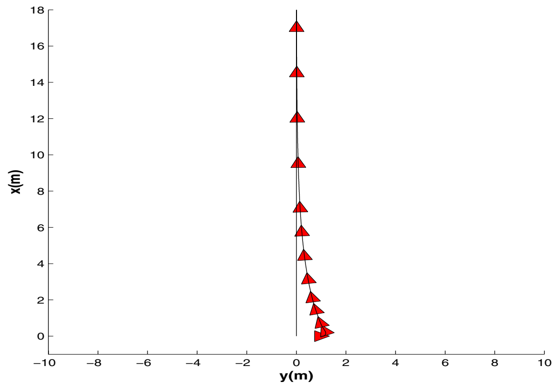

Fig. 2. $x - y$ trajectory for a sampling period of $T = 0.05s$

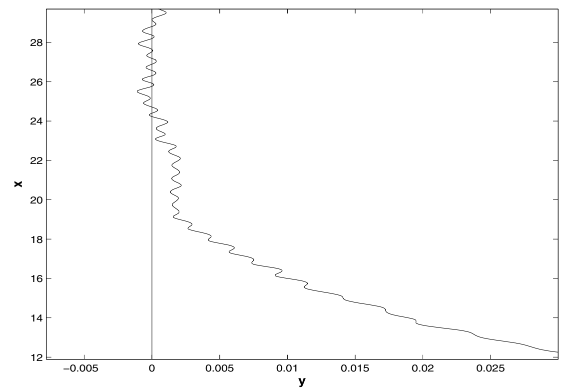

Fig. 3. $x - y$ trajectory zoomed for a sampling period of $T = 0.2s$

The trajectory in the $x - y$ plane using this controller is shown in figure 1 where it is clear that the controller makes the cart trajectory converge to the desired straight line. For a sampling period of $T = 0.05s$ the same controller was emulated in discrete-time between a sampler and a zero-order-hold and the results of the corresponding $x - y$ trajectory are shown in figure 2. It can be seen that the trajectory still follows approximately the one obtained with the continuous-time controller. When the sampling period is further increased to $T = 0.2s$ the simulation of the $x - y$ trajectory close to the line is zoomed in figure 3. It is clear that the trajectory converges to a region around the desired straight line, as predicted by the results of this paper.

V. Conclusions

This paper has presented for the first time stability results for closed-loop sampled-data PWA systems under state feedback. It was shown that the emulation of a state feedback controller designed in continuous-time to exponentially stabilize the system to a target point would still exponentially stabilize the system to a region around the target point. The size of this region was related to the sampling period. It was shown that when the sampling period converges to zero the exponential stability results for the closed-loop continuous-time system are recovered.

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