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Error Feedback Output Regulation of Bounded Uniformly Continuous Signals for Infinite-Dimensional Linear Systems

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Abstract— **In this article we present sufficient conditions for the existence of an error feedback controller which solves an output regulation problem for infinite-dimensional systems and bounded uniformly continuous reference/disturbance signals. The given sufficient conditions involve strong stability of the closed loop semigroup and solvability of two pairs of regulator equations; one for the plant and one for the error feedback controller. The proof of this sufficient condition relies on an argument originally due to Francis, in which the error feedback output regulation problem is solved as a feedforward control problem for the extended system. A delay-differential equation example is presented to illustrate the theory.**

I. Introduction

The term output regulation is usually associated to stabilization of a dynamical system and asymptotic tracking of a given class of reference signals under a given class of disturbances. Controllers which achieve output regulation are often either of feedforward or feedback type. For finite-dimensional linear systems and simple reference/disturbance signals generated by systems of linear ordinary differential equations, such output regulation problems were studied intensively in the 1970s. A complete solution now exists for both feedback and feedforward controllers, e.g. in the work of Francis, Wonham and Davison [7], [9], [10], [22].

The work of Francis and Wonham initiated what is nowadays known as geometric output regulation theory. This terminology stems from the fact that they studied output regulation problems in geometric terms, such as subspace inclusions. During the past two decades several authors have extended this theory for infinite-dimensional systems. In the early 1980s Schumacher [21] constructed finite-dimensional controllers for infinite-dimensional plants in which the system operator has compact resolvent and a complete set of generalized eigenfunctions. His solution of the output regulation problem is also expressed in geometric terms (cf. Theorem 3.1 in [21]). Several years later Byrnes et al. [3] generalized Francis' theory [9] for infinite-dimensional systems in such a way that the geometric conditions were replaced by the so called regulator equations. These equations (which are in fact also present in the finite-dimensional case [9]) express the geometric conditions in an algebraic way, hence simplifying solution of the output regulation problem. Byrnes et al. [3] showed that solvability of the feedforward regulation problem, solvability of the error feedback regulation problem and solvability of the regulator equations are all equivalent to each other provided that the plant and the finite-dimensional exogenous signal generator have sufficient stabilizability properties.

In the above approaches towards solving output regulation problems the class of admissible signals is rather small. The reference/disturbance signals are assumed to be generated by finite-dimensional exogenous systems, and they can be e.g. ramps, constant signals or sinusoids. Recently the feedforward output regulation theory of Byrnes et al. [3] has been generalized for infinite-dimensional exogenous systems in [4], [5], [11], [13], [14]. In [4], [5] the 1-D wave equation is generating reference signals. In the present paper, as well as in [11], [13], [14], the exogenous system is *purpose-built* so that any bounded and uniformly continuous reference functions can be treated. This class of reference signals is essentially larger than those resulting from the use of finite-dimensional exosystems; for example it is possible to consider the asymptotic tracking of arbitrary sufficiently smooth periodic reference signals.

In this article we provide sufficient conditions for the solvability of the error feedback regulation problem for bounded uniformly continuous signals. The sufficient conditions that we obtain in our main result (Theorem 4.1) are closely related to the well-known ones: Suitable closed loop stabilty and solvability of certain regulator equations guarantee solvability of the output regulation problem. The fundamental idea in our proof of these sufficient conditions is originally due to Francis [9]. We first provide sufficient conditions for the solvability of a related feedforward regulation problem in Theorem 3.1. As we observe that the error feedback regulation problem can be considered a feedforward regulation problem for *the extended system*, the sufficient conditions in Theorem 3.1 are easily adapted for the error feedback regulation problem to yield Theorem 4.1.

The sufficient conditions in Theorem 4.1, which guarantee solvability of the error feedback regulation problem, are more general than the existing ones, e.g. those provided in [3] for finite-dimensional exogenous systems. Instead of exponential stability, we only require the closed semigroup to be strongly stable. Furthermore, we do not fix the parameters F, G and J of the error feedback controller, but we show that they only need to satisfy another set of regulator equations — the regulator equations for the error feedback controller. We shall show in Corollary 4.2 that one possible choice then is an observer-based error feedback controller, as shown by Byrnes et al. [3] for the case of finite-dimensional exogenous

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systems.

The reason for our requiring only strong stability of the closed loop semigroup lies in the fact that the controller dynamics may contain a stabilized copy of the exogenous system dynamics in accordance with the internal model principle [9], [10], [22]. In the case of a finite-dimensional exogenous system [3] this is not a problem — it is reasonable to require the closed loop semigroup to be exponentially stable. However, in our setting one may be required to stabilize an operator having an unbounded spectrum on the imaginary axis (see the example in Section V). Such an operator cannot be exponentially stabilized using compact feedback operators (cf. Corollary 3.58 in [16]). However, it is well-known that in many cases such operators can be strongly stabilized [2].

A. Notation and preliminaries

For Banach spaces E and F, $\mathcal{L}(E,F)$ denotes the space of bounded linear operators $E \to F$. The resolvent set of a closed linear operator $A : E \to F$ is denoted by $\rho(A)$. $R(\lambda, A)$ denotes (whenever it exists) the resolvent operator $(\lambda I - A)^{-1}$. If $\widetilde{E}$ is a subspace of E, then $A|_{\widetilde{E}}$ denotes the restriction of A to $\widetilde{E}$.

A strongly continuous (or C_0) semigroup $T_A(t)$ in E is strongly stable if $\lim_{t\to\infty} \|T_A(t)x\| = 0$ for each $x \in E$. $T_A(t)$ is weakly stable if $\lim_{t\to\infty} f(T_A(t)x) = 0$ for each $x \in E$ and every functional $f \in \mathcal{L}(E, \mathbb{C})$. If A generates a C_0-semigroup in E and $B \in \mathcal{L}(F,E)$, then the pair (A,B) is strongly stabilizable if there exists $K \in \mathcal{L}(E,F)$ such that $A+BK$ generates a strongly stable semigroup in E. If $C \in \mathcal{L}(E,F)$, then the pair (A,C) is approximately observable if $\{\, x \in E \mid CT_A(t)x = 0 \text{ for all } t \geq 0 \,\} = \{0\}$. $\Re(z)$ denotes the real part of a complex number z. $BUC(\mathbb{R},E)$ denotes the Banach space (with respect to sup-norm) of bounded uniformly continuous functions $f : \mathbb{R} \to E$.

II. The Output Regulation Problems

In this section we define the infinite-dimensional plant and the infinite-dimensional exogenous system which we assume is generating the reference and disturbance signals. We also define two output regulation problems: The feedforward and the error feedback regulation problems. We shall use the former to solve the latter in Section IV.

A. The plant

We consider a plant described by the following possibly infinite-dimensional control system (for $t \geq 0$):

$$\dot{z}(t) = Az(t) + Bu(t) + \mathcal{U}_{dist}(t), \quad z(0) \in Z \tag{1a}$$

$$y(t) = Cz(t) + Du(t) \tag{1b}$$

Here A generates a C_0-semigroup $T_A(t)$, $t \geq 0$, in a complex Banach space Z. The continuous input $u : \mathbb{R}_+ \to H$ and continuous output $y : \mathbb{R}_+ \to H$ take values in a complex Banach space H. The control operator $B \in \mathcal{L}(H,Z)$, the observation operator $C \in \mathcal{L}(Z,H)$ and the feedthrough operator $D \in \mathcal{L}(H)$. The continuous function $\mathcal{U}_{dist}$ is a disturbance (to be defined shortly). Equation (1a) is to be considered in the mild sense.

B. The exogenous system

Let $\mathcal{H}$ be a Banach space which is continuously embedded in $BUC(\mathbb{R},H)$ (observe that $\mathcal{H}$ need not have the sup$-$norm). In symbols, let $\mathcal{H} \hookrightarrow BUC(\mathbb{R},H)$. We let $T_S(t)$ denote the shift C_0-group in $BUC(\mathbb{R},H)$ defined as $T_S(t)f = f(\cdot + t)$; its infinitesimal generator is $S = \frac{d}{dx}$ with a suitable domain of definition $\mathcal{D}(S) \subset BUC(\mathbb{R},H)$. If $\mathcal{H}$ is invariant for $T_S(t)$ and the restrictions $T_S(t)|_{\mathcal{H}}$ to $\mathcal{H}$ constitute an isometric C_0-group we denote this by $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R},H)$. In this case the generator of $T_S(t)|_{\mathcal{H}}$ is denoted by $S|_{\mathcal{H}}$.

Let $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R},H)$, and let Q denote the point evaluation at the origin in $\mathcal{H}$, i.e $Qf = f(0)$ for $f \in \mathcal{H}$. Clearly Q is linear and $\|Qf\|_H = \|f(0)\|_H \leq \|f\|_\infty \leq c\|f\|_{\mathcal{H}}$ for some $c \geq 0$ (because $\mathcal{H} \hookrightarrow BUC(\mathbb{R},H)$) and so $Q \in \mathcal{L}(\mathcal{H},H)$. Moreover, $QT_S(t)|_{\mathcal{H}} f = f(x+t)|_{x=0} = f(t)$ for every $f \in \mathcal{H}$ and $t \in \mathbb{R}$. This leads us to the following definition.

Definition 2.1 (The exogenous system): Assume that $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R},H)$ and let $P \in \mathcal{L}(\mathcal{H},Z)$ be a known disturbance operator. The exogenous system generating reference signals y_{ref} and disturbance signals $\mathcal{U}_{dist}$ is defined on the state space $\mathcal{H}$ as

$$\dot{w}(t) = S|_{\mathcal{H}} w(t), \quad w(0) = w_0 \in \mathcal{H} \tag{2a}$$

$$y_{ref}(t) = Qw(t) \tag{2b}$$

$$\mathcal{U}_{dist}(t) = Pw(t) \tag{2c}$$

where equation (2a) is to be considered in the mild sense.

By the above discussion it is easy to see that every reference signal y_{ref} in $\mathcal{H}$ can be obtained from (2b) by choosing $w(0) = y_{ref}$. Throughout the rest of this article, we assume that $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R},H)$ is fixed.

C. The feedforward regulation problem (FRP)

The task in the feedforward regulation problem is to find operators $K \in \mathcal{L}(Z,H)$ and $L \in \mathcal{L}(\mathcal{H},H)$ having the following properties.

1) The pair (A,B) is strongly stabilizable using K, i.e. $A + BK$ generates a strongly stable C_0-semigroup $T_{A+BK}(t)$ on Z.
2) As the control law $u(t) = Kz(t) + Lw(t)$ is applied, in the extended system on $Z \times \mathcal{H}$ given for $t \geq 0$ by

$$\dot{z}(t) = (A+BK)z(t) + (BL+P)w(t) \tag{3a}$$

$$\dot{w}(t) = S|_{\mathcal{H}} w(t) \tag{3b}$$

the tracking error $e(t) = y(t) - y_{ref}(t) = (C + DK)z(t) + (DL - Q)w(t) \to 0$ as $t \to \infty$ regardless of the initial conditions $z(0) \in Z$ and $w(0) \in \mathcal{H}$.

We hasten to emphasize that in the formulation of the feedforward regulation problem above, we do not require exponential stabilizability of the pair (A,B), as in [3], [14]. The concept of strong stability is more general than that of exponential stability, and it may be the more realistic one in applications. For example, for many partial differential equations it is known that solutions are stable, but no uniform decay rate exists [17]. Furthermore, that we do not require exponential stabilizability of (A, B) turns out to be important in solution of the error feedback regulation problem described in the next subsection.

D. The error feedback regulation problem (EFRP)

In practice the state $z(t)$ of the plant, which is used in the FRP, is usually not directly available for measurement. In this subsection we pose an output regulation problem which only relies on information on the outputs of the systems.

In the EFRP we seek an error feedback controller

$$\dot{x}(t) = Fx(t) + Ge(t), \quad x(0) \in X, \quad t \geq 0 \tag{4a}$$

$$u(t) = Jx(t) \tag{4b}$$

on some Banach state space X where F generates a C_0-semigroup, $G \in \mathcal{L}(H, X)$ and $J \in \mathcal{L}(X, H)$. The controller must satisfy the following requirements:

1) In the closed loop system for $t \geq 0$

$$\dot{z}(t) = Az(t) + BJx(t) + Pw(t) \tag{5a}$$

$$\dot{x}(t) = GCz(t) + (F + GDJ)x(t) - GQw(t) \tag{5b}$$

$$\dot{w}(t) = S|_{\mathcal{H}} w(t) \tag{5c}$$

$$e(t) = Cz(t) + DJx(t) - Qw(t) \tag{5d}$$

the semigroup $T_{\mathcal{A}}(t)$ generated by $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix}\right)$ in $Z \times X$ is strongly stable.

2) The tracking error $e(t) \to 0$ as $t \to \infty$ for any initial conditions $z(0) \in Z$, $x(0) \in X$ and $w(0) \in \mathcal{H}$.

Byrnes et al. [3] considered a similar error feedback regulation problem for finite-dimensional exosystems. However, they require exponential stability of $T_{\mathcal{A}}(t)$ and $D = 0$. It turns out that in our case exponential stability of $T_{\mathcal{A}}(t)$ is difficult to achieve if we plan to use the controller proposed by Byrnes et al. [3]. This is because their operator F contains a copy of $S|_{\mathcal{H}}$ which in our case is difficult to stabilize (see the example in Section V). In fact, it is well known that if $\mathcal{H}$ is infinite-dimensional, then $S|_{\mathcal{H}} + \Delta$ does not generate an exponentially stable C_0-semigroup for any compact operator $\Delta \in \mathcal{L}(\mathcal{H})$ (cf. Corollary 3.58 in [16]).

III. Sufficient conditions for the solvability of the FRP

In this section we show that if the so called regulator equations can be solved, and the pair (A, B) can be strongly stabilized, then the FRP can be solved.

Theorem 3.1: Assume that the pair (A, B) is strongly stabilizable using $K \in \mathcal{L}(H, Z)$. If there exist $\Pi \in \mathcal{L}(\mathcal{H}, Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H}, H)$ such that $\Pi(\mathcal{D}(S|_{\mathcal{H}})) \subset \mathcal{D}(A)$ and the following regulator equations are satisfied:

$$A\Pi + B\Gamma + P = \Pi S|_{\mathcal{H}} \quad \text{in } \mathcal{D}(S|_{\mathcal{H}}) \tag{6a}$$

$$C\Pi + D\Gamma = Q \quad \text{in } \mathcal{H} \tag{6b}$$

then the control law $u(t) = Kz(t) + (\Gamma - K\Pi)w(t)$ solves the FRP.

Proof: Since by assumption $A + BK$ generates the strongly stable C_0-semigroup $T_{A+BK}(t)$, we only need to verify the condition 2 in the definition of the FRP. Let $L = \Gamma - K\Pi \in \mathcal{L}(\mathcal{H}, H)$. Since $\Pi S|_{\mathcal{H}} = (A + BK)\Pi + BL + P$ in $\mathcal{D}(S|_{\mathcal{H}})$, we have that $\Pi S|_{\mathcal{H}} - (A + BK)\Pi = BL + P$ in $\mathcal{D}(S|_{\mathcal{H}})$ and hence

$$\int_0^t T_{A+BK}(t-\tau)(BL + P)T_S(\tau)|_{\mathcal{H}} w \, d\tau = \tag{7}$$

$$\int_0^t \frac{d}{d\tau} T_{A+BK}(t-\tau)\Pi T_S(\tau)|_{\mathcal{H}} w \, d\tau = \tag{8}$$

$$\Pi T_S(t)|_{\mathcal{H}} w - T_{A+BK}(t)\Pi w \tag{9}$$

for every $w \in \mathcal{D}(S|_{\mathcal{H}})$ and $t \geq 0$. By suitable density arguments it is in fact true that

$$\int_0^t T_{A+BK}(t-\tau)(BL + P)T_S(\tau)|_{\mathcal{H}} w \, d\tau = \tag{10}$$

$$\Pi T_S(t)|_{\mathcal{H}} w - T_{A+BK}(t)\Pi w \tag{11}$$

for every $w \in \mathcal{H}$ and $t \geq 0$.

Consider then the composite operator $\mathcal{A}$ on the extended state space $Z \times \mathcal{H}$ (see (3)) defined as

$$\mathcal{A} = \begin{pmatrix} A + BK & BL + P \\ 0 & S|_{\mathcal{H}} \end{pmatrix} \tag{12}$$

Since $A + BK$ generates the C_0-semigroup $T_{A+BK}(t)$ on Z and $S|_{\mathcal{H}}$ generates the C_0-(semi)group $T_S(t)|_{\mathcal{H}}$ on $\mathcal{H}$, it is clear that $\mathcal{A}$ generates a C_0-semigroup $T_{\mathcal{A}}(t)$ on $Z \times \mathcal{H}$, because $BL + P \in \mathcal{L}(\mathcal{H}, Z)$ (see also [6] Lemma 3.2.2). An easy calculation reveals that this semigroup is given by

$$T_{\mathcal{A}}(t) = \tag{13}$$

$$\begin{pmatrix} T_{A+BK}(t) & \int_0^t T_{A+BK}(t-\tau)(BL + P)T_S(\tau)|_{\mathcal{H}} d\tau \\ 0 & T_S(t)|_{\mathcal{H}} \end{pmatrix} = \tag{14}$$

$$\begin{pmatrix} T_{A+BK}(t) & \Pi T_S(t)|_{\mathcal{H}} - T_{A+BK}(t)\Pi \\ 0 & T_S(t)|_{\mathcal{H}} \end{pmatrix} \tag{15}$$

Let $z(0) = z_0 \in Z$ and $w(0) = w_0 \in \mathcal{H}$ be arbitrary. Then

$$T_{\mathcal{A}}(t)\left(\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right) = \begin{pmatrix} T_{A+BK}(t)(z_0 - \Pi w_0) + \Pi T_S(t)|_{\mathcal{H}} w_0 \\ T_S(t)|_{\mathcal{H}} w_0 \end{pmatrix} \tag{16}$$

Since by (6b) we have $(C + DK)\Pi + DL - Q = C\Pi + D\Gamma - Q = 0$, the explicit expression for the norm of the tracking error $e(t)$, $t \geq 0$, is as follows:

$$\|e(t)\| = \|(C + DK)T_{A+BK}(t)(z_0 - \Pi w_0) \tag{17}$$

$$+ (C\Pi + D\Gamma - Q)T_S(t)|_{\mathcal{H}} w_0\| \tag{18}$$

$$= \|(C + DK)T_{A+BK}(t)(z_0 - \Pi w_0)\| \tag{19}$$

Since $T_{A+BK}(t)$ is strongly stable and $C + DK \in \mathcal{L}(Z, H)$, we have that $e(t) \to 0$ as $t \to \infty$. This shows that the tracking condition in the definition of the FRP is also satisfied. The proof is then complete. ■

Remark 3.2: In the SISO case it is sufficient in Theorem 3.1 for asymptotic tracking to occur that the pair (A, B) is merely weakly stabilizable. In this case $C + DK \in \mathcal{L}(Z, \mathbb{C})$ and so for every $z \in Z$ $\lim_{t\to\infty}(C + DK)T_{A+BK}(t)z = 0$ (see (19)).

Remark 3.3: If the pair (A,B) can be exponentially stabilized, then also the converse of Theorem 3.1 holds [3], [11].

IV. Sufficient conditions for the solvability of the EFRP

In this section we present out main results, namely sufficient conditions for the solvability of the EFRP. In Theorem 4.1 we obtain a rather complete description of suitable error feedback controllers (4) in terms of solutions to another regulator equations — the so called regulator equations for the error feedback controller — which have the same form as the regulator equations (6) for the plant. We point out that to our knowledge no such description has been available for infinite-dimensional plants: Byrnes et al. [3] for example give *one possible choice* for the stabilizing and regulating dynamic controller (4) for a finite-dimensional exosystem. Notable in the proof of Theorem 4.1 is the use of a marvelous argument due to Francis (see also Byrnes et al. [3]): The error feedback regulation problem is interpreted as a feedforward regulation problem for the extended system and this problem is then solved in terms of the FRP and Theorem 3.1.

Theorem 4.1: Assume that there exist $\Pi \in \mathcal{L}(\mathcal{H},Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H},H)$ such that $\Pi(\mathcal{D}(S|_{\mathcal{H}})) \subset \mathcal{D}(A)$ and the regulator equations (6) are satisfied. If the parameters F, G, and J of the controller (4) can be chosen such that

1) The operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}}(t)$ on $Z \times X$.
2) There exists $\Lambda \in \mathcal{L}(\mathcal{H},X)$ such that $\Lambda(\mathcal{D}(S|_{\mathcal{H}})) \subset \mathcal{D}(F)$ and the following regulator equations for the error feedback controller are satisfied:

$$F\Lambda = \Lambda S|_{\mathcal{H}} \quad \text{in } \mathcal{D}(S|_{\mathcal{H}}) \tag{20a}$$

$$J\Lambda = \Gamma \quad \text{in } \mathcal{H} \tag{20b}$$

Then with this triplet (F,G,J) the EFRP is solvable.

Proof: Let $\Theta(t) = \begin{pmatrix} z(t) \\ x(t) \end{pmatrix} \in Z \times X$ and define

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{21}$$

$$\mathcal{P} = \begin{pmatrix} P \\ -GQ \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix}, \quad \mathcal{D} = 0 \tag{22}$$

Then consider the following representation of the closed loop system for $t \geq 0$:

$$\dot{\Theta}(t) = \mathcal{A}\Theta(t) + \mathcal{B}u(t) + \mathcal{P}w(t), \quad \Theta(0) \in Z \times X \tag{23a}$$

$$y(t) = \mathcal{C}\Theta(t) + \mathcal{D}u(t) \tag{23b}$$

with $\dot{w}(t) = S|_{\mathcal{H}}w(t)$, $w(0) \in \mathcal{H}$. Since the regulator equations (6) and the regulator equations (20) for the error feedback controller are satisfied, we have $\Pi S|_{\mathcal{H}} = A\Pi + BJ\Lambda + P$ and $\Lambda S|_{\mathcal{H}} = F\Lambda = GC\Pi + (F+GDJ)\Lambda - GQ$ in $\mathcal{D}(S|_{\mathcal{H}})$. Hence

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S|_{\mathcal{H}} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \tag{24}$$

$$Q = \begin{pmatrix} C & DJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \tag{25}$$

where equation (24) is satisfied in $\mathcal{D}(S|_{\mathcal{H}})$ and equation (25) is satisfied in $\mathcal{H}$. This shows that the regulator equations (6) for the system (23) have a solution. Consequently, by Theorem 3.1 and the strong stability assumption, for the system (23) the FRP is solvable for the control law $u(t) = \mathcal{K}\Theta(t) + \mathcal{L}w(t) \equiv 0$ (recall that $\mathcal{B} = 0$ and $\mathcal{D} = 0$). This implies that in (5d) the tracking error $e(t) = Cz(t) + DJx(t) - Qw(t) = (\mathcal{C} + \mathcal{D}\mathcal{K})\Theta(t) + (\mathcal{D}\mathcal{L} - Q)w(t) \to 0$ as $t \to \infty$ regardless of the initial conditions $z(0)$, $x(0)$ and $w(0)$. ■

The following Corollary generalizes Theorem IV.2 in [3]. It presents one possible choice for the triplet (F,G,J) in the solution of EFRP under two stability assumptions and the assumption $D = 0$.

Corollary 4.2: Let $X = Z \times \mathcal{H}$. Assume that $D = 0$ and that the following conditions hold:

1) There exist $\Pi \in \mathcal{L}(\mathcal{H},Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H},H)$ which solve the regulator equations (6).
2) The pair (A,B) is exponentially stabilizable using $K \in \mathcal{L}(Z,H)$.
3) There exist $G_1 \in \mathcal{L}(H,Z)$ and $G_2 \in \mathcal{L}(H,\mathcal{H})$ such that the operator $\mathcal{A}_s = \begin{pmatrix} A-G_1C & P+G_1Q \\ -G_2C & S|_{\mathcal{H}}+G_2Q \end{pmatrix}$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}_s}(t)$ in X.

Let

$$F = \begin{pmatrix} A+BK-G_1C & P+B(\Gamma-K\Pi)+G_1Q \\ -G_2C & S|_{\mathcal{H}}+G_2Q \end{pmatrix} \tag{26a}$$

$$J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix} \tag{26b}$$

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \tag{26c}$$

Then with this choice of (F,G,J) the EFRP is solvable.

Proof: Let $\Lambda = \begin{pmatrix} \Pi \\ I \end{pmatrix} \in \mathcal{L}(\mathcal{H}, Z \times \mathcal{H})$. Then $J\Lambda = K\Pi + \Gamma - K\Pi = \Gamma$ and it is elementary to verify that $\Lambda S|_{\mathcal{H}} = F\Lambda$ in $\mathcal{D}(S|_{\mathcal{H}})$. Hence Λ solves the regulator equations (20) for the error feedback controller having parameters as in (26).

Consider the operator

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix} = \tag{27}$$

$$\begin{pmatrix} A & BK & B(\Gamma-K\Pi) \\ G_1C & A+BK-G_1C & P+B(\Gamma-K\Pi)+G_1Q \\ G_2C & -G_2C & S|_{\mathcal{H}}+G_2Q \end{pmatrix} \tag{28}$$

If we can establish that $\mathcal{A}$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}}(t)$, then the error feedback controller having parameters as in (26) solves the EFRP by Theorem 4.1.

Applying a similarity transform U given as

$$U = \begin{pmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \tag{29}$$

on $Z \times Z \times \mathcal{H}$ to $\mathcal{A}$ we obtain the operator $\widetilde{\mathcal{A}} = U\mathcal{A}U$ having the expression

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A+BK & -BK & -B(\Gamma - K\Pi) \\ 0 & A - G_1C & P + G_1Q \\ 0 & -G_2C & S|_{\mathcal{H}} + G_2Q \end{pmatrix} \tag{30}$$

$$:= \begin{pmatrix} A+BK & \Delta \\ 0 & \mathcal{A}_s \end{pmatrix} \tag{31}$$

By our assumption $\mathcal{A}_s$ generates a strongly stable C_0-semigroup in X and $A+BK$ generates an exponentially stable C_0-semigroup in Z. Clearly the C_0-semigroup generated by $\widetilde{\mathcal{A}}$ on $Z \times X$ is given by

$$T_{\widetilde{\mathcal{A}}}(t) = \begin{pmatrix} T_{A+BK}(t) & \int_0^t T_{A+BK}(t-s)\Delta T_{\mathcal{A}_s}(s)ds \\ 0 & T_{\mathcal{A}_s}(t) \end{pmatrix} \tag{32}$$

Consequently $T_{\widetilde{\mathcal{A}}}(t)$ is strongly stable if

$$\lim_{t\to\infty} \int_0^t T_{A+BK}(t-s)\Delta T_{\mathcal{A}_s}(s)x ds = 0 \quad \forall x \in X \tag{33}$$

But (33) holds by Proposition 5.6.1 in [1]. This proves that also $T_{\mathcal{A}}(t)$ is strongly stable. ■

V. A DELAY-DIFFERENTIAL EQUATION EXAMPLE

Let $a > 0$, $r \neq 0$, $\tau_1 > \tau_2 > 0$. Consider the following scalar delay differential equation with control and observation :

$$\dot{x}(t) = -ax(t) - b\big[x(t-\tau_1) + x(t-\tau_2)\big] + u(t) \tag{34a}$$

$$y(t) = rx(t), \quad t \geq 0 \tag{34b}$$

Our goal is to build, using Corollary 4.2, a dynamic controller (4) which solves the EFRP for this plant and $p-$periodic reference signals in the standard Sobolev spaces $\mathcal{H} = H^{\alpha}_{per}(0,p)$ for $\alpha > \frac{1}{2}$ [15].

Taking initial conditions for $x(\cdot)$ into account, the pair (34) can be formulated as a plant of the form (1) in which $D = 0$ and $\mathcal{U}_{dist} = 0$ [6]. Moreover, it can be shown (see e.g. [6] Lemma 4.3.9) that the transfer function $H(s) = CR(s,A)B$ of this plant is given by

$$H(s) = \frac{r}{s + a + b(e^{-s\tau_1} + e^{-s\tau_2})} \tag{35}$$

for those $s \in \mathbb{C}$ at which the denominator is not equal to zero.

The semigroup generated by A is exponentially stable if and only if $s + a + b(e^{-s\tau_1} + e^{-s\tau_2}) \neq 0$ for all $s \in \{\, z \in \mathbb{C} \mid \Re(z) \geq 0 \,\}$ ([6] Theorem 5.1.7). Ruan and Wei [20] give a complete characterization (in terms of a, b, τ_1 and τ_2) of those instances in which all roots of equation $s+a+b(e^{-s\tau_1} + e^{-s\tau_2}) = 0$ have negative real parts. In their characterization, the parameter b lies on an interval (b_0^-, b_0^+). We assume that the semigroup generated by A is exponentially stable. By the above discussion, then $i\omega_n = i\frac{2\pi n}{p} \in \rho(A)$ and $H(i\omega_n) \neq 0$ for every $n \in \mathbb{Z}$.

It is evident that for every $\alpha > \frac{3}{2}$, $\sum n = -\infty^{\infty} |H(i\omega_n)^{-1}|^2 (1+\omega_n^2)^{-\alpha} < \infty$. Hence for $\alpha > \frac{3}{2}$ we can solve the regulator equations (6) for bounded operators Π and Γ as in [12], [14]. We obtain

$$\Gamma y = \sum y_n \frac{Q\theta_n}{H(i\omega_n)}, \quad \forall y \in \mathcal{H} \tag{36}$$

$$\Pi y = \sum y_n R(i\omega_n, A)B\Gamma\theta_n, \quad \forall y \in \mathcal{H} \tag{37}$$

where $(\theta_n)_{n\in\mathbb{Z}}$ is the natural orthonormal basis of (weighted) exponentials for $\mathcal{H}$ and y_n is the nth Fourier coefficient of y with respect to this basis.

Now $\mathcal{H}$ is a Hilbert space [15] and the left shift in $\mathcal{H}$ is $p-$periodic, i.e. $T_S(t)|_{\mathcal{H}} = T_S(t+p)|_{\mathcal{H}}$ for every $t \in \mathbb{R}$. Furthermore, the adjoint operator $S|_{\mathcal{H}}^* = -S|_{\mathcal{H}}$ and $S|_{\mathcal{H}}$ has compact resolvent [8]. By Corollary 1 in [2], if the pair $(S|_{\mathcal{H}}, Q^*)$ is approximately controllable, then $S|_{\mathcal{H}} - Q^*Q$ generates a strongly stable C_0-semigroup. But the pair $(S|_{\mathcal{H}}, Q^*)$ is approximately controllable if and only if the pair $(-S|_{\mathcal{H}}, Q)$ is approximately observable [6]. If $y \in \mathcal{H}$ is such that $QT_S(-t)|_{\mathcal{H}}y = 0$ for each $t \geq 0$, then $y(-t) = 0$ for each $0 \leq t \leq p$. By periodicity, $y(t)$ must be identically zero, and hence the pair $(-S|_{\mathcal{H}}, Q)$ is approximately observable. In conclusion, $S|_{\mathcal{H}} - Q^*Q$ generates a strongly stable C_0-semigroup.

Finally, we use Corollary 4.2 to deduce that an error feedback controller (4) with

$$F = \begin{pmatrix} A & B\Gamma \\ Q^*C & S|_{\mathcal{H}} - Q^*Q \end{pmatrix} \tag{38a}$$

$$J = \begin{pmatrix} 0 & \Gamma \end{pmatrix} \tag{38b}$$

$$G = \begin{pmatrix} 0 \\ -Q^* \end{pmatrix} \tag{38c}$$

solves the EFRP.

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