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Controllability Issues for the Navier-Stokes Equation on a Rectangle.

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***Abstract*— We study controllability issues for the Navier-Stokes Equation on a two dimensional rectangle with so-called Lions boundary conditions. Rewriting the Equation using a basis of harmonic functions we arrive to an infinite-dimensional system of ODEs. Methods of Geometric/Lie Algebraic Control Theory are used to prove controllability by means of low modes forcing of finite-dimensional Galerkin approximations of that system. Proving the continuity of the "control $\mapsto$ solution" map in the so-called relaxation metric we use it to prove both solid controllability on observed component and $\mathbf{L}^2$-approximate controllability of the Equation (full system) by low modes forcing.**

I. INTRODUCTION

Following part of the work iniciated by A. Agrachev and A. Sarychev in [2], where controllability issues for the Equation considered on the two-dimensional Torus are studied, we can ask ourselves about what can be done in a general two dimensional domain. We have decided to start with a rectangle as the first step to see how boundary conditions change the problem.

We study controllability, by means of low modes forcing, of incompressible 2D Navier-Stokes Equations (NSE) on a two dimensional rectangle with Lions Boudary Conditions. We compare this study with that done in [2] and refer what are the addictional dificulties carried by the boundary conditions.

We deal with the following 2D NS system

$$u_t + (u\cdot\nabla)u + \nabla p = \nu\Delta u + F(x_1,x_2) + v(t,x_1,x_2) \tag{1}$$

$$\nabla\cdot u = 0 \quad \text{in} \quad R \tag{2}$$

$$u\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial R; \tag{3}$$

$$\nabla^\perp\cdot u = 0 \quad \text{on} \quad \partial R \tag{4}$$

Where $R := \{(x_1,x_2)\in\mathbb{R}^2 \mid a_1 < x_1 < a_2; b_1 < x_2 < b_2\}$ and $\nabla^\perp := \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1}\end{pmatrix}$ and $\mathbf{n}$ is the unit normal to the boundary. Our control will be v we suppose to be a degenerate forcing, i.e., v is a finite sum of the form:

$$v = \sum_{k\in\mathcal{K}^1} v_k(t)W_k,$$

where W_k are eigenfunctions of the Stokes operator. So the components $v_k(t),\ k\in\mathcal{K}^1$, $t\in[0,\,T]$ will be our controls we suppose to be measurable essentially bounded functions.

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A natural way to study the NSE is to study its evolution on subspaces of Sobolev spaces, such subspaces depend on the boundary conditions.

We denote by $L^2(R)$ the space of Lebesgue measurable square integrable real functions defined on R and by $\mathbf{L}^2(R)$ the product space $L^2(R)^2$. Similarly $H^1(R) := \{f \in L^2(R) \mid \frac{\partial f}{\partial x_j}\in L^2(R),\ j=1,2\}$ and, $\mathbf{H}^1(R) := H^1(R)^2$.

For the boundary conditions (3)-(4) the spaces

$$\begin{aligned} H &:= \{u\in\mathbf{L}^2(R) \mid \nabla\cdot u = 0 \ \ \&\ \ u\cdot\mathbf{n} = 0 \,\text{on}\, \partial R\};\\ V &:= \text{closure of } \mathcal{D}_1(R) \text{ on } \mathbf{H}^1(R); \qquad (5)\\ D(A) &:= \{u\in\mathbf{H}^2(R) \mid \nabla\cdot u = 0 \ \ \& \qquad (6)\\ &\qquad (u\cdot\mathbf{n} = 0 \wedge \nabla^\perp\cdot u = 0)\,\text{on}\,\partial R\}; \end{aligned}$$

where $\mathcal{D}_1(R) := \{u \in C^\infty(\overline{R}) \mid \nabla\cdot u = 0 \ \ \&\ \ (u\cdot\mathbf{n} = 0 \wedge \nabla^\perp\cdot u = 0)\,\text{on}\, R\}$, are those where we shall consider the evolution of the NSE on.

Mainly we prove that if we control the finite set of modes $\{n\in\mathbb{N}_0^2 \mid 1\le n_1,n_2\le 3\}\setminus\{(3,\,3)\}$ then we obtain

1) Controllability of Galerkin approximations of the infinite-dimensional system associated with (1)-(4) and;
2) So-called *Solid Controllability on Observed Component* and $\mathbf{L}^2$-*Approximate Controllability* for the (full) Equation.

In the midlle some results on the dependence of the solution on initial data are achieved.

We shall not present here some of the proofs because they are long. The interested reader may find more details from those proofs in the preprint [9].

II. CONTROLLABILITY OF FINITE-DIMENSIONAL GALERKIN APPROXIMATIONS

Since the Equation is invariant under translations, from now we consider the rectangle $[0,\,a]\times[0,\,b]$.

A. An Advantage of Lions Boundary Conditions

The eigenfunctions of the Stokes operator depend on the boudary conditions and, it is not always possible to write down those eigenfunctions explicitely as a combination of well-known functions. It turns out that for Lions Boundary Conditions the Stokes operator coincides with the symmetric of the Laplacian and, for the Laplacian we find the basis of eigenfunctions

$$\mathcal{W} := \{W_k \mid k\in\mathbb{N}_0^2\}$$

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where $W_k := \begin{pmatrix} \frac{-k_2\pi}{b}\sin\left(\frac{k_1\pi x_1}{a}\right)\cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a}\cos\left(\frac{k_1\pi x_1}{a}\right)\sin\left(\frac{k_2\pi x_2}{b}\right)\end{pmatrix}$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

B. The Infinite System

Writing u in the basis of eigenfunctions $\mathcal{W}$:

$$u = \sum_{k\in\mathbb{N}_0^2} u_k W_k$$

and, projecting the Equation onto H:

$$P^{\nabla}(u_t + (u\cdot\nabla)u + \nabla p) = P^{\nabla}(\nu\Delta u + F + v)$$

[that, as usually is done, we may rewrite as

$$u_t + Bu = -Au + F + v \quad]$$

we arrive to the infinite-dimensional system of ODEs

$$\begin{aligned}
&\dot{u}_k = \nu\bar{k}u_k + F_k + v_k\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(++)m)^+=k}} -\frac{C^{\wedge}_{m,n}}{\bar{k}}(\bar{n}-\bar{m})\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(--)m)^+=k}} \frac{C^{\wedge}_{m,n}}{\bar{k}}(\bar{n}-\bar{m})\mathrm{sign}(n_1-m_1)\mathrm{sign}(n_2-m_2)\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(-+)m)^+=k}} -\frac{C^{\vee}_{m,n}}{\bar{k}}(\bar{n}-\bar{m})\mathrm{sign}(n_1-m_1)\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(+-)m)^+=k}} \frac{C^{\vee}_{m,n}}{\bar{k}}(\bar{n}-\bar{m})\mathrm{sign}(n_2-m_2)
\end{aligned} \tag{7}$$

where $m \vee n := m_1n_2 + n_1m_2$ $\;m \wedge n := m_1n_2 - n_1m_2$, $C^{\vee}_{m,n} = u_m u_n \frac{\pi^2}{4ab} m\vee n$ and, $C^{\wedge}_{m,n} = u_m u_n \frac{\pi^2}{4ab} m \wedge n$. Under the sum sign for $\sigma, \mu \in \{+,-\}$, $(n(\sigma\mu)m)^+ := (|n_1\sigma m_1|, |n_2\mu m_2|)$. In order to not repeat the long expressions inside the sums, we rewrite the previous system as

$$\begin{aligned}
\dot{u}_k =& \nu\bar{k}u_k + F_k + v_k\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(++)m)^+=k}} C^{++}_{m,n} + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(--)m)^+=k}} C^{--}_{m,n}\\
&+ \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(-+)m)^+=k}} C^{-+}_{m,n} + \sum_{\substack{m,n\in\mathbb{N}_0^2\\ m<n\\ (n(+-)m)^+=k}} C^{+-}_{m,n}.
\end{aligned}$$

C. Galerkin Approximations

Given a finite subset $\mathcal{G} \subset \mathbb{N}_0^2$ of modes, a $\mathcal{G}$-Galerkin Approximation of system (7) is the same system with the addictional condition $m, n, k \in \mathcal{G}$.

In system (7) we can already see an addictional difficulty carried by the boundary conditions: In the periodic case treated in [2] the method used to prove controllability of Galerkin approximations is an Induction procedure, i.e., starting from a system controlled in few modes, after a REC procedure (Reduction+Extraction+Convexification) is obtained the same system with a bigger set of controlled modes with the same closure of attainable sets. In our case such a procedure does not work so well because, contrary to the periodic case where we can prove that controlling two given modes we can control a third one without changing closure of attainable set, in our case controlling two modes we can only see that we can control a third direction (not a direction of a mode) without changing the closure of attainable set. That carries a dificulty: After a REC procedure we do not arrive to a similar system. Hence the REC procedure does not lead to a possible proof by Induction.

D. The FCE Procedure

Here we present the FCE (Factorization+Convexification+Extraction) procedure that allows us to prove the controllability of Galerkin approximaxions by Induction.

First write a $\mathcal{G}$-Galerkin approximation of system (7) in the followin concise form

$$\dot{u} = f(u) + gv, \quad u \in \mathbb{R}^N,\ v \in \mathbb{R}^r \tag{8}$$

where $\mathcal{K}^1$ is the finite set of controlled modes, $N := \#\mathcal{G}$, $r := \#\mathcal{K}^1$ and g is a matrix whose columns are vectors spanning $span(\mathcal{K}^1)$.

1) Factorization: It turns out that factorizing system (8) as

$$\dot{u} = f(u) + g(v^1 + v^2), \quad u \in \mathbb{R}^N,\ v^1, v^2 \in \mathbb{R}^r$$

the closure of the attainable set at time t of system (8) contains the closure of attainable set at time t of the system

$$\dot{u} = f(u + gV^2) + gv^1, \quad u \in \mathbb{R}^N,\ v^1, V^2 \in \mathbb{R}^r \tag{9}$$

Hence we have

Proposition 2.1: System (8) is approximately controllable at time t if so is system (9).

2) Convexification: It is known that a step of convexification does not change closure of attainable set at time t (see [7]), then

Corollary 2.2: System (8) is approximately controllable at time t if so is system $\dot{u} = \tilde{f}(u)$, where $\tilde{f}(u) \in \tilde{C} := Conv\{f(u + gV^2) + gv^1 \mid V^2,\ v^1 \in \mathbb{R}^r\}$.

3) Extraction: An obvious corollary of Corollary 2.2 is that if we select a family $\mathcal{F}_D := \{\tilde{f}_d(u) \mid d \in D\} \subseteq \tilde{C}$ we have

Corollary 2.3: System (8) is approximately controllable at time t if so is system $\dot{u} = \tilde{f}(u)$, where $\tilde{f}(u) \in \mathcal{F}_D$.

Defining for each $j \in \mathbb{N}_0$ the finite set of modes

$$\mathcal{K}^j := \{n \in \mathbb{N}_0^2 \mid 1 \le n_1, n_2 \le j+2\} \setminus \{(j+2,\ j+2)\}$$

and, iterating the FCE procedures we can prove the following theorem

Theorem 2.4: For each $j \in \mathbb{N}_0$ the $\mathcal{K}^j$-Galerkin approximation

$$\dot{u} = f(u) + gv, \quad u \in \mathbb{R}^N,\ v \in \mathbb{R}^r \tag{10}$$

with $\mathcal{K}^1$ as set of controlled modes, $N := \#\mathcal{K}^j,\ r := \#\mathcal{K}^1$ and g being a matrix whose columns are vectors spanning $span(\mathcal{K}^1)$, is approximately controllable at time t.

III. Proof of Theorem 2.4 (Sketch)

Applying the FCE procedure we arrive to Corollary 2.3. Setting $V^2 = 0$ in (9) we obtain $f(u) + gv^1 \in \tilde{C}$. We start by selecting all the directions in $\{f(u) + gv^1,\quad v^1 \in \mathbb{R}^r\}$ from $\tilde{C}$.
For $v^1 = 0$ and $V^2 \in \{v^\lambda_{n,m},\, w^\lambda_{m,n}\} \subset \mathbb{R}^r$ where

$$\begin{aligned}
(w^\lambda_{m,n})_n &= (v^\lambda_{m,n})_n = \lambda;\\
-(w^\lambda_{m,n})_m &= (v^\lambda_{m,n})_m = 1;\\
(w^\lambda_{m,n})_k &= (v^\lambda_{m,n})_k = 0, \quad k \in \mathcal{K}^1 \setminus \{n,\, m\};
\end{aligned}$$

we have that

$$\frac{f_{v^\lambda_{m,n}}(u) + f_{-v^\lambda_{m,n}}(u)}{2} \;=\; f(u) + \lambda\delta_{m,n}$$

and,

$$\frac{f_{w^\lambda_{m,n}}(u) + f_{-w^\lambda_{m,n}}(u)}{2} \;=\; f(u) - \lambda\delta_{m,n},$$

where

$$\begin{aligned}
\delta_{m,n} = \quad & C^{--}_{m,n} e_{(n(--)m)^+} + C^{-+}_{m,n} e_{(n(-+)m)^+}\\
+ \quad & C^{+-}_{m,n} e_{(n(+-)m)^+} + C^{++}_{m,n} e_{(n(++)m)^+},
\end{aligned} \tag{11}$$

belong to $\tilde{C}$ for every $\lambda \in \mathbb{R}$. Now we select, from $\tilde{C}$ the family $\{f(u) + \lambda\delta_{m,n} \mid \lambda \in \mathbb{R}, (m,\, n) \in S_1 \subseteq (\mathcal{K}^1)^2\}$, where

$$\begin{aligned}
S_1 = \{&((1,2),(2,1)); ((1,1),(2,3)); ((1,2),(2,2));\\
&((1,1),(3,2)); ((2,1),(2,2));\\
&((1,1),(1,3)); ((1,1),(3,1))\}.
\end{aligned} \tag{12}$$

So, the union $D := \{f(u) + gv^1,\quad v^1 \in \mathbb{R}^r\} \cup \{f(u) + \lambda\delta_{m,n} \mid \lambda \in \mathbb{R}, (m,\, n) \in S_1\}$ is a subset of $\tilde{C}$. Our final extraction from $\tilde{C}$ is $Conv(D)$.

It turns out that the family $\{e_i, \delta_{m,n} \mid i \in \mathcal{K}^1, \lambda \in \mathbb{R}, (m,\, n) \in S_1\}$ is a family of linearly independent vectors[1] spanning $span(\mathcal{K}^2)$ and that $Conv(D) = span(D)$. Hence system (10) is approximately controllable at time t if so is the system

$$\dot{u} = f(u) + g_1 v, \quad u \in \mathbb{R}^N,\ v \in \mathbb{R}^{r_1} \tag{13}$$

with $\mathcal{K}^2$ as set of controlled modes, $N := \#\mathcal{K}^j,\ r_1 := \#\mathcal{K}^2$ and g_1 being a matrix whose columns are vectors spanning $span(\mathcal{K}^2)$.

Repeating this procedure, we can prove that starting with system (10), but now with $\mathcal{K}^p \quad (p < j)$ as set of controlled modes, we can arrive by a FCE procedure to the same system with $\mathcal{K}^{p+1}$ as set of controlled modes. The proof of this inductive step is a bit technical and envolves quite complicated expressions (as we may guess from the form of the coeficients of system (7)) that we do not present here.

Therefore, after $j-1$ steps of FCE procedure we can arrive to the system

$$\dot{u} = f(u) + g_{j-1} v, \quad u \in \mathbb{R}^N,\ v \in \mathbb{R}^{r_{j-1}}$$

with $\mathcal{K}^j$ as set of controlled modes, $N := \#\mathcal{K}^j = r_{j-1}$ and g_{j-1} being a matrix whose columns are vectors spanning $span(\mathcal{K}^j)$. Such a system is approximately controllable at time t then, so is system (10).

Controllability at time t. The exact controllability at time t follows from approximate controllability at time t and some results from Lie-Algebraic Control Theory we can find in [7].

IV. Continuity of the NSE on the Initial Data

Consider the data

$$(u_0,\, F,\, v,\, \nu) \in H \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, V') \times]0,\, +\infty[.$$

Following the proof of existence and uniqueness of a weak solution for the NSE (with No-Slip Boundary Conditions) presented in [11], we can prove similarly the existence and uniqueness of a weak solution in $L^2(0,\, T,\, V) \cap L^\infty(0,\, T,\, H)$ for our problem. Asking some more regularity on the initial data, namely $(u_0,\, F,\, v,\, \nu) \in V \times L^2(0,\, T,\, H) \times L^\infty(0,\, T,\, H) \times]0,\, +\infty[$, we can prove the existence and uniqueness of a strong solution in $L^2(0,\, T,\, D(A)) \cap L^\infty(0,\, T,\, V)$.

The continuous dependence of the weak and strong solutions on the initial data follows by a standard procedure: we consider two close (in the product topology) quadruples of initial data and prove that the corresponding solutions are close as well. For that we use (like in the proofs of existence and uniqueness) mainly Young inequalities, Gronwall Lemma and some estimates for the bilinear term of the Equation.

V. Continuity on Relaxation Metric

In the study of controllability issues for the full NSE we shall need the continuous dependence of the Equation on the so-called relaxation metric defined as follows:

Definition 5.1: The **relaxation metric** in $L^1([0,\, T],\, \mathbb{R}^d)$ is defined by the norm

$$\|g\|_{rx} := \max_{t_1,\, t_2 \in [0,\, T]} \left\| \int_{t_1}^{t_2} g(\tau)\, d\tau \right\|_{\mathbb{R}^d}, \tag{14}$$

[1] If the rectangle is a square, this is not true ($\delta_{(1,2),(2,1)} \in span\mathcal{K}^1$) but, we can also arrive to a set of linearly independent vectors spanning $span(\mathcal{K}^2)$ iterating two steps of FCE procedure (see preprint [9] for details).

where, and if nothing in contrary is stated, we consider the spaces $\mathbb{R}^d$ $(d \in \mathbb{N}_0)$ endowed with l_1-norm — $\|x\|_{\mathbb{R}^d} = \|x\|_{l_1} := \sum_{i=1}^{d} |x_i|$.

Remark 5.1: It is easy to check that (14) is a semi-norm and, since functions in $L^1([0, T], \mathbb{R}^d)$ coinciding on a set of measure T are identified we can conclude that (14) is a norm.

A. Change of variables

We make the change of variables

$$u = y + \mathbb{I}v$$

where $\mathbb{I}$ is the primitive operator $[\mathbb{I}v](t) := \int_0^t v(\tau)\, d\tau$. From

$$u' = -\nu A u - Bu + F + v$$

we arrive to the equation

$$y' = -\nu A(y + \mathbb{I}v) - B(y + \mathbb{I}v) + F. \tag{15}$$

Note that the function v appears only implicitly in the last equation. Now we forget that $\mathbb{I}v$ is a primitive of an essentially bounded function and replace it by P in the equation. Note that v being a low modes forcing and $\mathbb{I}v$ being a primitive we have $\mathbb{I}v \in C([0, T], D(A))$. But we take P in the larger space $L^4(0, T, D(A))$.

Analogously as we prove existence, uniqueness and continuity on the initial data of a weak, or strong, solution for the NSE we can prove the same for equation (15). In particular, and since we are interested in the controllability of strong solutions, we have that the solution $\mathbb{Y}_s \in L^\infty(0, T, V) \cap L^2(0, T, D(A))$ of (15), with $y(0) = y_0 \in V, F \in L^2(0, T, H), P \in L^4(0, T, D(A))$ and $\nu \in]0, +\infty[$, is continuous as a map from

$$V \times L^2(0, T, H) \times L^4(0, T, D(A)) \times]0, +\infty[$$

to $C([0, T], V)$.

Since our control takes values in a finite-dimensional subspace of $D(A)$, say a space $\mathbb{F}$ of dimension d, from this continuity follows the continuity of the strong solution of the NSE Equation when controls vary on relaxation metric, i.e., the map $\mathbb{S}_{rx}$ giving us the strong solution of the NSE is continuous as a map from

$$V \times L^2(0, T, H) \times L^\infty_{rx}(0, T, \mathbb{F}) \times]0, +\infty[$$

to $C([0, T], V)$, where $L^\infty_{rx}(0, T, \mathbb{F})$"="$L^\infty_{rx}(0, T, \mathbb{R}^d)$ stays for the space resulting from the set $L^\infty(0, T, \mathbb{R}^d)$ endowed with relaxation metric.

VI. Solid Controllability on Observed Component

Definition 6.1: Let $\phi^0 : M^1 \to M^2$ be a continuous map between two finite dimensional C^0-manifolds, $\Omega \subset M^1$ be an open subset with compact closure and, $S \subseteq M^2$ be any subset. We say that $\phi^0(\Omega)$ **covers** S **solidly**, if for some C^0-neighborhood $\mathcal{N}$ of $\phi^0 \mid_{\overline{\Omega}}$ there holds: $S \subseteq \phi(\Omega)$.

Let $\mathcal{O} \subset \mathbb{N}_0^2$ be the finite set of modes we want to observe and, $\Pi_{\mathcal{O}}$ be the projection map from V onto $span\{W_k \mid k \in \mathcal{O}\}$. Define, for each $T > 0$ and each finite subset $\mathbb{F} \subset \mathbb{N}_0^2$, the "end point" map

$$\begin{aligned} \mathbb{E}_T : V \times L^\infty([0, T], \mathbb{R}^{\#\mathbb{F}}) &\to \mathcal{O} \\ (u_0, v) &\mapsto \Pi_{\mathcal{O}} \circ \mathbb{S}_s(u_0, F, v, \nu)(T). \end{aligned}$$

Write system (7) (with $\mathcal{K}^1$ as set of controlled modes) in the form

$$\begin{cases} \dot{u}_k = \mathcal{B}_k(u) + \nu\bar{k}u_k + F_k + v_k & k \in \mathcal{K}^1 \\ \dot{u}_k = \mathcal{B}_k(u) + \nu\bar{k}u_k + F_k & k \notin \mathcal{K}^1 \end{cases} \tag{16}$$

and, for any $N \in \mathbb{N}_0$ define, also, the system

$$N : \begin{cases} \dot{u}_k = \mathcal{B}_k(u) + \nu\bar{k}u_k + F_k + v_k; & k \in \mathcal{K}^N \\ \dot{u}_k = \mathcal{B}_k(u) + \nu\bar{k}u_k + F_k; & k \notin \mathcal{K}^N. \end{cases} \tag{17}$$

that is the same as system (16) with $\mathcal{K}^N$ as the finite set of controlled modes.

Definition 6.2: We shall say that system [(17).N] is **time-**T **solidly controllable in observed component** if for any $u_0 \in V$ and $R > 0$ there exists a family

$$\mathcal{V}_{u_0,R} := \{v_b \in L^\infty([0, T], \mathbb{R}^{\kappa_N}) \mid b \in B_{u_0,R}\}$$

such that $\mathbb{E}_T(u_0, B_{u_0,R}) := \mathbb{E}_T(u_0, \mathcal{V}_{u_0,R})$ covers $\overline{\mathcal{O}}_R(u_0^{\#\mathcal{O}})$ solidly. Where, by $y^{\#\mathcal{O}}$ we mean the projection of y onto $\mathbb{R}^{\#\mathcal{O}} = \mathcal{O}$, $B_{u_0,R}$ is an open relatively compact subset of a C^0-manifold and, $\overline{\mathcal{O}}_R(y)$ is the closed ball

$$\{x \in \mathcal{O} \mid \|x - y\|_{l_1} \le R\} := \{x \in \mathbb{R}^{\#\mathcal{O}} \mid \|x - y\|_{l_1} \le R\}.$$

We may also define open ball $\mathcal{O}_R(y)$ by

$$\{x \in \mathcal{O} \mid \|x - y\|_{l_1} < R\} := \{x \in \mathbb{R}^{\#\mathcal{O}} \mid \|x - y\|_{l_1} < R\}.$$

Proposition 6.1: System [(17).1] is time-T solidly controllable on observed component.

Remark 6.1: Proposition 6.1 implies controllability on observed component and, it follows from Proposition 6.2 (with $N = 1$) below. Indeed given $R > 0$ and $u_0 \in V$, if $T \le T^0$ it is included in the statement of Proposition 6.2 (with $N = 1$), otherwise if $T > T^0$ we apply any control $\bar{v} \in L^\infty([0, T], \mathbb{R}^{\kappa_1})$ (for example $\bar{v} = 0$ — no control) up to time $T - T^0$ arriving to some point $y \in V$. Put $\bar{R} := R + \|y^{\kappa_1} - u_0^{\kappa_1}\|$. Then apply first part of Proposition 6.2 (with $N = 1$ and $T = T^0$) to the pair $(y, \bar{R}) \in V \times]0, +\infty[$. The family "$\mathcal{V}_{y,\bar{R}} \circ \bar{v}$" will do.

Proposition 6.2:

1) For some $T^0 > 0$, every $0 < T \le T^0$ and every $N \in \mathbb{N}_0$ the system [(17).N] is time-T solid controllable on observed component;
2) For each pair $(u_0, R) \in V \times [0, +\infty[$ the family

$$\mathcal{V}_{u_0,R} := \{v_b \mid b \in B_{u_0,R}\}$$

can be chosen satisfying:

- The map $b \mapsto v_b$ is $(B, L^2(0, T, \mathbb{R}^{\kappa_N}))$-continuous and;
- The controls $v_b(t)$ are uniformly (w.r.t. b and t) l_1-bounded: $\|v_b(t)\|_{l_1} \leq A = A(T, R, u_0)$.

Fix $M \in \mathbb{N}_0$ such that $\mathcal{O} \subseteq \mathcal{K}^M$. We prove Proposition 6.2 in two steps. Prove it in the case $N \geq M$ and prove the "back-induction" step: " it holds for N implies it holds for $N-1$" $(N = 2, \ldots, M)$. These steps are the following subsections VI-A and VI-B.

A. First Step. Proposition 6.2: N Big

The proof that the statement of Proposition 6.2 holds for $N \geq M$ is similar to the first step of the proof of Lemma 12.2 of [2] and is based in a rescaling of time and a Degree Theory argument.

B. Second Step. Proposition 6.2: "Back-Induction".

In this subsection we "imitate" a driving using controls on $\mathbb{R}^{\kappa_N}$ by a driving using controls on $\mathbb{R}^{\kappa_{N-1}}$, $N = 2, \ldots, M$, M is fixed and satisfies $\mathcal{O} \subseteq \mathcal{K}^M$. Both drivings leading to the same projection on $\mathbb{R}^{\kappa_{N-1}}$ at final time but, possibly going by paths with projections "far from each other" in the middle. The projection on $V \setminus \mathbb{R}^{\kappa_{N-1}}$ of the paths will be H-close to each other so, at time T the two drivings lead to points close in H-metric. Hence the end points of the projection onto the finite dimensional observed space $\mathcal{O}$ are close. Solid controllability will follow from this closeness and (again) from a Degree Theory argument.
Such imitation is then the key for the prove that if the system [(17).N] is solid controllable in observed component then so is system [(17).N-1].
After we prove this "$N \to N-1$" step it will be clear, from the fact that [(17).M] is solid controllable in observed component (see subsection VI-A), that system (16) is solid controllable in observed component, we just note that the systems [(17).1] and (16) are the same system.

To prove the "back-induction" step "$N \to N-1$" we shall need some lemmas:

1) Useful Lemmas:

Lemma 6.3: Given:

- A finite subset $\mathbb{J} \subset \mathbb{N}_0^2$
- A function $q \in W^{1,\infty}([t_i, t_f], \mathbb{R}^J)$, where $J := \#\mathbb{J}$, such that $q(t_i) = q_i$
- An element $Q_i \in V \setminus \mathbb{R}^J$; $F \in L^\infty(t_i, t_f, H)$. [2]

Then there exists a control $v^J(q, Q_i) \in L^\infty([t_i, t_f], \mathbb{R}^J)$ depending on q and Q_i such that the projection onto $\mathbb{J}$ of the solution of the NSE

$$u_t = -\nu Au - Bu + F + v^J(q, Q_i), \qquad u(t_i) = q_i + Q_i$$

equals q on $[t_i, t_f]$.
Moreover the map $v^J : (q, Q_i) \mapsto v^J(q, Q_i)$ is $(W^{1,2} \times V \setminus \mathbb{R}^J, L^2(t_i, t_f, \mathbb{R}^J)$-continuous.

Another Lemma we shall need is a corollary, not hard to derive, from of the Approximation Lemma we can find in [6]:

Corollary 6.4 (Approximation Corollary): Let $A \subseteq \mathbb{R}^d$ be the convexification of a finite set of points:

$$A := Conv\{p_1, p_2, \ldots, p_r\}$$

and, $\mathcal{V} := \{v(t, b) \in L^\infty([0, T], A) \mid b \in B\}$ be a L^1-continuous family of A-valued functions. Then for each $\varepsilon > 0$ there is $\theta^\varepsilon > 0$ and a family $\mathcal{Z}^\varepsilon := \{z^\varepsilon(t, b) \in L^\infty([0, T], \{p_1, p_2, \ldots, p_r\}) \mid b \in B\}$ of $\{p_1, p_2, \ldots, p_r\}$-valued functions such that

- $\mathcal{Z}^\varepsilon$ is δ-continuous;
- $\mathcal{Z}^\varepsilon$ ε-approximates, uniformly w.r.t. b, the family $\mathcal{V}$ in relaxation metric, i.e., $\forall b \in B \quad \|z^\varepsilon(\cdot, b) - v(\cdot, b)\|_{rx} < \varepsilon$;
- The elements of $\mathcal{Z}^\varepsilon$ are piecewise constant and the number of intervals of constancy is the same for all $b \in B$ and,
- For all $b \in B$ all the intervals of constancy of $z^\varepsilon(\cdot, b)$ have a length not less than $\theta^\varepsilon > 0$.

The only difference from the Approximation Lemma is addiction of the last item.

The last Lemma we want to refer is the following

Lemma 6.5: For $w \in \mathbb{R}, \quad w \geq 3$ we can define in $[0, T]$ a function $\phi_w(\cdot, b)$ depending on the parameter $b \in B$, of our family of controls, with the following properties

- $\phi_w(\cdot, b)$ vanishes at the switching points of the control $v(\cdot, b)$;
- $\phi_w(\cdot, b)$ is $(B, W^{1,2}(0, T, \mathbb{R})$-continuous and;
- $\phi_w(t, b)$ coincides with $\sin(wt)$ in a set of measure not less than $T - \frac{2T}{w}$.

2) Imitation.: We "imitate" a control $z(\cdot, b) \in \mathcal{Z}$ taking values in $\{\pm\Xi e_k, \pm\Xi\delta_{m,n} \mid k \in \mathcal{K}^{N-1}, (m, n) \in S_{N-1}\}$ by a control $z^w(\cdot, b)$ taking values in $\mathbb{R}^{\kappa_{N-1}}$.

Take the solution $u^\infty(\cdot, b)$ of the equation

$$u_t^\infty(\cdot, b) = -\nu Au^\infty - Bu^\infty + F + z(\cdot, b), \quad u(0) = u_0$$

and, consider its projection onto $\mathbb{R}^{\kappa_{N-1}}$:

$$q^\infty(\cdot, b) = P^{\kappa_{N-1}} u^\infty(\cdot, b).$$

Let $\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = T\}$ be the end-points of the intervals of constancy of $z(\cdot, b)$. For $w \geq 3$ define the control $z^w(\cdot, b)$ by recursion in the following way:

[2]It is enough to have $F \in L^2(t_i, t_f, H)$ & $P^J F \in L^\infty(t_i, t_f, span\mathbb{J})$. This last extra condition on F is needed to guarantee that the control v^J, obtained below, is essentially bounded.

- In the first interval of constancy $[\alpha_0,\, \alpha_1]$:

$$z^w(\cdot,\, b) := \begin{cases} z(\cdot,\, b) \\ \qquad \text{if} \quad z(\cdot,\, b) \in \{\pm\Xi e_k \mid k \in \mathcal{K}^{N-1}\}; \\ v^{\kappa_{N-1}}(q_1^\infty(\cdot,\, b) + \sqrt{2\Xi}\phi^w(\cdot,\, b)(e_m \pm e_n),\, U_0) \;^{3} \\ \qquad \text{if} \quad z(\cdot,\, b) \in \{\pm\Xi\delta_{m,n} \mid (m,\, n) \in S_{N-1}\}. \end{cases}$$

where U_0 is the projection of u_0 onto $V \setminus \mathbb{R}^{\kappa_{N-1}}$ and, $q_1^\infty(\cdot,\, b)$ is the restriction of $q^\infty(\cdot,\, b)$ to $[\alpha_0,\, \alpha_1]$;
- If the control $z^w(\cdot,\, b)$ is already defined in the first $p-1$ intervals of constancy (up to α_{p-1}), we define it in the p^{th} interval $[\alpha_{p-1},\, \alpha_p]$ by:

$$z^w(\cdot,\, b) := \begin{cases} z(\cdot,\, b) \\ \qquad \text{if} \quad z(\cdot,\, b) \in \{\pm\Xi e_k \mid k \in \mathcal{K}^{N-1}\} \\ v^{\kappa_{N-1}}(q_p^\infty(\cdot,\, b) + \sqrt{2\Xi}\phi^w(\cdot,\, b)(e_m \pm e_n),\, U^w_{\alpha_{p-1}}) \\ \qquad \text{if} \quad z(\cdot,\, b) \in \{\pm\Xi\delta_{m,n} \mid (m,\, n) \in S_{N-1}\}. \end{cases}$$

where $U^w_{\alpha_{p-1}} := U^w(\alpha_{p-1}))$ and U^w is the projection onto $V \setminus \mathbb{R}^{\kappa_{N-1}}$ of the solution of the equation

$$u_t^w(\cdot,\, b) = -\nu A u^w - P^\nabla B u^w + F + z^w(\cdot,\, b), \qquad u(0) = u_0,\, t \in [0,\, \alpha_{p-1}]$$

and, $q_p^\infty(\cdot,\, b)$ is the restriction of $q^\infty(\cdot,\, b)$ to $[\alpha_{p-1},\, \alpha_p]$.

We shall prove that at time T, $u^w(T)$ goes, uniformly w.r.t. b, to $u^\infty(T)$ in $\mathbf{L}^2$-norm as w goes to ∞, i.e.,

Lemma 6.6: For any $\varepsilon > 0$ there exists $w_\varepsilon \geq 3$ such that

$$\forall b \in B \forall w \geq w_\varepsilon \quad |u^w(T,\, b) - u^\infty(T,\, b)| < \varepsilon.$$

The controllability on observed component will follows from this Lemma and by a Degree Theory argument. The proof of this lemma is quite long and techical and we shall do not present it here. The proof is a variation of that of Proposition 12.4 presented in [2] for the case of periodic conditions.

VII. $\mathbf{L}^2$-Approximate Controllability

The following Proposition says that for any $T > 0$, system (16) is **time-T approximately controllable** in $\mathbf{L}^2$-norm.

Proposition 7.1: For any $u_0 \in V$ and $T > 0$, the attainable set at time T from u_0 of system (16) is dense in H.

[3]Here $v^{\kappa_{N-1}}$ is the control given by Lemma 6.3, for $\mathbb{J} = \mathcal{K}^{N-1}$.

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IX. References

[1] A. A. Agrachev, Y. L. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopaedia of Mathematical Sciences, 87, Springer,2004

[2] A. A. Agrachev, A. V. Sarychev, *Navier-Stokes Equations: Controllability by Means of Low Modes Forcing*, J. math. fluid mech. **7** (2005) 108-152.

[3] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications*, Masson, 1993.

[4] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes Equations and Turbulence*, Encyclopedia of Mathematics and its Applications, Cambridge university Press, 2001.

[5] I. Fonseca, W. Gangbo, *Degree Theory in Analysis and Applications*; Oxford Lectures Series in Mathematics and its Applications, Oxford University Press, 1995.

[6] R.V. Gamkrelidze, *Principles of Optimal Control Theory*, Plenum Press, 1978.

[7] V. Jurdjevic, *Geometric Control Theory*, Cambridge Studies in Advanced Mathematics 51, Cambridge University Press, 1997.

[8] P.L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, vol.I, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, band 181, Springer-Verlag, 1972.

[9] S.S. Rodrigues, *Navier-Stokes Equation on the Rectangle*, Preprint SISSA 23/2005/M (April 2005); and, arXive:math.OC/0504323 v1 15 Apr 2005.

[10] R. Temam, *Infinite-Dimensional dynamical Systems in Mechanics and Physics*, 2nd ed., Applied Mathematical Sciences, 68, Springer, 1997.

[11] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, 2001.

[12] E. Weinam, J.C. Mattingly *Ergodicity for the Navier-Stokes Equation with Degenerate Random Forcing: Finite Dimensional Approximation*, Comm. on Pure and Applied Math., Vol. 54, **1386-1402**, 2001.