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Uniqueness conditions for the infinite-planning horizon Open-Loop Linear Quadratic Differential Game.

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Abstract— **In this note we consider the open-loop Nash linear quadratic differential game with an infinite planning horizon. The performance function is assumed to be indefinite and the underlying system affine. We derive both necessary and sufficient conditions under which this game has a unique Nash equilibrium.**

Keywords: **linear-quadratic games, open-loop Nash equilibrium, affine systems, solvability conditions, Riccati equations.**

I. Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games. In particular in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner et al. [4] and Engwerda [10]). In these problems, the open-loop Nash strategy is often used as one of the benchmarks to evaluate outcomes of the game. In optimal control theory it is well-known that, e.g., the issue to obtain robust control strategies can be approached as a dynamic game problem (see e.g. [2]).

In this note we consider the open-loop linear quadratic differential game. This problem has been considered by many authors and dates back to the seminal work of Starr and Ho in [17] (see, e.g., [15], [16], [5], [12], [11], [1], [18], [6], [7], [3] and [13]). More specifically, we study in this paper the (regular indefinite) infinite-planning horizon case. The corresponding regular definite (that is the case that the state weighting matrices Q_i (see below) are semi-positive definite) problem has been studied, e.g., extensively in [6] and [7]. [13] (see also [14]) studied the regular indefinite case using a functional analysis approach, under the assumption that the uncontrolled system is stable. In particular, these papers show that, in general, the infinite-planning horizon problem does not have a unique equilibrium. Moreover [13] shows that whenever the game has more than one equilibrium, there will exist an infinite number of equilibria. Furthermore the existence of a unique solution is related to the existence of a so-called strongly stabilizing solution of the set of coupled algebraic Riccati equations, see (4) below.

In [9] these results were generalized for stabilizable systems using a state-space approach, for a performance criterion that is a pure quadratic form of the state and control variables. In this note we generalize this result for performance criteria that also include ”cross-terms”, i.e. products of the state and control variables. Performance criteria of this type often naturally appear in economic policy making and have been studied, e.g., in [8] and [13]. In this paper we, moreover, assume that the linear system describing the dynamics is affected by a deterministic variable. For a finite-planning horizon the corresponding open-loop linear quadratic game has been studied in [3].

The outline of this note is as follows. Section two introduces the problem and contains some preliminary results. The main results of this paper are stated in Section three, whereas Section four contains some concluding remarks. The proofs of the main theorems are included in the Appendix.

II. Preliminaries

In this paper we assume that the performance criterion player $i = 1, 2$ likes to minimize is:

$$J_i(u_1, u_2) := \int_0^{\infty} [x^T(t),\ u_1^T(t),\ u_2^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (1)$$

where $M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}$ and $R_{ii} > 0,\ i = 1, 2,$ and $x(t)$ is the solution from the linear differential equation

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) + c(t),\ x(0) = x_0. \quad (2)$$

The variable $c(.) \in L_2$ here is some given vector. Notice that we make no definiteness assumptions w.r.t. matrix Q_i.

We assume that the matrix pairs $(A, B_i),\ i = 1, 2,$ are stabilizable. So, in principle, each player is capable to stabilize the system on his own.

The open-loop information structure of the game means that we assume that both players only know the initial state of the system and that the set of admissible control actions are functions of time, where time runs from zero to infinity. We assume that the players choose control functions belonging to the set

$$\mathcal{U}_s = \Big\{ u \in L_2 \mid J_i(x_0,u) \text{ exists in } I\!R \cup \{-\infty,\infty\}, \lim_{t\to\infty} x(t) = 0 \Big\}.$$

Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of both players to stabilize the system (see e.g. [10] for a discussion). The next shorthand notation will be used.

$$S_i := B_i R_{ii}^{-1} B_i^T;$$

$$G := \begin{bmatrix} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} R_{11} & N_1 \\ N_2^T & R_{22} \end{bmatrix};$$

where we assume throughout that this matrix G is invertible,

$$A_2 := \operatorname{diag}\{A, A\};\ B := [B_1,\ B_2];\ \widetilde{B}^T := \operatorname{diag}\{B_1^T, B_2^T\};$$

$$\widetilde{B}_1^T := \begin{bmatrix} B_1^T \\ 0 \end{bmatrix};\ \widetilde{B}_2^T := \begin{bmatrix} 0 \\ B_2^T \end{bmatrix};\ Q := \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix};$$

$$Z := \begin{bmatrix} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_1^T \\ W_2^T \end{bmatrix};$$

$$Z_i := [I\ 0\ 0] M_i \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} = [V_i,\ W_i],\ i = 1,2;$$

$$\widetilde{A} := A - BG^{-1}Z;\ \widetilde{A}_2^T := A_2^T - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} G^{-1} \widetilde{B}^T;$$

$$\widetilde{Q}_i := Q_i - Z_i G^{-1} Z;\ \widetilde{Q} := \begin{bmatrix} \widetilde{Q}_1 \\ \widetilde{Q}_2 \end{bmatrix};\ \widetilde{S}_i := BG^{-1}\widetilde{B}_i^T;$$

$$\widetilde{S} := [\widetilde{S}_1,\ \widetilde{S}_2], \quad \text{and } M := \begin{bmatrix} \widetilde{A} & -\widetilde{S} \\ -\widetilde{Q} & -\widetilde{A}_2^T \end{bmatrix}.$$

Notice that

$$M = \begin{bmatrix} A & 0 & 0 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix} + \begin{bmatrix} -B \\ Z_1 \\ Z_2 \end{bmatrix} G^{-1} \begin{bmatrix} Z, & \widetilde{B}_1^T, & \widetilde{B}_2^T \end{bmatrix}.$$

In the rest of the paper the algebraic Riccati equations

$$A^T K_i + K_i A - (K_i B_i + V_i) R_{ii}^{-1} (B_i^T K_i + V_i^T) + Q_i = 0,\ i = 1,2, \tag{3}$$

and the set of (coupled) algebraic Riccati equations

$$0 = \widetilde{A}_2^T P + P\widetilde{A} - PBG^{-1}\widetilde{B}^T P + \widetilde{Q} \tag{4}$$

or, equivalently,

$$0 = A_2^T P + PA - (PB + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}) G^{-1} (\widetilde{B}^T P + Z) + Q$$

play a crucial role.

Definition 2.1: A solution $P^T =: (P_1^T, P_2^T)$, with $P_i \in I\!R^n$, of the set of algebraic Riccati equations (4) is called

a. *stabilizing*, if $\sigma(\tilde{A} - BG^{-1}\tilde{B}^T P) \subset \mathbb{C}^-$; [1]

b. *strongly stabilizing* if

i. it is a stabilizing solution, and

ii. $\sigma(-\tilde{A}_2^T + PBG^{-1}\tilde{B}^T) \subset \mathbb{C}_0^+$; □

[1] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \geq 0\}$.

The next relationship between certain invariant subspaces of matrix M and solutions of the Riccati equation (4) is well-known (see e.g. Engwerda et al. [8]). This property can also be used to calculate the (strongly) stabilizing solutions of (4).

Lemma 2.2: Let $V \subset I\!R^{3n}$ be an n-dimensional invariant subspace of M, and let $X_i \in I\!R^{n\times n},\ i = 0,1,2,$ be three real matrices such that

$$V = \operatorname{Im}\left[X_0^T,\ X_1^T,\ X_2^T\right]^T.$$

If X_0 is invertible, then $P_i := X_i X_0^{-1},\ i = 1,2,$ solves (4) and $\sigma(A - BG^{-1}(Z + \tilde{B}^T P)) = \sigma(M|_V)$. Furthermore, (P_1, P_2) is independent of the specific choice of basis of V. □

Lemma 2.3:

1. The set of algebraic Riccati equations (4) has a strongly stabilizing solution (P_1, P_2) if and only if matrix M has an n-dimensional stable graph subspace and M has $2n$ eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.

2. If the set of algebraic Riccati equations (4) has a strongly stabilizing solution, then it is unique.

Proof.

1. Assume that (4) has a strongly stabilizing solution P. Then with $T := \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$,

$$TMT^{-1} = \begin{bmatrix} \tilde{A} - \tilde{S}P & -\tilde{S} \\ 0 & -\tilde{A}_2^T + P\tilde{S} \end{bmatrix}.$$

Since P is a strongly stabilizing solution, by Definition 2.1, matrix M has exact n stable eigenvalues and $2n$ eigenvalues (counted with algebraic multiplicities) in $\mathbb{C}_0^+$. Furthermore, obviously, the stable subspace is a graph subspace.

The converse statement is obtained similarly using the result of Lemma 2.2.

2. See, e.g., Kremer [13, Section 3.2]. □

III. Main results

Using the previous results, in the Appendix the following theorem is proved.

Theorem 3.1: If the differential game (1,2) has an open-loop Nash equilibrium for every initial state, then

1. M has at least n stable eigenvalues (counted with algebraic multiplicities). More in particular, there exists a p-dimensional stable M-invariant subspace S, with $p \geq n$, such that

$$\operatorname{Im} \begin{bmatrix} I \\ \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} \subset S,$$

for some $\tilde{V}_i \in I\!R^{n\times n}$.

2. the two algebraic Riccati equations (3) have a stabilizing solution.

Conversely, if the two algebraic Riccati equations (3) have a stabilizing solution and $v^T(t) =: [x^T(t), \psi_1^T(t), \psi_2^T(t)]$ is an asymptotically stable solution of

$$\dot{v}(t) = Mv(t) + \begin{bmatrix} c(t) \\ 0 \\ 0 \end{bmatrix},\ x(0) = x_0,$$

then,

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1} \begin{bmatrix} B_1^T\psi_1(t) + V_1^T x(t) \\ B_2^T\psi_2(t) + W_2^T x(t) \end{bmatrix}, \tag{5}$$

provides an open-loop Nash equilibrium for the linear quadratic differential game (1,2). □

Remark 3.2: Similar conclusions as in [9] can be drawn now. A general conclusion is that the number of equilibria depends critically on the eigenstructure of matrix M. With s denoting the number (counting algebraic multiplicities) of stable eigenvalues of M we have.

1. If $s < n$, still for some initial state there may exist an open-loop Nash equilibrium.

2. In case $s \geq 2$, the situation might arise that for some initial states there exists an infinite number of equilibria.

3. If M has a stable graph subspace, S, of dimension $s > n$, for every initial state x_0 there exists, generically, an infinite number of open-loop Nash equilibria. □

The next theorem shows that in case the set of coupled algebraic Riccati equations (4) have a stabilizing solution, the game always has at least one equilibrium.

Theorem 3.3: Assume that

1. the set of coupled algebraic Riccati equations (4) has a set of stabilizing solutions $P_i,\ i = 1, 2$; and
2. the two algebraic Riccati equations (3) have a stabilizing solution $K_i(.),\ i = 1, 2$.

Then the linear quadratic differential game (1,2) has an open-loop Nash equilibrium for every initial state.

Moreover, one set of equilibrium actions is given by:

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1}(Z + \tilde{B}^T P)\tilde{\Phi}(t,0)x_0 - G^{-1}\tilde{B}^T m(t), \tag{6}$$

where $\tilde{\Phi}(t,0)$ is the solution of the transition equation

$$\dot{\tilde{\Phi}}(t,0) = (A - BG^{-1}(Z + \tilde{B^T}P))\tilde{\Phi}(t,0);\ \tilde{\Phi}(0,0) = I$$

and $m(t) = \int_t^\infty e^{(-\tilde{A}_2 + PBG^{-1}\tilde{B}^T)(t-s)}Pc(s)ds$. □

Corollary 3.4: An immediate consequence of Lemma 2.2 and Theorem 3.3 is that if M has a stable invariant graph subspace and the two algebraic Riccati equations (3) have a stabilizing solution, the game will have at least one open-loop Nash equilibrium. □

Remark 3.5: In case $c(.) = 0$ it can be shown, similar to [6], that the costs by using the actions (6) for the players are

$$x_0^T \bar{M}_i x_0,\ i = 1, 2,$$

where, with $A_{cl} := A - BG^{-1}(Z + \tilde{B}^T P)$, $\bar{M}_i$ is the unique solution of the Lyapunov equation

$$[I,\ -G^{-1}(Z + \tilde{B}^T P)]M_i[I,\ -G^{-1}(Z + \tilde{B}^T P)]^T \tag{7}$$
$$+ A_{cl}^T \bar{M}_i + \bar{M}_i A_{cl} = 0.$$
□

Notice that in case the set of algebraic Riccati equations (4) has more than one set of stabilizing solutions, there exists more than one open-loop Nash equilibrium. Matrix M has then a stable subspace which dimension is larger than n. Consequently (see Remark 3.2, item 3) for every initial state there will exist, generically, an infinite number of open-loop Nash equilibria. This point was first noted by Kremer in [13] in case matrix A is stable.

The above reflections raise the question whether it is possible to find conditions under which the game has a unique equilibrium for every initial state. The next Theorem 3.6 gives such conditions. Moreover, it shows that in case there is a unique equilibrium the corresponding actions are obtained by those described in Theorem 3.3. The proof of this theorem is provided in the Appendix.

Theorem 3.6: Consider the differential game (1,2) with $c(.) = 0$.

This game has a unique open-loop Nash equilibrium for every initial state if and only if

1. The set of coupled algebraic Riccati equations (4) has a strongly stabilizing solution, and
2. the two algebraic Riccati equations (3) have a stabilizing solution.

Moreover, in case this game has a unique equilibrium, also the corresponding affine linear quadratic differential game, where $c(.) \in L_2$, has a unique equilibrium and the unique equilibrium actions are given by (6). □

IV. Concluding Remarks

In this note we considered the affine regular indefinite infinite-planning horizon linear-quadratic differential game. Both necessary conditions and sufficient conditions were derived for the existence of an open-loop Nash equilibrium. Moreover, conditions were presented that are both necessary and sufficient for the existence of a unique equilibrium.

The prove our results we basically proceeded along the lines of the proofs of the paper [9]. By adapting those proofs (in a not always trivial way) we showed that the results obtained in that paper carry over to this extended model.

The above results can be generalized straightforwardly to the N-player case. Furthermore, since Q_i are assumed to be indefinite, the obtained results can be directly used to (re)derive properties for the zero-sum game. If players discount their future loss, similar to [6], it follows from Theorem 3.6 that if the discount factor δ is "large enough" the game has generically a unique open-loop Nash equilibrium (all that changes is that matrix A has to be replaced by $A - \frac{1}{2}I$ everywhere). Finally we conclude from (5) that the conclusion in [13], that if the game has an open-loop Nash equilibrium for every initial state either there is a unique equilibrium or an infinite number of equilibria, applies in general.

APPENDIX

Theorem 4.1: Let $S := BR^{-1}B^T$. Consider the minimization of the linear quadratic cost function

$$\int_0^\infty x^T(t)Qx(t) + 2p^T(t)x(t) + u^T(t)Ru(t)dt \tag{8}$$

subject to the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + c(t, x_0),\ x(0) = x_0, \tag{9}$$

and $u \in \mathcal{U}_s(x_0)$. Then,
1. with $c(.) = p(.) = 0$, (8,9) has a solution for all $x_0 \in I\!R^n$ if and only if the algebraic Riccati equation

$$A^TK + KA - KSK + Q = 0 \tag{10}$$

has a symmetric stabilizing solution $K(.)$ (i.e. $A - SK$ is a stable matrix).
2. for every x_0, (8,9) with $c(., x_0),\ p(.) \in L_2$, has a solution iff. item 1 has a solution. Moreover if this problem has a solution then the problem has the unique solution

$$u^*(t) = -R^{-1}B^T(Kx^*(t) + m(t)).$$

Here $m(t)$ is given by

$$m(t) = \int_t^\infty e^{-(A-SK)^T(t-s)}(Kc(s) + p(s))ds, \tag{11}$$

and $x^*(t)$ satisfies

$$\dot{x}^*(t) = (A - SK)x^*(t) - Sm(t) + c(t),\ x^*(0) = x_0.$$

Proof. Similar to the proof of [10, Theorem 5.16]. □

Proof of Theorem 3.1.
"⇒ part" Suppose that u_1^*, u_2^* are a Nash solution. That is,

$$J_1(u_1, u_2^*) \geq J_1(u_1^*, u_2^*) \text{ and } J_2(u_1^*, u_2) \geq J_2(u_1^*, u_2^*).$$

From the first inequality we see that for every $x_0 \in I\!R^n$ the (nonhomogeneous) linear quadratic control problem to minimize $J_1 =$

$$\int_0^\infty \{x^T(t)Q_1x(t) + 2u_1^T(t)V_1^Tx(t) + 2{u_2^*}^T(t)W_1^Tx(t) + u_1^T(t)R_{11}u_1(t) + u_1^T(t)N_1u_2^*(t) + {u_2^*}^T(t)R_{21}u_2^*(t)\}dt, \tag{12}$$

subject to the (nonhomogeneous) state equation

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2^*(t) + c(t),\ x(0) = x_0, \tag{13}$$

has a solution. Or, equivalently, with

$$v_1(t) := u_1(t) + R_{11}^{-1}V_1^Tx_1(t) + R_{11}^{-1}N_1u_2^* \tag{14}$$

the minimization of $J_1 =$

$$\int_0^\infty \{x_1^T(t)(Q_1 - V_1R_{11}^{-1}V_1^T)x_1(t) + v_1^T(t)R_{11}v_1(t) + 2({u_2^*}^T(t)W_1^T - {u_2^*}^T(t)N_1^TR_{11}^{-1}V_1^T)x_1(t) + {u_2^*}^T(t)(R_{21} - N_1^TR_{11}^{-1}N_1)u_2^*(t)\}dt, \tag{15}$$

subject to the (nonhomogeneous) state equation

$$\dot{x}_1(t) = (A - B_1R_{11}^{-1}V_1^T)x_1(t) + B_1v_1(t) + (B_2 - B_1R_{11}^{-1}N_1)u_2^*(t) + c(t),\ x(0) = x_0, \tag{16}$$

has a solution. This implies, see Theorem 4.1, that the algebraic Riccati equation

$$(A - B_iR_{ii}^{-1}V_i^T)^TK_i + K_i(A - B_iR_{ii}^{-1}V_i^T) - K_iS_iK_i + Q_i - V_iR_{ii}^{-1}V_i^T = 0$$

has a stabilizing solution. It is easily verified that this equation can be rewritten as (3), with $i = 1$. Similarly we get that also the second algebraic Riccati equation must have a stabilizing solution. Which completes the proof of point 2. To prove point 1 we consider Theorem 4.1 in some more detail. According Theorem 4.1 the minimization problem (15,16) has a unique solution. Its solution is

$$\tilde{v}_1(t) = -R_{11}^{-1}B_1^T(K_1x_1(t) + m_1(t))\text{with } m_1(t) = \int_t^\infty e^{-(A-B_1R_{11}^{-1}V_1^T-S_1K_1)^T(t-s)}(K_1n_1(s) + p_1(s))ds, \tag{17}$$

where $p_1^T(s) = {u_2^*}^T(s)(W_1^T - N_1^TR_{11}^{-1}V_1^T)$, $n_1(s) = (B_2 - B_1R_{11}^{-1}N_1)u_2^*(s) + c(s)$ and K_1 the stabilizing solution of the algebraic Riccati equation (3), with $i = 1$. Consequently, see (14),

$$\tilde{u}_1(t) := \tilde{v}_1(t) - (R_{11}^{-1}V_1^Tx_1(t) + R_{11}^{-1}N_1u_2^*) \tag{18}$$

solves the original optimization problem. Notice that, since the optimal control for this problem is uniquely determined, and by definition the equilibrium control u_1^* solves the optimization problem, $u_1^*(t) = \tilde{u}_1(t)$. Consequently,

$$\begin{aligned}\frac{d(x(t) - x_1(t))}{dt} &= Ax(t) + B_1u_1^*(t) + B_2u_2^*(t) - \\ &\quad (A - B_1R_{11}^{-1}V_1^T - S_1K_1)x_1(t) + S_1m_1(t) - \\ &\quad (B_2 - B_1R_{11}^{-1}N_1)u_2^*(t) \\ &= Ax(t) - S_1(K_1x_1(t) + m_1(t)) - (B_1^TR_{11}^{-1}V_1^Tx_1(t) + \\ &\quad B_1R_{11}^{-1}N_1u_2^*) - Ax_1(t) + S_1(K_1x_1(t) + m_1(t)) + \\ &\quad B_1R_{11}^{-1}V_1^Tx_1(t) + B_1R_{11}^{-1}N_1u_2^*(t) \\ &= A(x(t) - x_1(t)).\end{aligned}$$

Since $x(0) - x_1(0) = 0$ it follows that $x_1(t) = x(t)$.
Analogously we obtain from the minimization of J_2, with u_1^* now entering into the system as an external signal, that

$$u_2^*(t) := -R_{22}^{-1}B_2^T(K_2x(t) + m_2(t)) - (R_{22}^{-1}W_2^Tx(t) + R_{22}^{-1}N_2^Tu_1^*) \tag{19}$$

with $m_2(t) = \int_t^\infty e^{-(A-B_2R_{22}^{-1}V_2^T-S_2K_2)^T(t-s)}(K_2c_2(s) + p_2(s))ds$, $p_2^T(s) = {u_1^*}^T(s)(W_2^T - N_2^TR_{22}^{-1}V_2^T)$, $n_2(s) = (B_1-B_2R_{22}^{-1}N_2)u_1^*(s)+c(s)$ and K_2 the stabilizing solution of the algebraic Riccati equation (3), with $i=2$.
Differentiation of $m_1(t)$ in (17) gives

$$\dot{m}_1(t) = -(A - B_1R_{11}^{-1}V_1^T - S_1K_1)^Tm_1(t)- \quad (20)$$
$$(K_1B_2 - K_1B_1R_{11}^{-1}N_1 + W_1 - V_1R_{11}^{-1}N_1)u_2^*(t) - K_1c(s).$$

Next, introduce $\psi_1(t) := K_1x(t)+m_1(t)$. Using (16,17) and (20) we get

$$\begin{aligned}\dot{\psi}_1(t) &= K_1\dot{x}(t) + \dot{m}_1(t)\\ &= K_1(A - B_1R_{11}^{-1}V_1^T - S_1K_1)x(t) - K_1S_1m_1(t)+\\ &\quad K_1(B_2 - B_1R_{11}^{-1}N_1)u_2^*(t) + K_1c(s)-\\ &\quad (A - B_1R_{11}^{-1}V_1^T - S_1K_1)^Tm_1(t) - (K_1B_2-\\ &\quad K_1B_1R_{11}^{-1}N_1 + W_1 - V_1R_{11}^{-1}N_1)u_2^*(t) - K_1c(s)\\ &= -Q_1x(t) - A^T(K_1x(t) + m_1(t))+ \quad (21)\\ &\quad (V_1R_{11}^{-1}B_1^TK_1 + V_1R_{11}^{-1}V_1^T)x(t)+\\ &\quad V_1R_{11}^{-1}B_1^Tm_1(t) + V_1R_{11}^{-1}N_1u_2^*(t) - W_1u_2^*(t)\\ &= -Q_1x(t) - A^T\psi_1(t) - V_1u_1^*(t) - W_1u_2^*(t). \quad (22)\end{aligned}$$

Similarly it follows that $\dot{\psi}_2(t) = -Q_2x(t) - A^T\psi_2(t) - V_2u_1^*(t) - W_2u_2^*(t)$.
From (17,19) it follows that (u_1^*, u_2^*) satisfy

$$R_{11}u_1^* + N_1u_2^*(t) = -B_1^T\psi_1(t) - V_1^Tx(t)$$
$$N_2^Tu_1^* + R_{22}u_2^*(t) = -B_2^T\psi_2(t) - W_2^Tx(t),$$

respectively. Due to our invertibility assumption on matrix G we can rewrite this as (5). Consequently,

$$v^T(t) = [v_1^T(t),\ v_2^T(t),\ v_3^T(t)] := [x^T(t),\ \psi_1^T(t),\ \psi_2^T(t)],$$

satisfies $\dot{v}(t) = Mv(t) + \begin{bmatrix} c(t)\\ 0\\ 0\end{bmatrix}$, with $v_1(0) = x_0$.
Since by assumption, for arbitrary x_0, $v_1(t)$ converges to zero it is clear from [10, Lemma 7.36] by choosing consecutively $x_0 = e_i$, $i = 1, \cdots, n$, that matrix M must have at least n stable eigenvalues (counting algebraic multiplicities). Moreover, the other statement follows from the second part of this lemma. Which completes this part of the proof.
"⇐ part" Let u_2^* be as defined in (5) where $x(t)$ satisfies

$$\dot{x}(t) = (A - BG^{-1}Z)x(t) - BG^{-1}\tilde{B}_1^T\psi_1(t)- BG^{-1}\tilde{B}_2^T\psi_2(t),\ x(0) = x_0.$$

We next show that then necessarily u_1^* solves the minimization problem (12,13). Since, by assumption, the algebraic Riccati equation (3) has a stabilizing solution, according Theorem 4.1, the minimization problem (12,13) has a solution. Following the notation of the "⇒" part of the proof this solution is given by (see (18,17))

$$\begin{aligned}\tilde{u}_1(t) &= -R_{11}^{-1}B_1^T(K_1x_1(t) + m_1(t))\\ &\quad - (R_{11}^{-1}V_1^Tx_1(t) + R_{11}^{-1}N_1u_2^*)\end{aligned}$$

Next, introduce $\tilde{\psi}_1(t) := K_1x_1(t) + m_1(t)$. Then, similar to (22) we obtain

$$\dot{\tilde{\psi}}_1(t) = -Q_1x_1(t) - A^T\tilde{\psi}_1(t) - V_1\tilde{u}_1(t) - W_1u_2^*(t).$$

Consequently, with $x_d(t) := x(t) - x_1(t)$, $\psi_d(t) := \psi_1(t) - \tilde{\psi}_1(t)$ and $\psi^T := [\psi_1^T\ \psi_2^T]$ we have:

$$\begin{aligned}\dot{x}_d(t) &= \dot{x}(t) - \dot{x}_1(t)\\ &= (A - BG^{-1}Z)x(t) - BG^{-1}\tilde{B}_1^T\psi_1(t)-\\ &\quad BG^{-1}\tilde{B}_2^T\psi_2(t) - (A - B_1R_{11}^{-1}V_1^T)x_1(t) + S_1\tilde{\psi}_1(t)\\ &\quad - (B_2 - B_1R_{11}^{-1}N_1)u_2^*(t)\\ &= (A - BG^{-1}Z)x(t) - BG^{-1}\widetilde{B}^T\psi(t)-\\ &\quad (A - B_1R_{11}^{-1}V_1^T)x_1(t) + S_1\tilde{\psi}_1(t)+\\ &\quad (B_2 - B_1R_{11}^{-1}N_1)[0\ I]G^{-1}(\widetilde{B}^T\psi(t) + Zx(t))\\ &= (A - [B_1\ 0]G^{-1}Z)x(t) - [B_1\ 0]G^{-1}\widetilde{B}^T\psi(t)\\ &\quad - (A - B_1R_{11}^{-1}V_1^T)x_1(t) + S_1\tilde{\psi}_1(t)-\\ &\quad [0\ B_1R_{11}^{-1}N_1]G^{-1}(\widetilde{B}^T\psi(t) + Zx(t))\\ &= Ax(t) - B_1R_{11}^{-1}[R_{11}\ N_1]G^{-1}(\widetilde{B}^T\psi(t)+\\ &\quad Zx(t)) - (A - B_1R_{11}^{-1}V_1^T)x_1(t) + S_1\tilde{\psi}_1(t)\\ &= (A - B_1R_{11}^{-1}V_1^T)x_d(t) - S_1\tilde{\psi}_d(t).\end{aligned}$$

Furthermore, using (21),

$$\begin{aligned}\dot{\psi}_d(t) &= \dot{\psi}_1(t) - \dot{\tilde{\psi}}_1\\ &= -Q_1x(t) - (A^T - V_1R_{11}^{-1}B_1^T)\psi_1(t) + V_1R_{11}^{-1}V_1^Tx(t)\\ &\quad + V_1R_{11}^{-1}N_1u_2^*(t) - W_1u_2^*(t) + Q_1x_1(t)+\\ &\quad A^T\tilde{\psi}_1 + V_1\tilde{u}_1(t) + W_1u_2^*(t)\\ &= -Q_1x(t) - (A^T - V_1R_{11}^{-1}B_1^T)\psi_1(t) + V_1R_{11}^{-1}V_1^Tx(t)\\ &\quad + V_1R_{11}^{-1}N_1u_2^*(t) + Q_1x_1(t) + A^T\tilde{\psi}_1-\\ &\quad V_1R_{11}^{-1}B_1^T\tilde{\psi}_1 - V_1R_{11}^{-1}V_1^Tx_1(t) - V_1R_{11}^{-1}N_1u_2^*(t)\\ &= (-Q_1 + V_1R_{11}^{-1}V_1^T)x_d(t) - (A - B_1R_{11}^{-1}V_1^T)^T\psi_d(t).\end{aligned}$$

Now, let $H := \begin{bmatrix} A - B_1R_{11}^{-1}V_1^T & -S_1\\ -Q_1 + V_1R_{11}^{-1}V_1^T & -(A - B_1R_{11}^{-1}V_1^T)^T\end{bmatrix}$ and $e^T := [x_d^T,\ \psi_d^T]$. Then for some $p \in I\!R^n$,

$$\dot{e}(t) = He(t),\ \text{with}\ e^T(0) = [0,\ p].$$

Notice that matrix H is the Hamiltonian matrix associated with the algebraic Riccati equation (3). The rest of the proof follows now along the lines of the corresponding part of the proof of [10, Theorem 7.11]. □.

Proof of Theorem 3.3.
Since (4) has a stabilizing solution, we can factorize M as in the proof of Lemma 2.3. That is, $M =$

$$T^{-1}\begin{bmatrix} A - BG^{-1}(Z + \tilde{B}^TP) & -BG^{-1}\tilde{B}^T\\ 0 & -\tilde{A}_2^T + PBG^{-1}\tilde{B}^T\end{bmatrix}T.$$

Next consider

$$\psi(t) := Px(t) + m(t) \text{ with}$$
$$m(t) = \int_t^{\infty} e^{(-\tilde{A}_2^T + PBG^{-1}\tilde{B}^T)(t-s)} Pc(s)ds,$$

and $x(.)$ the solution of the differential equation

$$\dot{x}(t) = (A - BG^{-1}(Z + \tilde{B}^T P))x(t) - BG^{-1}\tilde{B}^T m(t) + c(t), \; x(0) = x_0.$$

Notice that both $x(t)$ and $\psi(t)$ converges to zero if $t \to \infty$. By direct substitution of this $x(t)$ and $\psi(t)$ into

$$\dot{v}(t) = Mv(t) + \begin{bmatrix} c(t) \\ 0 \\ 0 \end{bmatrix}, \; x(0) = x_0,$$

it is straightforwardly verified (using the above decomposition of M) that $v(t) := [x^T(t) \; \psi^T(t)]$ is an asymptotically solution of this differential equation. So, by Theorem 3.1

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = -G^{-1} \begin{bmatrix} B_1^T \psi_1(t) + V_1^T x(t) \\ B_2^T \psi_2(t) + W_2^T x(t) \end{bmatrix} = -G^{-1}((Z + \tilde{B}^T P)x(t) + \tilde{B}^T m(t)),$$

provides an open-loop Nash equilibrium for the linear quadratic differential game (1,2). □

Proof of Theorem 3.6.

"⇒ part" With some small straightforward modifications this part of the proof can be copied from the corresponding part of the proof of [10, Theorem 7.16].

"⇐ part" Since by assumption the stable subspace, E^s, is a graph subspace we know that every initial state, x_0, can be written uniquely as a combination of the first n entries of the basisvectors in E^s. Consequently, with every x_0 there corresponds a unique ψ_1 and ψ_2 for which the solution of the differential equation $\dot{z}(t) = Mz(t)$, with $z_0^T = [x_0^T, \; \psi_1^T, \; \psi_2^T]$, converges to zero. So, by Theorem 3.1, for every x_0 there is a Nash equilibrium. On the other hand the proof of Theorem 3.1 shows that all Nash equilibrium actions (u_1^*, u_2^*) satisfy (5), where $\psi_i(t)$ solves

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix}, \text{ with } x(0) = x_0.$$

Now, with $z^T := [x^T \; \psi_1^T \; \psi_2^T]$ and $y^T := [x^T \; {u_1^*}^T \; {u_2^*}^T]$ consider the system

$$\dot{z}(t) = Mz(t); \; y(t) = Cz(t), \text{ where}$$
$$C := \begin{bmatrix} I & 0 & 0 \\ -[I \; 0]G^{-1}Z & -[I \; 0]G^{-1}\tilde{B}_1^T & -[I \; 0]G^{-1}\tilde{B}_2^T \\ -[0 \; I]G^{-1}Z & -[0 \; I]G^{-1}\tilde{B}_1^T & -[0 \; I]G^{-1}\tilde{B}_2^T \end{bmatrix}.$$

Then, rank $\begin{bmatrix} M - \lambda I \\ C \end{bmatrix} =$

$$\text{rank} \begin{bmatrix} A - \lambda I & 0 & 0 \\ -Q_1 & -A^T - \lambda I & 0 \\ -Q_2 & 0 & -A^T - \lambda I \\ I & 0 & 0 \\ Z & \tilde{B}_1^T & \tilde{B}_2^T \end{bmatrix}.$$

Since $(A, B_i), \; i = 1, 2$, is stabilizable, it is easily verified from the above expression that the pair (C, M) is detectable. Consequently, due to our assumption that $x(t)$ and $u_i^*(t), \; i = 1, 2$, converge to zero, we have from [19, Lemma 14.1] that $[x^T(t), \; \psi_1^T(t), \; \psi_2^T(t)]$ converges to zero. Therefore, $[x^T(0), \; \psi_1^T(0), \; \psi_2^T(0)]$ has to belong to the stable subspace of M. However, as we argued above, for every x_0 there is exactly one vector $\psi_1(0)$ and vector $\psi_2(0)$ such that $[x^T(0), \; \psi_1^T(0), \; \psi_2^T(0)] \in E^s$. So we conclude that for every x_0 there exists exactly one Nash equilibrium.

Notice that in case the conditions 1. and 2. of this theorem are satisfied, Theorem 3.3 implies that the unique equilibrium actions are given by (6).

Finally, it will be clear that with $c(.) \neq 0$ one can pursue the same analysis as above. Since this analysis brings on only some additional technicalities and distracts the attention from the basic reasoning we skipped that analysis here. □

References

[1] Abou-Kandil H., Freiling G. and Jank G., 1993, Necessary and sufficient conditions for constant solutions of coupled Riccati equations in Nash games, *Systems&Control Letters*, Vol.21, pp.295-306.

[2] Başar T. and Bernhard P., 1995, *H_∞-Optimal Control and Related Minimax Design Problems*, Birkhäuser, Boston.

[3] Başar T. and Olsder G.J., 1999, *Dynamic Noncooperative Game Theory*, SIAM, Philadelphia.

[4] Dockner E., Jørgensen S., Long N. van and Sorger G., 2000, *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge.

[5] Eisele T., 1982, Nonexistence and nonuniqueness of open-loop equilibria in linear-quadratic differential games, *Journal of Optimization Theory and Applications*, Vol.37, no.4, pp.443-468.

[6] Engwerda J.C., 1998, On the open-loop Nash equilibrium in LQ-games, *Journal of Economic Dynamics and Control*, Vol.22, pp.729-762.

[7] Engwerda J.C., 1998, Computational aspects of the open-loop Nash equilibrium in LQ-games, *Journal of Economic Dynamics and Control*, Vol.22, pp.1487-1506.

[8] Engwerda J.C., van Aarle B. and Plasmans J.E.J., 1999, The (in)finite horizon open-loop Nash LQ game: An application to EMU, *Annals of Operations Research*, Vol.88, pp.251-273.

[9] Engwerda J.C., 2004, The Open-Loop linear quadratic differential game revisited, Internal report Tilburg University, The Netherlands, submitted for publication.

[10] Engwerda J.C., 2005, *LQ Dynamic Optimization and Differential Games*, John Wiley & Sons.

[11] Feucht M., 1994, *Linear-quadratische Differentialspiele und gekoppelte Riccatische Matrixdifferentialgleichungen*, Ph.D. Thesis, Universität Ulm, Germany.

[12] Haurie A. and Leitmann G., 1984, On the global asymptotic stability of equilibrium solutions for open-loop differential games, *Large Scale Systems*, Vol.6, 107-122.

[13] Kremer D., 2002, *Non-Symmetric Riccati Theory and Noncooperative Games*, Ph.D. Thesis, RWTH-Aachen, Germany.

[14] Kremer D. and Stefan R., 2002, *Non-symmetric Riccati theory and linear quadratic Nash games*, Proceedings MTNS 2002 Conference, University of Notre Dame.

[15] Lukes D.L. and Russell D.L., 1971, Linear-quadratic games, *Journal of Mathematical Analysis and Applications*, Vol.33, pp.96-123.

[16] Meyer H.-B., 1976, The matrix equation $AZ + B - ZCZ - ZD = 0$, *SIAM Journal Applied Mathematics*, Vol.30, 136-142.

[17] Starr A.W. and Ho Y.C., 1969, Nonzero-sum differential games, *Journal of Optimization Theory and Applications*, Vol.3, pp.184-206.

[18] Weeren A.J.T.M., 1995, *Coordination in Hierarchical Control*, Ph.D. Thesis, Tilburg University, The Netherlands.

[19] Zhou K., J.C. Doyle, and K. Glover, 1996, *Robust and Optimal Control*, Prentice Hall, New Jersey.