Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Adaptive H_∞-Control Design for a Class of LPV Systems

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Abstract— **In this paper, we focus on the issue of adaptive H_∞-control design for a class of linear parameter-varying (LPV) systems based on the Hamiltonian-Jacobi-Isaac (HJI) method. By combining the idea of polynomially parameter-dependent quadratic functions and vector projection method to derive an adaptive H_∞-control, sufficient conditions with high precision are given to guarantee both robust asymptotic stability and disturbance attenuation of the LPV systems with unknown constant parameters. The applicability of the proposed design method is illustrated on a simple example.**

Index Terms - **Adaptive H_∞-control; LPV systems.**

I. INTRODUCTION

Over the last three decades, considerable attention has been paid to robustness analysis and control of linear systems affected by structured real parameters. For linear parameter-varying (LPV) systems, establishing stability via the use of constant Lyapunov functions is conservative. Therefore, to investigate the stability of LPV systems one needs to resort the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability, see [1, 2, 3, 4, 5].

Generally, the H_∞–control problem for dynamical systems with exogenous inputs as disturbances is to design feedback controllers which make the resulting systems to have small L_2–gains such that the disturbances are attenuated onto the controlled output [6]. In this paper, since there exist unknown parameters in the system matrices, we design an adaptive H_∞–control based on a parameter adaptation law to estimate the unknown parameters. In this case, both the parameter-dependent controller and the parameter adaptation law are designed properly such that the augmented system is stable. In literature, there are some works related to the adaptive H_∞–control with different approaches, see [7, 8, 9, 10].

A new technique for robust stability problem of LPV systems with unknown parameter vector has been investigated in [2] and also the stabilization technique of the parameter-dependent systems was developed in [11]. Recently, a systematic way for the use of polynomial parameter-dependent Lyapunov functions in robust disturbance attenuation and robust asymptotic stability problem of finite-dimensional linear parameter-dependent systems with time-delay in the state vector proposed in [12].

In this paper, we investigate the issue of adaptive H_∞-control design with full information feedback for a class of LPV systems, which additionally depend on unknown parameters in terms of the solutions of parameter-dependent Hamiltonian-Jacobi-Isaac (HJI) inequality. By combining the idea of polynomially parameter-dependent quadratic functions and vector projection method, sufficient conditions with high precision are given to guarantee robust asymptotic stability and disturbance attenuation of the LPV systems with unknown constant parameters. Sufficient conditions of increasing precision for the existence of polynomial parameter-dependent quadratic functions are given by LMI formulations. This paper makes three specific contributions. First, it suggests polynomially parameter-dependent quadratic functions, which can be applied to derive sufficient and necessary stability conditions for LPV systems. Second, both robust stabilization and disturbance attenuation of such systems are investigated based on sufficient conditions produced by the Hamiltonian-Jacobi-Isaac (HJI) approach. Then, an adaptive H_∞-control can be constructed from the positive-definite solution to a certain linear matrix inequality (LMI). Finally, the applicability of the proposed design method is illustrated on a simple example.

The notations used throughout the paper are fairly standard. The matrices I_n, 0_n, $0_{n\times p}$ are the identity matrix and the $n\times n$ and $n\times p$ zero matrices, respectively. The symbol * denotes the elements below the main diagonal of a symmetric block matrix. Also, the symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes}=1$, $M^{p\otimes}:=M^{(p-1)\otimes}\otimes M$. Let $\hat{J}_k$, $\tilde{J}_k \in \Re^{k\times(k+1)}$ and $\vartheta^{[k]}$ be defined by $\hat{J}_k := [I_k \quad 0_{k\times 1}]$, $\tilde{J}_k := [0_{k\times 1} \quad I_k]$ and $\vartheta^{[k]} := [1 \quad \vartheta \quad \cdots \quad \vartheta^{k-1}]^T$, respectively, which have essential roles for polynomial manipulations [2]. Finally, given a signal $x(t)$, $\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; i.e., $\|x(t)\|_2^2 = \int_0^\infty x(t)^T x(t)\,dt$. According to [13] for convex set $\Theta \subset \Re^m$, the contingent cone $T_\Theta(\upsilon)$ and its normal cone $N_\Theta(\upsilon)$ are defined as $T_\Theta(\upsilon)=\{\, y : \exists t>0 : \upsilon + t\, y \in \Theta \}$ and

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$N_\Theta(\upsilon) = \{z \in \Re^m : y^T z \le 0, \forall y \in T_\Theta(\upsilon)\}$, respectively, and $N_\Theta(\upsilon) = \{0\}$ if $\upsilon \in \Theta$.

II. PROBLEM DESCRIPTION

In this section, we consider a class of LPV systems as

$$\begin{cases} \dot{x}(t) = A(\theta)\, x(t) + B_u(\theta)\, u(t) + B_w(\theta)\, w(t) \\ z(t) = C_1\, x(t) + C_2\, u(t) \end{cases} \quad (1)$$

where $\theta = [\theta_1, \theta_2, \cdots, \theta_m]^T$ is a m–dimensional vector of *unknown constant parameters* with $\theta \in \Theta \in \Re^m$ and dependency of the system matrices affinely on the parameter vector θ is shown as

$$[A(\theta) \quad B_u(\theta) \quad B_w(\theta)] = [A_0 \quad B_{0u} \quad B_{0w}] + \sum_{i=1}^{m} \theta_i\, [A_i \quad B_{iu} \quad B_{iw}], \quad (2)$$

and $x(t) \in \Re^n$, $u(t) \in \Re^l$, $w(t) \in \Re^s$ and $z(t) \in \Re^p$ are the state vector, the control input, the disturbance vector and the controlled output, respectively.

Assumption 1. In the system (1), we assume,

I. the parameter set Θ is convex and compact.

II. $C_1^T C_2 = 0$.

In this paper, the aim is to design an adaptive controller for attenuating the impact of the exogenous disturbance $w(t)$ and error induced by initial guess of the parameter onto the controlled output $z(t)$. The proposed controller has the following fashion

$$\begin{cases} \dot{\rho}(t) = \phi(x(t), \rho(t), u(t)) \\ u(t) = K\,(\rho(t))\, x(t) \end{cases} \quad (3)$$

where $K(.) \in \Re^{l \times n}$ and $\rho \in \Re^m$ is the estimation of the real parameter θ. Now, we shall make the following definitions for the system (1).

Definition 1. The adaptive H_∞ – control design (3) is said to achieve robust global asymptotic stability of the system (1) if for $w = 0$ and any $\theta \in \Theta$ the closed-loop system

$$\begin{cases} \dot{x}(t) = (A(\rho(t)) + B_u(\rho(t))\, K\,(\rho(t)))\, x(t) \\ \dot{\rho}(t) = \phi(x(t), \rho(t), u(t)) \end{cases} \quad (4)$$

is globally asymptotically stable in the Lyapunov sense.

Definition 2. By considering $\varepsilon > 0$, the adaptive H_∞ – control design (3) is said to achieve robust disturbance attenuation if under zero initial condition there exists $0 \le \gamma < \infty$ for which the performance bound is such that:

$$\int_0^T z^T(t)\, z(t)\; dt < \gamma^2 \int_0^T w^T(t)\, w(t)\; dt + \varepsilon\,, \quad (5)$$
$$\forall\; T > 0,\, w \in L^2,\; \rho(0) \in \Theta,\; \theta \in \Theta.$$

According to the definitions above, the main objective of the paper is to design the adaptive H_∞ – control (3) for achieving the robust global asymptotic stability and the robust disturbance attenuation of the linear parameter-dependent system (1), simultaneously.

Definition 3. We call a polynomially parameter-dependent quadratic function (PPDQ function for short) any quadratic function $x^T(t)\, S(\rho)\, x(t)$ such $S(\rho)$ is defined as

$$S(\rho) := (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^T S_k\, (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)$$

for $x(t) \in \Re^n$ and a certain $S_k \in \Re^{k^m n}$. The integer $k-1$ is called the degree of the PPDQ function of $S(\rho)$.

III. MAIN RESULTS

The main approach employed here is to design an adaptive state feedback control in the presence of the disturbances based on the standard HJI method. Hence, we define a quadratic energy function in the form

$$E_\theta(x; \rho_1, \rho_2, \cdots, \rho_m) = x^T(t)\, P_\rho\, x(t) + (\rho - \theta)^T Q\, (\rho - \theta) \quad (6)$$

where $P_\rho := P(\rho) > 0$, PPDQ Lyapunov function of degree $k-1$, and $Q > 0$ are to be determined.

Suppose that there exists the following HJI function

$$H[u, w; \rho_1, \rho_2, \cdots, \rho_m] = \frac{d}{dt} E_\theta(x; \rho_1, \rho_2, \cdots, \rho_m) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) \quad (7)$$

where derivative of $E_\theta(.)$ is evaluated along the trajectory of the closed-loop system (1), i.e.

$$\frac{d}{dt} E_\theta(x; \rho_1, \rho_2, \cdots, \rho_m) = \frac{\partial}{\partial x}(x^T(t)\, P_\rho\, x(t) + (\rho - \theta)^T Q\, (\rho - \theta))\; (A(\theta)\, x(t) + B_u(\theta)\, u(t) + B_w(\theta)\, w(t)) + \sum_{i=1}^{m} \frac{\partial\, (x^T(t)\, P_\rho\, x(t))}{\partial\, \rho_i}\; \dot{\rho}_i + \frac{\partial\, ((\rho - \theta)^T Q\, (\rho - \theta))}{\partial\, \rho}\; \dot{\rho}. \quad (8)$$

It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality $H[u, w; \rho_1, \rho_2, \cdots, \rho_m] < 0$ for all $w \in L^2$ results in a function $E_\theta(.)$, which is strictly radially unbounded (see for example [14, 15]), $E_\theta(.)$ may be regulated as a Lyapunov function for the closed-loop system (1). In this paper we will establish conditions under which

$$Inf_u \ Sup_{w\in L^2} H[u,w;\rho_1,\rho_2,\cdots,\rho_m]<0\,, \tag{9}$$

then for every T, taking the definite integral from 0 to T of both sides of (7) gives

$$\int_0^T z^T(t)\,z(t)\,dt - \gamma^2 \int_0^T w^T(t)\,w(t)\,dt < E_\theta(x;\rho_1,\rho_2,\cdots,\rho_m)\big|_{t=0} - E_\theta(x;\rho_1,\rho_2,\cdots,\rho_m)\big|_{t=T} = \varepsilon$$

i.e., constraint of (5).

Noting to the expression (7) and according to (8), we have:

$$\begin{aligned} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] &= 2x^T P_\rho (A(\rho)x(t)+B_u(\rho)u(t) \\ &+ B_w(\rho)w(t)) - 2x^T P_\rho (\sum_{i=1}^m (\rho_i-\theta_i)A_i x + \sum_{i=1}^m (\rho_i-\theta_i)B_{iu}u \\ &+ \sum_{i=1}^m (\rho_i-\theta_i)B_{iw}w) + \sum_{i=1}^m \frac{\partial (x^T(t)P_\rho x(t))}{\partial \rho_i}\dot\rho_i + 2(\rho-\theta)^T Q\,\dot\rho \\ &+ (C_1x+C_2u)^T(C_1x+C_2u) - \gamma^2 w^T w. \end{aligned} \tag{10}$$

Using Assumption 1 and structure of the matrix P_ρ, it is easy to show that

$$\begin{aligned} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] &= 2x^T P_\rho (A(\rho)x(t)+B_u(\rho)u(t) \\ &+ B_w(\rho)w(t)) + \sum_{i=1}^m \frac{\partial}{\partial \rho_i} x^T(t)(\rho_m^{[k]}\otimes\cdots\otimes\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)^T P_k \\ &\times(\rho_m^{[k]}\otimes\cdots\otimes\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)\,x(t)\,\dot\rho_i + x^T C_1^T C_1 x + u^T C_2^T C_2 u \\ &- \gamma^2 w^T w + 2(\rho-\theta)^T Q\,(\dot\rho - \Phi(x,w,u)) \end{aligned}$$

or

$$\begin{aligned} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] &= 2x^T P_\rho (A(\rho)x(t)+B_u(\rho)u(t) \\ &+ B_w(\rho)w(t)) + 2\sum_{i=1}^m x^T(t)(\rho_m^{[k]}\otimes\cdots\otimes\tilde\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)^T \\ &\times P_k(\rho_m^{[k]}\otimes\cdots\otimes\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)\,x(t)\,\dot\rho_i + x^T C_1^T C_1 x \\ &+ u^T C_2^T C_2 u - \gamma^2 w^T w + 2(\rho-\theta)^T Q\,(\dot\rho-\Phi(x,w,u)), \end{aligned} \tag{11}$$

where $\tilde\rho_i^{[k]} = (0,\,1,\,2\rho_i,\cdots,(k-1)\rho_i^{k-2})^T$ for $k>1$ and the vector function $\Phi:\Re^n\times\Re^s\times\Re^l\to\Re^m$ is defined as

$$\Phi(x,w,u) = Q^{-1}\begin{bmatrix} x^T P_\rho(A_1 x + B_{1u}u + B_{1w}w) \\ x^T P_\rho(A_2 x + B_{2u}u + B_{2w}w) \\ \vdots \\ x^T P_\rho(A_m x + B_{mu}u + B_{mw}w) \end{bmatrix}. \tag{12}$$

Therefore, one finds

$$\begin{aligned} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] &\le 2x^T P_\rho (A(\rho)x(t)+B_u(\rho)u(t)+B_w(\rho)w(t)) \\ &+ x^T C_1^T C_1 x + u^T C_2^T C_2 u - \gamma^2 w^T w \end{aligned} \tag{13}$$

if the following conditions are satisfied:

$$\begin{aligned} &\sum_{i=1}^m x^T(t)(\rho_m^{[k]}\otimes\cdots\otimes\tilde\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)^T P_k \\ &\times(\rho_m^{[k]}\otimes\cdots\otimes\rho_i^{[k]}\otimes\cdots\otimes\rho_1^{[k]}\otimes I_n)\,x(t)\,\dot\rho_i \le 0, \end{aligned} \tag{14}$$

$$\rho\in\Theta, \tag{15}$$

$$(\rho-\theta)^T Q\,(\dot\rho-\Phi(x,w,u)) \le 0. \tag{16}$$

As discussed in [13], the conditions (15) and (16) are satisfied by choosing the parameter update law as

$$\dot\rho = \mu_\Theta(\rho,\Phi(x,w,u)) \tag{17}$$

where $\mu_\Theta(\rho,\Phi(x,w,u))$ called the vector projection of the vector $\Phi(x,w,u)$ at a point $\rho\in\Theta$ on the contingent cone $T_\Theta(\rho)$ as follows:

$$\mu_\Theta(\rho,\Phi(x,w,u)) = \begin{cases} \Phi(x,w,u) & \text{if } \rho\in\Theta \text{ or } \Phi(x,w,u)\in T_\Theta(\rho); \\ 0 & \text{if } \Phi(x,w,u)\in N_\Theta(\rho); \\ \Phi^T(x,w,u)\,\nu\nu & \text{otherwise}; \end{cases} \tag{18}$$

where $\nu = \arg\max\{\Phi^T(x,w,u)\,\nu_0 \mid \nu_0\in T_\Theta(\rho), \|\nu_0\|=1\}$.

It can be shown that $w^*(t)=\gamma^{-2}B_w^T(\rho)P_\rho x(t)$ maximizes the right-hand side of the inequality (13), in other words,

$$\begin{aligned} Sup_{w\in L^2} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] &= H[u,w^*;\rho_1,\rho_2,\cdots,\rho_m] \\ &\le x^T(A^T(\rho)P_\rho + P_\rho A(\rho) + \gamma^{-2}P_\rho B_w(\rho)B_w^T(\rho)P_\rho \\ &+ C_1^T C_1)x(t) + 2\,x^T P_\rho B_u(\rho)u(t) + u^T C_2^T C_2 u. \end{aligned} \tag{19}$$

The optimal control law, which minimizes the right-hand side of (19), is given by

$$u(t) = -(C_2^T C_2)^{-1} B_u^T(\rho)P_\rho x(t). \tag{20}$$

As a result, one obtains

$$Inf_u \ Sup_{w\in L^2} H[u,w;\rho_1,\rho_2,\cdots,\rho_m] \le x^T M_\rho x \tag{21}$$

where the parameter-dependent matrix M_ρ is defined as

$$\begin{aligned} M_\rho &:= A^T(\rho)P_\rho + P_\rho A(\rho) - P_\rho(B_u(\rho)(C_2^T C_2)^{-1}B_u^T(\rho) \\ &- \gamma^{-2}B_w(\rho)B_w^T(\rho))P_\rho + C_1^T C_1. \end{aligned} \tag{22}$$

Consequently, for a prescribed performance bound γ if there exists a positive-definite solution $P_\rho>0$ to the matrix inequality $M_\rho<0$ then we have:

$$H[u,w;\rho_1,\rho_2,\cdots,\rho_m]<0, \qquad \forall\, w\in L^2. \tag{23}$$

In the sequel, we provide the robust global asymptotic stability and robust disturbance attenuation in the sense of Definition 1 and Definition 2 for the system (1), respectively. This may be done by converting the parameter-dependent inequality $M_\rho<0$ into the associated LMI and then we are able to determine the positive-definite solution

P_ρ. At first, we state a technical lemma, called *Schur Complement Lemma*.

Lemma 1. Given constant matrices Ψ_1, Ψ_2 and Ψ_3 where $\Psi_1 = \Psi_1^T$ and $\Psi_2 = \Psi_2^T > 0$, then $\Psi_1 + \Psi_3^T \Psi_2^{-1} \Psi_3 < 0$ if and only if

$$\begin{bmatrix} \Psi_1 & \Psi_3^T \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0,$$

or equivalently,

$$\begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^T & \Psi_1 \end{bmatrix} < 0.$$

Using Schur Complement Lemma, the inequality $M_\rho < 0$ holds if and only if

$$\begin{bmatrix} A^T(\rho) P_\rho + P_\rho A(\rho) + C_1^T C_1 & P_\rho B_u(\rho) & P_\rho B_w(\rho) \\ * & -(C_1^T C_1)^{-1} & 0_{l\times s} \\ * & * & -\gamma^2 I_s \end{bmatrix} < 0. \tag{24}$$

The PPDQ function of degree k for the positive-definite matrix $R_\rho = A^T(\rho) P_\rho + P_\rho A(\rho)$ is defined, as

$$R_\rho := (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n)^T R_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n) \tag{25}$$

and according to Remark 1 and some matrix manipulations, the matrix $R_k \in \Re^{(k+1)^m n}$ in (25) which depends linearly on the matrix P_k is obtained as follows:

$$\begin{aligned} R_k = &\left((\hat{J}_k^{m\otimes} \otimes A_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i) \right)^T P_k (\hat{J}_k^{m\otimes} \otimes I_n) \\ &+ (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes A_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i) \right) \end{aligned} \tag{26}$$

where

$$P_\rho := (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^T P_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n) \tag{27}$$

is the PPDQ Lyapunov function of degree $k-1$ for the matrix P_ρ [2].

It is clear that the LMI (24) is parameter-dependent and this dependency makes hard to find the matrix P_ρ using the toolbox LMITOOL of the MATLAB software. To overcome this disadvantage, one lemma is stated to find a parameter-independent LMI from (24).

Lemma 2. Let the degree of the PPDQ function P_ρ is $k-1$. The non-quadratic matrix $P_\rho B_u(\rho)$ can be represented as

$$P_\rho B_u(\rho) := (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n)^T H_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l),$$

the matrix $H_k \in \Re^{((k+1)^m n) \times ((k+1)^m l)}$ which depends linearly on the matrix P_k is defined as

$$H_k = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes B_{0u}) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_{iu}) \right).$$

According to Lemma 2, representation of the matrix $P_\rho B_w(\rho)$ will be as follows:

$$P_\rho B_w(\rho) := (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n)^T F_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_s), \tag{28}$$

where the matrix $F_k \in \Re^{((k+1)^m n) \times ((k+1)^m s)}$ is represented as

$$F_k = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes B_{0w}) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_{iw}) \right). \tag{29}$$

For quadratic parameter-independent matrices $C_1^T C_1$ and $C_2^T C_2$ the PPDQ function representations of degree $k-1$ are as

$$\begin{aligned} C_1^T C_1 &= (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^T \overline{C}_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n) \\ &= (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n)^T (\hat{J}_k^{m\otimes} \otimes I_n)^T \overline{C}_k (\hat{J}_k^{m\otimes} \otimes I_n) \\ &\times (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n), \end{aligned} \tag{30}$$

and

$$\begin{aligned} C_2^T C_2 &= (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l)^T \tilde{C}_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l) \\ &= (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l)^T (\hat{J}_k^{m\otimes} \otimes I_l)^T \tilde{C}_k (\hat{J}_k^{m\otimes} \otimes I_l) \\ &\times (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l), \end{aligned} \tag{31}$$

where the parameter-independent certain matrices $\overline{C}_k$ and $\tilde{C}_k$ are defined, respectively, as

$$\overline{C}_k := Block\ diagonal\ \left(\begin{bmatrix} C_{01}^T \\ C_{11}^T \\ \vdots \\ C_{m1}^T \end{bmatrix} \begin{bmatrix} C_{01} & C_{11} & \cdots & C_{m1} \end{bmatrix},\ \underbrace{0_n, \cdots, 0_n}_{(k^m - m - 1)\, elements} \right) \tag{32}$$

and

$$\tilde{C}_k := Block\ diagonal\ \left(\begin{bmatrix} C_{02}^T \\ C_{12}^T \\ \vdots \\ C_{m2}^T \end{bmatrix} \begin{bmatrix} C_{02} & C_{12} & \cdots & C_{m2} \end{bmatrix},\ \underbrace{0_l, \cdots, 0_l}_{(k^m - m - 1)\, elements} \right).. \tag{33}$$

By substituting the relations (25-33) into the parameter-dependent LMI (24), we conclude the following theorem.

Theorem. Consider the LPV system (1) with the non-singular matrix $C_2^T C_2$. For a prescribed performance bound γ if there exist the positive-definite solution $P_k \in \Re^{(k^m n) \times (k^m n)}$ and the positive definite multipliers $\left(\hat{Q}_{i,k}^{(1)}, \hat{Q}_{i,k}^{(2)}, \hat{Q}_{i,k}^{(3)}\right) \in \Re^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}$ to the following parameter-independent LMI,

$$\begin{bmatrix} \Sigma_{11} & H_k & F_k \\ * & \Sigma_{22} & 0_{(k^m l)\times(k^m s)} \\ * & * & \Sigma_{33} \end{bmatrix} < 0\,, \tag{34}$$

where

$$\Sigma_{11} = R_k + (\hat{J}_k^{m\otimes} \otimes I_n)^T \overline{C}_k (\hat{J}_k^{m\otimes} \otimes I_n) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \times \hat{Q}_{i,k}^{(1)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right) - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(1)} \times \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Sigma_{22} = -(\hat{J}_k^{m\otimes} \otimes I_l)^T \tilde{C}_k (\hat{J}_k^{m\otimes} \otimes I_l) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(2)} \times \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right) - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(2)} \times \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

and

$$\Sigma_{33} = -\gamma^2 (\hat{J}_k^{m\otimes} \otimes I_s)^T \bar{I}_k (\hat{J}_k^{m\otimes} \otimes I_s) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(3)} \times \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right) - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(3)} \times \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

and $\bar{I}_k := Block\ diagonal\ (I_s, \underbrace{0_s, \cdots, 0_s}_{(k^m-1)\,elements})$, then, the optimal control law

$$\begin{cases} \dot{\rho} = \mu_\Theta(\rho, \Phi(x, w, -(C_2^T C_2)^{-1} B_u^T(\rho) P_\rho\ x(t))) \\ u(t) = -(C_2^T C_2)^{-1} B_u^T(\rho) P_\rho\ x(t) \end{cases} \tag{35}$$

achieves both robust global asymptotic stability and robust disturbance attenuation with the attenuation bound γ in the sense of Definition 1 and Definition 2.

Remark 2. It is essential in this result that P_k is calculated independently from the parameter vector ρ and after that P_ρ and the control law are found analytically by (27) and (35), respectively.

Remark 3. It is observed that the inequality (34) is linear in P_k and the positive definite multipliers $\hat{Q}_{i,k}^{(1)}, \hat{Q}_{i,k}^{(2)}, \hat{Q}_{i,k}^{(3)}$ and thus the standard LMI techniques can be exploited to find the positive definite solutions [16]. It is also seen from the above results that the choice of appropriate parameter $k-1$ as the degree of the PPDQ functions plays the role of degree of freedom (DOF) in the design of the adaptive H_∞–control law.

Remark 4 (*Matching condition*). To remove restriction of the condition (14), we use a control with the following form

$$u = u^*(x,\rho) + v(x,\rho,\dot{\rho}) \tag{36}$$

where $u^*(x,\rho) = -(C_2^T C_2)^{-1} B_u^T(\rho) P_\rho\ x(t)$. By replacing the control (36) in the main system (1), one has the following representation of the plant

$$\begin{cases} \dot{x}(t) = A(\theta)\, x(t) + B_u(\theta)\, u^*(x,p) + B_u(\theta)\, v(x,p,\dot{p}) + B_w(\theta)\, w(t) \\ z^*(t) = C_1\, x(t) + C_2\, u^*(x,\rho). \end{cases} \tag{37}$$

Since

$$\|z(t)\|^2 \le \|z^*(t)\|^2 + \|C_2\, v(x,\rho,\dot{\rho})\|^2;$$

the inequality (5) for the controlled output $z(t) = z^*(t) + C_2\, v(x,\rho,\dot{\rho})$ would be guaranteed if we consider $\varepsilon = \varepsilon^* - \int_0^T \|C_2\, v(x,\rho,\dot{\rho})\|^2\, dt$ such ε^* is an arbitrary positive scalar. Using the quadratic energy function (6) and HJI function (7), it is straightforward to find the inequality condition for the system (37) as

$$H[u^*(x,\rho) + v(x,\rho,\dot{\rho}), w; \rho_1, \rho_2, \cdots, \rho_m] < 0, \quad \forall\, w \in L^2 \tag{38}$$

if the following conditions are satisfied:

$$\sum_{i=1}^{m} x^T(t) (\rho_m^{[k]} \otimes \cdots \otimes \tilde{\rho}_i^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^T P_k \times (\rho_m^{[k]} \otimes \cdots \otimes \rho_i^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n) x(t)\, \dot{\rho}_i + x^T(t) P_\rho\, B_u(\rho)\, v(x(t),\rho,\dot{\rho}) \le 0\,, \tag{39}$$

$$\rho \in \Theta\,, \tag{40}$$

$$(\rho - \theta)^T Q\ (\dot{\rho} - \Phi(x, w, u^*(x,\rho) + v(x,\rho,\dot{\rho}))) \le 0\,. \tag{41}$$

The condition (39) is called the *matching condition* that will be assumed to hold for some function [13].

IV. SIMULATION RESULTS

In this section, we illustrate the methodology proposed on a second order LPV system as presented in [13].

The LPV plant in the state-space form is given by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -(1+\theta_1) & -(1+\theta_2) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -1+\theta_2 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{cases} \tag{42}$$

where $x = [x_1 \quad x_2]^T$, $w = [w_1 \quad w_2]^T$ and $x(0) = [1 \quad -1]^T$ are state vector, disturbance vector and initial condition, respectively. Also, the parameter vector θ is defined as

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \in \Theta = [-0.04, 5] \times [-5, 0.04].$$

By solving the matrix inequality (34) for $\gamma = 1$ using the Lmitool toolbox of the Matlab software [16], the positive definite matrices can be calculated for $k = 2$.

Robust stability and disturbance attenuation of the states in the presence of disturbance have been depicted in Figure 1.

Therefore, we conclude that system (42) can be stabilized by the control law (35), which has been depicted in Figure 2. Since the parameters θ_1 and θ_2 are unknown, the adaptive H_∞–control (35) is used. The simulation results for two parameters θ_1 and θ_2 are shown in Figure 3.

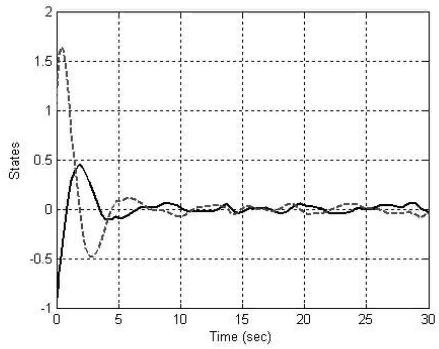

Fig. 1. Robust stability of the controlled output: First state (solid), and Second state (dotted line).

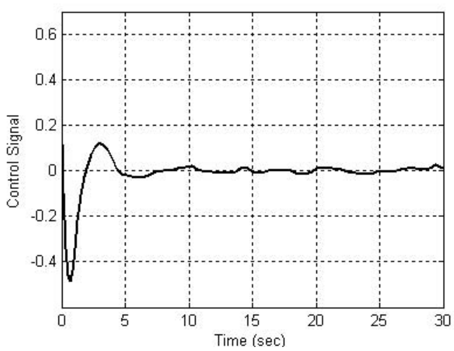

Fig. 2. Adaptive H_∞ -control.

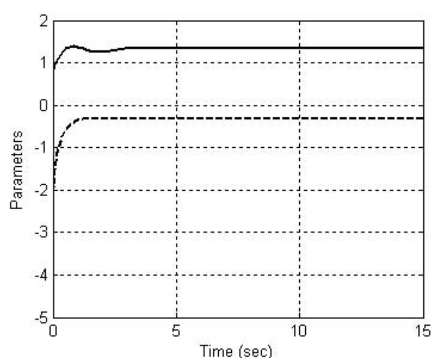

Fig. 3. Time behaviours of Parameters: $\rho_1(t)$ (solid), and $\rho_2(t)$ (dotted line).

V. CONCLUSION

In this paper, the issue of adaptive H_∞-control design for a class of LPV systems based on the Hamiltonian-Jacobi-Isaac (HJI) method was investigated. By combining the idea of polynomially parameter-dependent quadratic functions and vector projection method to derive the adaptive H_∞-control, sufficient conditions with high precision were given to guarantee robust asymptotic stability and disturbance attenuation of the LPV systems with unknown constant parameters. Finally, a simulation example was given to illustrate the applicability of the proposed method.

Acknowledgment

The German Academic Exchange Service (DAAD) financially supported this research project.

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