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Discrete-Time Control Systems Approach for Optimal Smoothing Splines

Hiroyuki Kano, Hiroyuki Fujioka and Keiichi Inoue

Abstract— **We consider the problem of designing optimal smoothing spline curves by employing an approach based on linear control systems. First, the problem is formulated using continuous-time, time-invariant systems with piecewise constant inputs. Then by introducing discrete time-varying systems, the solutions for optimal splines including periodic splines are derived. The existence conditions for unique optimal solutions are established, where the concepts of controllability and observability play central roles. The computational procedures for the optimal splines are straightforward. The design method for periodic splines is applied to a shape synthesizing problem using jellyfish as the example.**

I. INTRODUCTION

Spline functions have been used in various fields including computer graphics, numerical analysis, image processing, trajectory planning of robot and aircraft, and data analysis in general [1]. Recently, using B-splines [2], the authors studied smoothing splines and applied to generating cursive characters [3], [4] and modeling of Dow-Jones industrial data [5].

The studies on splines have a long history, but there have been relatively new developments , called dynamic splines (e.g. [6], [7]). Namely, it has been shown that, by considering linear continuous-time control systems, various types of spline functions can be generated and that spline interpolation and smoothing problems can be treated in a unified framework [8], [9]. Also B-splines have been studied from the viewpoints of optimal control theory [10]. Moreover, by introducing a class of discrete-time systems, the problem of 'multilevel' interpolation is considered in [11]. Namely, not only the function value but also its derivatives are interpolated.

In this paper, we study the problems of smoothing splines by considering a linear SISO continuous-time, time-invariant system. We restrict the control inputs to be piecewise constant and no other conditions are imposed. We design optimal smoothing splines and periodic splines by formulating the problems as optimal control problems. The continuous-time system is sampled to yield a discrete-time system, which is time-varying when the specified knot points are not equally spaced. Necessary and sufficient conditions are established for the existence of unique optimal solutions to the smoothing and periodic smoothing spline problems. We see that the conditions are closely related to, and in fact in some special cases are exactly, the controllability and observability of the sampled system. The periodic splines can be used for example to model contours or shapes [12] of various objects.

The smoothing splines and periodic splines have been treated in various papers. In fact, the periodic splines are treated in [13] by dynamic splines approach and in [14] by using B-splines as the basis functions. The present approach by means of discrete-time control systems provides a new framework for studying spline interpolation and smoothing problems. The analytical results on the existence of unique optimal solutions are such examples, and moreover we can easily compute the optimal splines by a combination of the discrete- and continuous-time systems.

The rest of this paper is organized as follows. In Section II, we formulate the smoothing and periodic smoothing spline problems. In Section III, we establish necessary and sufficient conditions for the existence of unique optimal solutions. Some special cases including the case of the standard splines are treated in Section IV. In Section V, we apply the results on periodic splines to synthesizing the shape of jellyfish from its image data. Concluding remarks are given in Section VI.

II. PROBLEM STATEMENT

Let $y(t),\ t \in [0,T]$, be a polynomial spline of degree n with the knot points t_k,

$$t_0(=0) < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_m(=T). \tag{1}$$

Since $y(t)$ is a piecewise polynomial of degree n and is $(n-1)$-times continuously differentiable, it can be written as

$$y^{(n+1)}(t) = 0, \quad t \in [t_k, t_{k+1}),\ k = 0,1,\cdots,m-1, \tag{2}$$

with the continuity conditions

$$y^{(i)}(t_k-) = y^{(i)}(t_k+),\ i = 0,1,\cdots,n-1, \tag{3}$$

for $k = 1,2,\cdots,m-1$. We rewrite (2) as

$$y^{(n)}(t) = \text{const.} = u_k, \tag{4}$$

for $t \in [t_k, t_{k+1}),\ k = 0,1,\cdots,m-1$.

Then, letting $x \in \mathrm{R}^n$ be

$$x = \begin{bmatrix} y & y^{(1)} & \cdots & y^{(n-1)} \end{bmatrix}^T, \tag{5}$$

$y(t)$ is expressed as the output of the following continuous-time linear system:

$$\begin{aligned} \dot{x} &= Ax + bu,\ x(t_0) = x_0 \\ y &= cx. \end{aligned} \tag{6}$$

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The authors are with the Department of Information Sciences, Tokyo Denki University, Hatoyama, Hiki-gun, Saitama 350-0394, Japan. kano@j.dendai.ac.jp, fujioka@j.dendai.ac.jp, keiichi@eyes.j.dendai.ac.jp

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Here $u \in$U is a piecewise constant control input with

$$\mathscr{U} = \{u(t) : u(t) = u_k, t \in [t_k, t_{k+1}), k = 0, 1, \cdots, m-1\}, \quad (7)$$

and $A \in \mathrm{R}^{n\times n},\ b \in \mathrm{R}^n,\ c \in \mathrm{R}^{1\times n}$ are defined by

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \\ c &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \end{aligned} \quad (8)$$

Thus the design of spline functions is regarded as that of the control input $u \in \mathscr{U}$ and the initial state $x_0 \in \mathrm{R}^n$.

Now suppose that we are given a set of data

$$\mathscr{D} = \{(s_k, \alpha_k) : s_k \in [0, T], \alpha_k \in \mathrm{R}, k = 1, 2, \cdots, N\} \quad (9)$$

where we assume $s_i \neq s_j$ for $i \neq j$. Then we consider the following two problems.

Problem 1: (smoothing spline) Find an optimal control $u^\star$ and an optimal initial state $x_0^\star$ such that

$$\min_{u\in\mathscr{U}, x_0\in \mathrm{R}^n} J(u, x_0),$$

where

$$J(u, x_0) = \lambda \int_0^T u^2(t)dt + \sum_{k=1}^N w_k (y(s_k) - \alpha_k)^2, \quad (10)$$

$\lambda > 0$, and $w_k > 0\ \forall k$ with $w_1 + w_2 + \cdots + w_N = 1$.

Problem 2: (periodic smoothing spline) Find an optimal control $u^\star$ and an optimal initial state $x_0^\star$ such that

$$\min_{u\in\mathscr{U}, x_0\in \mathrm{R}^n} J(u, x_0),$$

subject to the constraints

$$y^{(i)}(t_0) = y^{(i)}(t_m),\ i = 0, 1, \cdots, n-1, \quad (11)$$

for $J(u, x_0)$ in (10).

Note that the constraint (11) together with (3) produces $(n-1)$-times continuously differentiable periodic curve $y(t)$ in $[0, +\infty)$ with the period $T\ (= t_m)$.

In the next section, we derive the optimal solutions. Note that we do not assume A, b, c of the forms in (8). Moreover, in Section IV, the three cases where the knot points t_k are equally-spaced, the data points s_k are equally-spaced, and A, b, c are in the forms in (8) are treated as the special cases.

In order to solve these problems, it is convenient to introduce a discrete-time system obtained by sampling (6) at the knot points. Namely, with $x_k = x(t_k),\ y_k = y(t_k)$ and the sampling interval

$$h_k = t_{k+1} - t_k, \quad (12)$$

we consider the following system,

$$\begin{aligned} x_{k+1} &= \Phi_{k+1,k} x_k + g_k u_k,\ k = 0, 1, \cdots \\ y_k &= c x_k, \end{aligned} \quad (13)$$

where

$$\Phi_{k+1,k} = \mathrm{e}^{Ah_k}, \quad (14)$$

$$g_k = \int_{t_k}^{t_{k+1}} \mathrm{e}^{A(t_{k+1}-\tau)} b d\tau = \int_0^{h_k} \mathrm{e}^{A(h_k-\tau)} b d\tau. \quad (15)$$

Obviously $\Phi_{k+1,k}$ is nonsingular for all k. Once the state x_k and the input u_k are determined for $k = 0, 1, \cdots, m-1$, the spline is obtained in each interval $[t_k, t_{k+1})$ by

$$\begin{aligned} x(t) &= \mathrm{e}^{A(t-t_k)} x_k + g(t - t_k) u_k, \\ y(t) &= c x(t), \end{aligned} \quad (16)$$

where $g(t)$ is defined by

$$g(t) = \int_0^t \mathrm{e}^{A(t-\tau)} b d\tau. \quad (17)$$

III. OPTIMAL SOLUTIONS

We establish optimal solutions for Problems 1 and 2.

A. Smoothing Splines

First we solve Problem 1. In the cost function $J(u, x_0)$ in (10), the integral term is expressed as

$$\int_0^T u^2(t)dt = \|\bar{u}\|_H^2,$$

where $\bar{u} \in \mathrm{R}^m$ and $H \in \mathrm{R}^{m\times m}$ are defined by

$$\begin{aligned} \bar{u} &= \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \end{bmatrix}^T, && (18) \\ H &= \mathrm{diag}\{h_0,\ h_1,\ \cdots,\ h_{m-1}\}. && (19) \end{aligned}$$

The second term in (10) is written as

$$\sum_{k=1}^N w_k (y(s_k) - \alpha_k)^2 = \|\bar{y} - \alpha\|_W^2,$$

where

$$\begin{aligned} \bar{y} &= \begin{bmatrix} y(s_1) & y(s_2) & \cdots & y(s_N) \end{bmatrix}^T, \\ \alpha &= \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{bmatrix}^T, \\ W &= \mathrm{diag}\{w_1,\ w_2,\ \cdots,\ w_N\}. \end{aligned} \quad (20)$$

Here $y(s_k)$ is obtained from (6) as

$$y(s_k) = c\mathrm{e}^{As_k} x_0 + \int_0^{s_k} c\mathrm{e}^{A(s_k-\tau)} b u(\tau) d\tau.$$

By introducing $f_t(\tau)$:

$$f_t(\tau) = \begin{cases} c\mathrm{e}^{A(t-\tau)} b & \tau < t \\ 0 & \tau \geq t \end{cases}, \quad (21)$$

$y(s_k)$ is expressed as

$$\begin{aligned} y(s_k) &= c\mathrm{e}^{As_k} x_0 + \int_0^T f_{s_k}(\tau) u(\tau) d\tau \\ &= c\mathrm{e}^{As_k} x_0 + d_k^T \bar{u}, \end{aligned} \quad (22)$$

where $d_k \in \mathrm{R}^m$ is defined by

$$\begin{aligned} d_k = \Big[& \textstyle\int_{t_0}^{t_1} f_{s_k}(\tau) d\tau \quad \int_{t_1}^{t_2} f_{s_k}(\tau) d\tau \quad \cdots \\ & \cdots \quad \textstyle\int_{t_{m-1}}^{t_m} f_{s_k}(\tau) d\tau \ \Big]^T. \end{aligned} \quad (23)$$

Thus $\bar{y}$ is obtained as

$$\bar{y} = C x_0 + D\bar{u}, \quad (24)$$

where the matrices $C \in \mathrm{R}^{N\times n}$ and $D \in \mathrm{R}^{N\times m}$ are defined by

$$C = \begin{bmatrix} c e^{As_1} \\ c e^{As_2} \\ \vdots \\ c e^{As_N} \end{bmatrix}, \tag{25}$$

$$D = \begin{bmatrix} d_1 & d_2 & \cdots & d_N \end{bmatrix}^T. \tag{26}$$

Thus the cost function $J(u, x_0)$ is expressed in terms of $\bar{u}$ and x_0 as

$$J(\bar{u}, x_0) = \lambda \|\bar{u}\|_H^2 + \|Cx_0 + D\bar{u} - \alpha\|_W^2. \tag{27}$$

Taking the derivatives with respect to $\bar{u}$ and x_0 and setting zero yield the following set of algebraic equations.

$$\begin{bmatrix} \lambda H + D^T W D & D^T W C \\ C^T W D & C^T W C \end{bmatrix} \begin{bmatrix} \bar{u} \\ x_0 \end{bmatrix} = \begin{bmatrix} D^T W \alpha \\ C^T W \alpha \end{bmatrix}. \tag{28}$$

The optimal solution of Problem 1, if it exists, is obtained as the solution of (28).

Theorem 1: Problem 1 has a unique optimal solution $u^\star$ and $x_0^\star$ if and only if rank $C = n$.

(Proof) Denoting the coefficient matrix in (28) by $\mathscr{A} \in \mathrm{R}^{(m+n)\times(m+n)}$, it may be expressed as

$$\mathscr{A} = \begin{bmatrix} \lambda H & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D & C \end{bmatrix}^T W \begin{bmatrix} D & C \end{bmatrix}, \tag{29}$$

and hence $\mathscr{A} \geq 0$. Moreover, suppose that $\mathscr{A} a = 0$ for a vector $a \in \mathrm{R}^{m+n}$, and let $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1 \in \mathrm{R}^m, a_2 \in \mathrm{R}^n$. Then we get

$$\begin{aligned} &(\lambda H + D^T W D) a_1 + D^T W C a_2 = 0 \\ &C^T W D a_1 + C^T W C a_2 = 0. \end{aligned}$$

Manipulating these equations, we obtain

$$\lambda \|a_1\|_H^2 + \|Da_1 + Ca_2\|_W^2 = 0. \tag{30}$$

Thus $a_1 = 0$ since $H > 0$, and hence $Ca_2 = 0$ since $W > 0$. It is then obvious that rank $C = n$ is necessary and sufficient for $a_2 = 0$ or regularity of $\mathscr{A}$. In other words, $J(\bar{u}, x_0)$ is strictly convex if and only if rank $C = n$ holds. (QED)

Remark 1: For any matrices $S = S^T \geq 0$, U and vector v of compatible dimensions, it holds that

$$\mathrm{rank}(S + U^T U,\ U^T v) \ = \ \mathrm{rank}(S + U^T U).$$

Hence, in view of (28) and (29), we easily see that (28) is always consistent. Thus if rank $C = n$ does not hold, there are infinitely many optimal solutions for Problem 1. Then the minimum-norm solution of (28) may be employed.

B. Periodic Smoothing Splines

Next we consider Problem 2. The constraints in (11) are nothing but

$$x_0 = x_m. \tag{31}$$

From (13), x_m is obtained as

$$\begin{aligned} x_m &= \Phi_{m,0} x_0 + \sum_{k=0}^{m-1} \Phi_{m,k+1} g_k u_k \\ &= \Phi_{m,0} x_0 + G\bar{u}, \end{aligned} \tag{32}$$

where the matrix $G \in \mathrm{R}^{n\times m}$ is given by

$$G = \begin{bmatrix} \Phi_{m,1} g_0 & \Phi_{m,2} g_1 & \cdots & g_{m-1} \end{bmatrix}. \tag{33}$$

Now, using the cost function in (27), we form the following Lagrangian

$$\begin{aligned} L(\bar{u}, x_0, \mu) &= J(\bar{u}, x_0) + \mu^T (x_0 - x_m) \\ &= \lambda \|\bar{u}\|_H^2 + \|Cx_0 + D\bar{u} - \alpha\|_W^2 \\ &\quad + \mu^T (x_0 - \Phi_{m,0} x_0 - G\bar{u}), \end{aligned} \tag{34}$$

where $\mu \in \mathrm{R}^n$ is the Lagrangian multiplier. Taking the derivatives with respect to $\bar{u}, x_0$ and μ yields

$$\begin{bmatrix} \lambda H + D^T W D & D^T W C & -G^T \\ C^T W D & C^T W C & I_n - \Phi_{m,0}^T \\ -G & I_n - \Phi_{m,0} & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ x_0 \\ \frac{1}{2}\mu \end{bmatrix} = \begin{bmatrix} D^T W \alpha \\ C^T W \alpha \\ 0 \end{bmatrix}. \tag{35}$$

Thus the optimal solution is obtained as the solution of this algebraic equation.

We now have the following theorem.

Theorem 2: Problem 2 has a unique optimal solution $u^\star$ and $x_0^\star$ if and only if rank $C = n$ and rank $G = n$.

(Proof) Denoting the coefficient matrix of (35) by $\hat{\mathscr{A}}$, we first examine its regularity. Letting

$$\hat{\mathscr{A}} a = 0,\ a \in \mathrm{R}^{m+2n},$$

and partitioning a into three vectors $a_1 \in \mathrm{R}^m,\ a_2, a_3 \in \mathrm{R}^n$ compatibly with $\hat{\mathscr{A}}$ yield

$$(\lambda H + D^T W D) a_1 + D^T W C a_2 - G^T a_3 = 0 \tag{36}$$

$$C^T W D a_1 + C^T W C a_2 + (I_n - \Phi_{m,0}^T) a_3 = 0 \tag{37}$$

$$-G a_1 + (I_n - \Phi_{m,0}) a_2 = 0. \tag{38}$$

Premultiplying a_1^T to (36), premultiplying a_2^T to (37), and using (38), we obtain

$$\lambda \|a_1\|_H^2 + \|Da_1 + Ca_2\|_W^2 = 0,$$

the same equation as (30). Thus we get $a_1 = 0$, $Ca_2 = 0$, and further $G^T a_3 = 0$ by (36).

We then see that (35) has a unique solution, i.e. $\hat{\mathscr{A}} a = 0$ implies $a = 0$, if and only if rank $C = n$ and rank $G = n$. Moreover, under these rank conditions, the cost function $J(\bar{u}, x_0)$ in (34) is strictly convex in $\bar{u}$ and x_0 as shown in the proof of Theorem 1. Thus the solution of (35) provides the optimal solution to Problem 2. (QED)

Remark 2: The condition rank $G = n$ is the controllability condition

$$V(0, m) = \sum_{i=0}^{m-1} \Phi_{m,i+1} g_i g_i^T \Phi_{m,i+1}^T > 0. \tag{39}$$

Finally, in this subsection, we examine in details the structure of the matrix G in (33). Denoting its i-th column by G_i, we obtain

$$\begin{aligned} G_i &= \Phi_{m,i} g_{i-1} = \int_{t_{i-1}}^{t_i} e^{A(t_m - \tau)} b d\tau \\ &= \left(\int_{t_{i-1}}^{t_m} - \int_{t_i}^{t_m} \right) e^{A(t_m - \tau)} b d\tau. \end{aligned} \tag{40}$$

Moreover, by changing the integration variable and using (17), we get

$$\begin{aligned}\int_{t_i}^{t_m} e^{A(t_m-\tau)} b d\tau &= \int_0^{t_m-t_i} e^{A(t_m-t_i-\tau)} b d\tau \\ &= g(t_m-t_i).\end{aligned}$$

Thus G_i is written as

$$G_i = g(\delta_{i-1}) - g(\delta_i),\ i=1,2,\cdots,m, \tag{41}$$

where δ_i is defined by

$$\delta_i = t_m - t_i = T - t_i,\ i=0,1,\cdots,m. \tag{42}$$

In particular, $\delta_m = 0$ and hence $G_m = g(\delta_{m-1})$.

Remark 3: For an optimal choice of the smoothing parameter λ, we may employ the so-called cross validation method (e.g. [1]): Let $\mathscr{D}_l$ be the data set obtained from $\mathscr{D}$ in (9) by deleting the l-th data, i.e.

$$\mathscr{D}_l = \mathscr{D} - \{(s_l, \alpha_l)\},\ l=1,2,\cdots,N, \tag{43}$$

and let $y^\star_{\lambda,l}(t)$ be the splines constituted from the above optimal solutions for the data $\mathscr{D}_l$. Then an optimal λ is obtained by minimizing the cross validation function,

$$V(\lambda) = \sum_{l=1}^{N} w_l (y^\star_{\lambda,l}(s_l) - \alpha_l)^2. \tag{44}$$

IV. SOME SPECIAL CASES

The results established in the previous section are examined in details for the following three special cases.

A. The Case of Equally-Spaced Knot Points

We consider the case where the knot points t_k are equally spaced, namely

$$t_{k+1} - t_k = \text{const.} = h\ \forall k. \tag{45}$$

Then, the sampled system in (13) is time-invariant, and is given by

$$\begin{aligned} x_{k+1} &= \Phi x_k + g u_k,\ k=0,1,\cdots \\ y_k &= c x_k, \end{aligned} \tag{46}$$

where Φ and g are given from (14) and (15) as

$$\Phi = \Phi_{k+1,k} = e^{Ah},\ g = g_k = \int_0^h e^{A(h-\tau)} b d\tau. \tag{47}$$

In this case, the matrix G in (33) is written as

$$G = \begin{bmatrix} \Phi^{m-1} g & \Phi^{m-2} g & \cdots & g \end{bmatrix}. \tag{48}$$

Thus the condition rank $G = n$ is the same as the controllability of the pair (Φ, g), and Theorem 2 can be restated as follows.

Corollary 1: Assume that the knot points t_k are equally spaced. Then, Problem 2 has a unique optimal solution $u^\star$ and $x_0^\star$ if and only if rank $C = n$ and the pair (Φ, g) is controllable.

Regarding the relation of controllability between a continuous-time system and the sampled discrete-time system, the following result holds (see e.g. [15], Theorem C-2), where $\Re[\cdot]$ and $\Im[\cdot]$ respectively denotes real and imaginary part of complex numbers.

Lemma 1: Assume that the pair (A, b) in (6) is controllable. Then the pair (Φ, g) in (46) is controllable if and only if $\Im[\lambda_i(A) - \lambda_j(A)] \neq 2\pi k/h$ for $k = \pm 1, \pm 2, \cdots$, whenever $\Re[\lambda_i(A) - \lambda_j(A)] = 0$.

Thus, unless the matrix A has complex conjugate eigenvalues $\gamma \pm j\delta$ satisfying $\delta = \pi k/h$ for some $k = \pm 1, \pm 2, \cdots$, the controllability condition of (Φ, g) in Corollary 1 can be replaced with the controllability of (A, b).

B. The Case of Equally-Spaced Data Points

Let us examine the case where the data points s_k in (9) are equally spaced as

$$s_{k+1} - s_k = \text{const.} = \beta\ \forall k. \tag{49}$$

Then the matrix C in (25) is written as

$$C = C_{0,N-1} e^{As_1}, \tag{50}$$

where

$$C_{0,N-1} = \begin{bmatrix} c \\ c\Psi \\ \vdots \\ c\Psi^{N-1} \end{bmatrix}, \tag{51}$$

and $\Psi = e^{A\beta}$. It is obvious that, when $N \geq n$, rank $C =$ rank $C_{0,N-1} =$ rank $C_{0,n-1}$. Thus the condition rank $C = n$ in Theorems 1 and 2 is the observability of the pair (c, Ψ):

Corollary 2: Assume that the data points s_k are equally spaced. Then, Problem 1 has a unique optimal solution $u^\star$ and $x_0^\star$ if and only if the pair (c, Ψ) is observable. Problem 2 has a unique optimal solution if and only if the pair (c, Ψ) is observable and rank $G = n$.

Again, using Lemma 1 and the duality, the observability of the pair (c, Ψ) is linked to that of (c, A) for continuous-time system (6): The observability condition of the pair (c, Ψ) is equivalent to that of (c, A) unless the matrix A has complex conjugate eigenvalues $\gamma \pm j\delta$ satisfying $\delta = \pi k/\beta$ for some $k = \pm 1, \pm 2, \cdots$.

C. The Case of A, b, c given in (8)

We consider the case where A, b, and c in (6) are given by (8). This case corresponds to the standard polynomial splines.

It is then possible to examine in details the existence conditions in Theorems 1 and 2, and moreover we can derive more explicit expressions for the matrices and vectors arising in the optimal solutions.

First of all, e^{At} becomes

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix}, \tag{52}$$

and $g(t)$ in (17) is computed as

$$g(t) = \begin{bmatrix} \frac{t^n}{n!} & \frac{t^{n-1}}{(n-1)!} & \cdots & t \end{bmatrix}^T. \quad (53)$$

We now show that the condition rank $C = n$ always holds when $N \geq n$. In (25), we get

$$ce^{As_k} = \begin{bmatrix} 1 & s_k & \frac{s_k^2}{2!} & \cdots & \frac{s_k^{n-1}}{(n-1)!} \end{bmatrix}. \quad (54)$$

Thus, denoting the matrix consisting of the first n rows of matrix C by $\bar{C}$, we see that

$$\bar{C} = \begin{bmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{n-1} \\ & \cdots & \cdots & & \\ 1 & s_n & s_n^2 & \cdots & s_n^{n-1} \end{bmatrix} S, \quad (55)$$

where $S = \mathrm{diag}\{1 \ \frac{1}{1!} \ \frac{1}{2!} \ \cdots \ \frac{1}{(n-1)!}\}$. Noting $s_i \neq s_j$ for $i \neq j$, the result on Vandermonde matrix shows that the matrix $\bar{C}$ is nonsingular, and hence rank $C = n$ always holds.

Next we consider the matrix G in (33). Since its i-th column G_i is given by (41) with $G_m = g(\delta_{m-1})$, it holds that

$$\begin{aligned} \mathrm{rank} G &= \mathrm{rank} \big[\ g(\delta_0) - g(\delta_1) \quad g(\delta_1) - g(\delta_2) \\ &\qquad \cdots \quad g(\delta_{m-2}) - g(\delta_{m-1}) \quad g(\delta_{m-1})\ \big] \\ &= \mathrm{rank} \big[\ g(\delta_0) \quad g(\delta_1) \quad \cdots \quad g(\delta_{m-1})\ \big] \end{aligned} \quad (56)$$

Now, assuming that $m \geq n$, let us consider the square submatrix $\bar{G} \in \mathrm{R}^{n \times n}$ defined as

$$\bar{G} = \begin{bmatrix} g(\delta_0) & g(\delta_1) & \cdots & g(\delta_{n-1}) \end{bmatrix}. \quad (57)$$

Then using the function $g(t)$ in (53), $\bar{G}$ may be written as

$$\bar{G} = \hat{S} \begin{bmatrix} \delta_0^{n-1} & \delta_1^{n-1} & \cdots & \delta_{n-1}^{n-1} \\ & \cdots & & \cdots \\ \delta_0 & \delta_1 & \cdots & \delta_{n-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \Delta,$$

where $\hat{S} = \mathrm{diag}\{\frac{1}{n!} \ \frac{1}{(n-1)!} \ \cdots \ \frac{1}{1!}\}$ and $\Delta = \mathrm{diag}\{\delta_0 \ \delta_1 \ \cdots \ \delta_{n-1}\}$. Since $\delta_i \neq \delta_j$ for $i \neq j$, we see that the matrix $\bar{G}$ is nonsingular, and hence rank $G = n$ always holds.

Thus, Theorems 1 and 2 may be written as follows.

Corollary 3: Assume that A, b, c are given by (8). The condition $N \geq n$ is necessary and sufficient for Problem 1 to have a unique optimal solution $u^\star$ and $x_0^\star$. On the other hand, the conditions $N \geq n$ and $m \geq n$ are necessary and sufficient for Problem 2 to possess a unique optimal solution.

Finally, in this subsection, we present the matrix D in (26) in a more explicit form. Consider the vector d_k in (23), and let j $(0 \leq j < m)$ be such that $t_j < s_k \leq t_{j+1}$. Then d_k is of the following form

$$d_k = \begin{bmatrix} d_{k,1} & \cdots & d_{k,j+1} & 0 & \cdots & 0 \end{bmatrix}^T. \quad (58)$$

Here, $d_{k,i}$ is computed as

$$\begin{aligned} d_{k,i} &= \int_{t_{i-1}}^{t_i} ce^{A(s_k-\tau)} bd\tau \\ &= \frac{1}{n!}\left[(s_k - t_{i-1})^n - (s_k - t_i)^n\right] \end{aligned} \quad (59)$$

for $i = 1, 2, \cdots, j$, and $d_{k,j+1}$ is obtained by

$$d_{k,j+1} = \int_{t_j}^{s_k} ce^{A(s_k-\tau)} bd\tau = \frac{1}{n!}(s_k - t_j)^n. \quad (60)$$

V. EXPERIMENTAL RESULTS

We apply the above method for designing optimal periodic smoothing splines to a contour synthesis problem. In particular, we model the shape of a jellyfish from its image data.

The data $\mathscr{D}$ in (9) is obtained as follows. First the image data are digitized into two levels by using an image processing technique and the gravity center is computed. Fixing an $O-pq$ plane with its origin O at the gravity center, the distance α (in pixels) is computed from the origin to the boundary pixel by increasing an angle s by $\delta s = 1$ (degree).

In the following design example, we selected the data consisting of 36 data points (i.e. $N = 36$) which correspond to the measurement at every 10 degrees. Thus the data points are equally spaced. The weights w_k are set as $w_k = 1/N$ $\forall k$. We used (A, b, c) of the form in (8) with $n = 3$ and equally spaced knot points with $t_{k+1} - t_k = h = 10$. Thus Corollary 3 guarantees the existence of unique optimal periodic solution.

Figure 1 shows the results of designed optimal periodic spline $y(t)$ (blue line) together with the data points ('$*$' mark). Green and red lines show $y^{(1)}(t)$ and $y^{(2)}(t)$ respectively. In Figure 2, the value of the cross validation function $V(\lambda)$ in (44) is plotted in the interval $[10^{-0.5}, 10^{3.5}]$. We confirmed that, outside of this interval, $V(\lambda)$ increases further, and very fast beyond $10^{3.5}$ in particular. The optimal value of the smoothing parameter was found to be $\lambda^\star = 251.1886$. Figure 3 shows the reconstructed contour in the $O-pq$ plane, and the optimal control input $u^\star(t)$ is plotted in Figure 4. The original image is overlaid with the constructed contour in Figure 5. From these figures, we may observe that we have recovered fairly good contour by the present model.

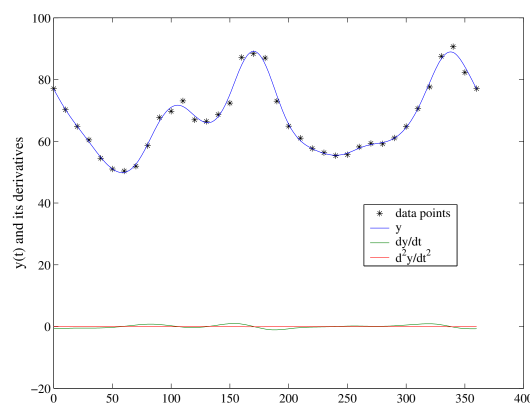

Fig. 1. Optimal periodic spline $y(t)$ and its derivatives.

VI. CONCLUDING REMARKS

We considered the problem of designing optimal smoothing spline curves by employing an approach based on linear control systems. The problem is formulated using

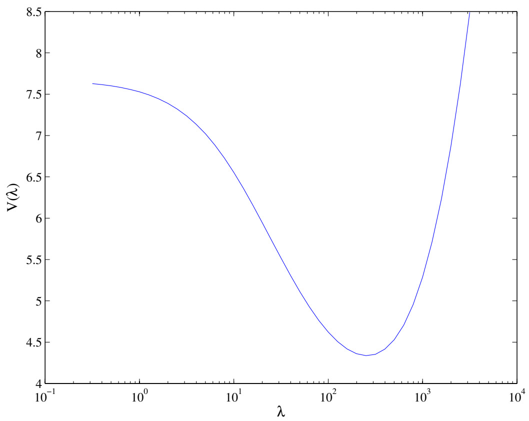

Fig. 2. Cross validation function.

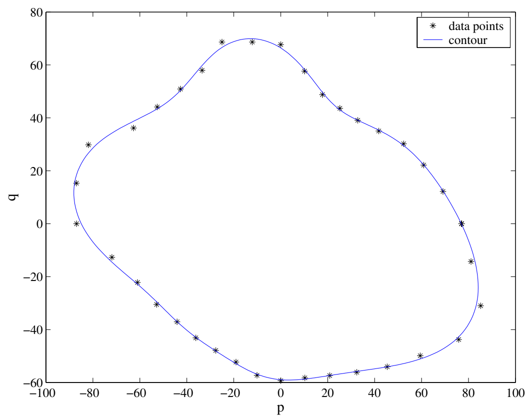

Fig. 3. Constructed contour.

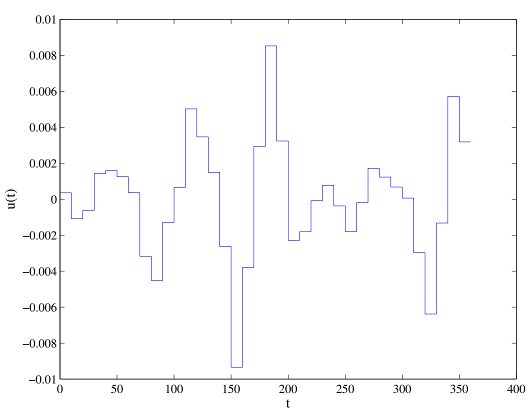

Fig. 4. Optimal control input.

continuous-time, time-invariant systems with piecewise constant inputs and the sampled discrete (in general, time-varying) systems. We established necessary and sufficient conditions for the existence of unique optimal smoothing and periodic smoothing splines. These conditions are closely related to the controllability and observability. We believe that this approach by means of continuous- and discrete-time control systems will allows us to see a various kind of spline problems more in details in a more systematic fashion. As the results, the computational procedures for the optimal splines are straightforward. The design method for periodic splines is applied successfully to synthesizing the shape of jellyfish from its image data.

Fig. 5. Original image and the constructed contour.

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