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A Locally Weighted Learning Method for Online Approximation Based Control

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Abstract— **This article is concerned with tracking control problems for nonlinear systems that are not affine in the control signal and that contain unknown nonlinearities in the system dynamic equations. This paper develops a piecewise linear approximation to the unknown functions during the system operation. New control and parameter adaptation algorithms are designed and analyzed using Lyapunov-like methods. The objectives are to achieve global stability of the state, accurate tracking of bounded reference signals contained within a known domain $\mathcal{D}$, and at least boundedness of the function approximation parameter estimates.**

Keywords: *Adaptive approximation based control, receptive field weighted regression, adaptive nonlinear control, locally weighted learning.*

I. Introduction

Several adaptive on-line approximation based controllers have been proposed that use cooperative learning methods to approximate unknown nonlinearities sufficiently well to achieve a specified level of command tracking, e.g. [2], [8], [9], [10], [11]. Cooperative learning adjusts the parameters of the linear combination so that the weighted linear combination of the basis elements jointly achieves the approximation accuracy sufficient to achieve the tracking objective. The total number of basis functions (and definition of each) directly affects the complexity of the functions that can be approximated over a compact set $\mathcal{D}$ that defines the region of operation of the system. Since the parameters of the approximator are jointly adjusted to achieve the tracking objective, increasing the number of basis elements may cause the learning approach to locally over-parameterize the approximation and fit noise in the data.

The idea of locally weighted regression (LWR) is discussed in [1]. Locally weighted regression uses approximators composed of local models each of which is adjusted independently to achieve local approximation accuracy over a small subregion S_k of the domain $\mathcal{D}$ such that $\mathcal{D} \subset \bigcup_{k=1}^{N} S_k$. Locally weighted regression decreases the effects of overfitting the data and facilitates the definition and adjustment of the approximator structure. The original LWR results did not include closed loop stability analysis. Stability analysis, of LWR based tracking control methods, is contained in [6], [7]. These initial LWR stability results contained two major limitations. First, the analysis considered systems of the form $\dot{x} = f(x) + g(x)u$ where $g(x) = 1$ and $x \in \Re$. Second, the approach assumed that both x and $\dot{x}$ were measured and available for use in the adaptive control law. The use of the state derivative in the implementation is undesirable and removal of this assumption is one of the issues addressed herein. In addition, we will develop an approach applicable to affine systems of the form $\dot{x} = f(x)+g(x)u$ where $g(x) \neq 1$ or (controllable) non-affine systems of the form $\dot{x} = f(x, u)$. Due to space limitations, we will still limit our analysis to scalar systems $x \in \Re$. Additional contributions relative to the existing literature include: addressing initial conditions which are outside of the compact set $\mathcal{D}$, analysis of existence and stability of solutions that result from the (switching) LWR controller, and addressing non-affine systems.

The problem statement and LWR approximator are formulated in Section II. The control law addressing initial conditions both within and outside of $\mathcal{D}$ is defined and analyzed in Section III. The resulting controller involves switching which can be analyzed in the sense of Filippov. The approximator parameter adaptation is defined in Section IV. Overall stability analysis is concluded in Section V. The extension to nonaffine systems is discussed in Section VI.

II. System Formulation

Sections II–V of this paper consider the scalar affine system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \Re$, $f(x)$ and $g(x)$ are unknown continuous nonlinear functions and u is a control input. We are interested in tracking problems, where the desired output $x_d(t)$ and its time derivative $\dot{x}_d(t)$ are available at any $t \geq 0$. We assume that $x_d(t) \in \mathcal{D}$ for all $t \geq 0$ where the compact region $\mathcal{D}$ is known and contains the origin $x = 0$. Finally, we assume existence of a constant $\gamma > 0$ such that

$$\gamma \leq \min_{x \in \{\Re - \mathcal{D}\}} (\|x_d(t) - x\|), \tag{2}$$

for any $t \geq 0$. This condition states that the desired trajectory is at least a distance γ from the boundary of $\mathcal{D}$. The region $\mathcal{D}$ also defines the largest region over which the functions f and g will be approximated. We assume that $f(x)$ and $g(x)$ are smooth on $\mathcal{D}$ and that for $x \in \Re$, $g(x)$ is nonzero and of known sign. Therefore, without loss of generality, we will invoke the following assumption.

Assumption 2.1: It is assumed that $g(x)$ has lower bound g_l such that $g(x) > g_l > 0$, $\forall x \in \Re$, where g_l is a known constant.

In Section VI, we extend the approach to (controllable) nonaffine systems of the form

$$\dot{x} = f(x, u). \tag{3}$$

A. Local Model Decomposition

Our goal for this study is to design a stabilizing tracking controller for a dynamic system containing unknown nonlinear functions. The controller will include adaptive approximations of the unknown nonlinear functions. The adaptive approximation will use the *locally weighted learning (LWL)* framework [1], [6]. Therefore, we first review the difference between cooperative learning and locally weighted learning.

The objective for a standard cooperative learning system is to minimize the least square criterion:

$$J(\theta) = \sum_{i=1}^{m} (f(x_i) - \hat{f}(x_i, \theta))^2 \tag{4}$$

over all m training data points $\{x_i, f(x_i)\}$ where $\theta = [\theta_1, \ldots, \theta_N]$. In eqn. (4), $\hat{f}(x, \theta)$ is the approximator for function $f(x)$, which is usually linear in the approximator parameter vector θ. In approximating every $f(x_i)$, cooperation is caused by this form of cost function because all θ_k are jointly optimized to minimize each $f(x_i) - \hat{f}(x_i)$. Adding one new training point x_{m+1} could affect all the approximator parameters.

In contrast to such cooperative strategies, LWL optimizes each local approximator independently to minimize the *locally weighted* error criterion:

$$J_k(\theta_k) = \sum_{i=1}^{m} \omega_k(x_i) \left(f(x_i) - \hat{f}_k(x_i, \theta_k)) \right)^2, \tag{5}$$

where $\hat{f}_k(x_i, \theta_k) = \phi_k^T(x_i)\theta_k$ and $\omega_k(x_i)$ is a localized kernel function that is discussed in Section II-A.1, and $\hat{f}_k(x)$ is an approximator of function $f(x)$ on the k-th local region (where ω_k is not zero). Using this cost function, it can be shown that θ_k adapts only when $\omega_k(x_i)$ is not zero, and that the adjustment of θ_k is independent of θ_j for $j \neq k$.

In LWL, the approximation of $f(x)$ for a point x_i is formed from the normalized weighted average of all local approximators $\hat{f}_k(x_i)$ such that[1]

$$\hat{f}(x_i) = \frac{\sum_{k=1}^{N} \omega_k(x_i) \hat{f}_k(x_i, \theta_k)}{\sum_{k=1}^{N} \omega_k(x_i)}. \tag{6}$$

Next, we will focus on a specific form of LWL algorithm and give all definitions required for the discussions that follow.

1) Weighting Functions: Denote the support of $\omega_k(x)$ by $S_k = \{x \in \mathcal{D} | \omega_k(x) \neq 0\}$. Define a set of continuous, positive, locally supported[2] functions $\omega_k(x)$ for $k = 1 \ldots, N$ such that each set S_k is convex and connected with $\mathcal{D} = \bigcup_{k=1}^{N} S_k$ where N is a finite integer. Let $\bar{S}_k$ denote the closure of S_k. Note that $\bar{S}_k$ is a compact set. This definition of $\omega_k(x)$ ensures that for any $x \in \mathcal{D}$, there exists at least one k such that $\omega_k(x) \neq 0$. The family of sets $\{S_k\}_{k=1}^{N}$ forms a finite cover of $\mathcal{D}$. An example of a weighting function satisfying the above conditions is the biquadratic kernel defined as

$$\omega_k(x) = \begin{cases} \left(1 - \left(\frac{||x - c_k||}{\mu_k}\right)^2\right)^2, & \text{if } ||x - c_k|| < \mu_k \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

where c_k is the center location of the k-th weighting function and μ_k is a constant which represents the radius of the region of support.

To simplify expressions such as eqn. (6), define

$$\bar{\omega}_k(x) = \frac{\omega_k(x)}{\sum_k \omega_k(x)}.$$

The set of non-negative functions $\{\bar{\omega}_k(x)\}_{k=1}^{N}$ forms a *partition of unity* on $\mathcal{D}$: $\sum_{k=1}^{N} \bar{\omega}_k(x) = 1, \quad \text{for all } x \in \mathcal{D}$. Note that the support of $\omega_k(x)$ is exactly the same as the support of $\bar{\omega}_k(x)$.

For use later in the definition of the control law, we decompose the operation region $\mathcal{D}$ into N subregions G_k. First, at any $x \in \mathcal{D}$, let

$$I(x) = \left\{ i \,\middle|\, \bar{\omega}_i(x) = \max_{1 \le j \le N} (\bar{\omega}_j(x)) \right\}. \tag{8}$$

When $x \notin \mathcal{D}$, $I(x) = \emptyset$. Since $\{S_k\}_{k=1}^{N}$ form a finite cover for $\mathcal{D}$, the set $I(x)$ is finite and not empty for any $x \in \mathcal{D}$. Next, define

$$G_k = \{x \in \mathcal{D} \mid \bar{\omega}_k(x) = \max_i (\bar{\omega}_i(x))\} \tag{9}$$

for $k = 1, \cdots, N$. G_k is defined as the set of $x \in \mathcal{D}$ such that $k \in I(x)$. Let ∂G_k and $\bar{G}_k$ denote the boundary and the closure of G_k. Note that $G_k \subset S_k$ and $\mathcal{D} = \bigcup_{k=1}^{N} G_k$.

2) Approximator: Let f_k and g_k be some continuous functions such that

$$f(x) - f_k(x) = \epsilon_{f_k}(x) \tag{10}$$

$$g(x) - g_k(x) = \epsilon_{g_k}(x) \tag{11}$$

where $|\epsilon_{f_k}(x)| \le \bar{\epsilon}_f$ and $|\epsilon_{g_k}(x)| \le \bar{\epsilon}_g$ for $x \in \bar{S}_k$. Note that the boundedness of $\max_{x \in \bar{S}_k} (|\epsilon_{f_k}(x)|)$ and $\max_{x \in \bar{S}_k} (|\epsilon_{g_k}(x)|)$ comes from the fact that $|\epsilon_{f_k}|$ and $|\epsilon_{g_k}|$ are continuous on compact $\bar{S}_k$. Therefore, f_k and g_k will be referred to as local approximators on $\bar{S}_k$. In order for ϵ_{f_k} and ϵ_{g_k} to be defined everywhere, let

$$\epsilon_{f_k}(x) = \begin{cases} f(x) - f_k(x), & \text{on } \bar{S}_k, \\ 0, & \text{otherwise.} \end{cases}$$

$$\epsilon_{g_k}(x) = \begin{cases} g(x) - g_k(x), & \text{on } \bar{S}_k, \\ 0, & \text{otherwise.} \end{cases}$$

For $x \in \mathcal{D}$, $f(x)$ and $g(x)$ can be represented as the weighted sum of the local approximators:

$$f(x) = \sum_k \bar{\omega}_k(x) f_k(x) + \delta_f(x) \tag{12}$$

$$g(x) = \sum_k \bar{\omega}_k(x) g_k(x) + \delta_g(x) \tag{13}$$

[1]Note that the form of eqn. (6) is similar to the definition of a Takagi-Sugeno (TS) fuzzy model [14]; however, the control law, parameter adaptation, LWR method, and stability analysis are all distinct from those used in TS fuzzy systems.

[2]If we define the size of a set S by $\rho(S) = \max_{x,y \in S} (\|x - y\|)$, then 'locally supported' means that $\rho(S_k)$ is small relative to $\rho(D)$.

where $|\delta_f(x)| \le \bar{\epsilon}_f$ and $|\delta_g(x)| \le \bar{\epsilon}_g$, since

$$\begin{aligned} |\delta_f| &= \left| f(x) - \sum_k \bar{\omega}_k(x) f_k(x) \right| \\ &= \left| \sum_k \bar{\omega}_k(x)(f(x) - f_k(x)) \right| \\ &\le \sum_k \bar{\omega}_k(x) |\epsilon_{f_k}(x)| \qquad (14) \\ |\delta_f| &\le \max_k(|\epsilon_{f_k}|) \sum_k \bar{\omega}_k(x) = \bar{\epsilon}_f. \qquad (15) \end{aligned}$$

Therefore, if each local model $f_k(x)$ has accuracy $\bar{\epsilon}_f$ on $\bar{S}_k$, then the global accuracy of $\sum_k \bar{\omega}_k(x) f_k(x)$ on $\mathcal{D}$ also achieves at least accuracy $\bar{\epsilon}_f$. A similar property also applies to $g(x)$ with a corresponding definition for $\bar{\epsilon}_g$. The terms δ_f and δ_g in (12)-(13) will be referred to as the *residual approximation errors* for $f(x)$ and $g(x)$, respectively. Note that with the previous definitions, $f(x) + g(x)u$ can be rewritten as

$$f(x) + g(x)u = \sum_k \bar{\omega}_k (f_k + g_k u) + \Delta \qquad (16)$$

where $\Delta = \delta_f + \delta_g u$.

3) Linear Parametrization: In the remainder of this paper we will assume f_k and g_k linearly parameterized. By this we mean that they are defined as:

$$f_k(x) = \bar{x}_k^T \theta^*_{f_k}, \quad g_k(x) = \bar{x}_k^T \theta^*_{g_k} \qquad (17)$$

where $\bar{x}_k$ is a prespecified vector of basis functions. We let $\theta^*_{f_k}$ and $\theta^*_{g_k}$ denote the (unknown) optimal parameter estimates for $x \in \bar{S}_k$ such that

$$\begin{aligned} \theta^*_{f_k} &= \arg\min_{\theta_{f_k}} \left(\int_{\mathcal{D}} \omega_k(x) \left| f(x) - \hat{f}_k(x) \right|^2 dx \right) \qquad (18) \\ \theta^*_{g_k} &= \arg\min_{\theta_{g_k}} \left(\int_{\mathcal{D}} \omega_k(x) \left| g(x) - \hat{g}_k(x) \right|^2 dx \right) \qquad (19) \end{aligned}$$

where

$$\hat{f}_k(x) = \bar{x}_k^\top \theta_{f_k}, \quad \hat{g}_k(x) = \bar{x}_k^\top \theta_{g_k}. \qquad (20)$$

For $x \in \bar{S}_k$, using equations (10) and (11), we can rewrite $f(x)$ and $g(x)$ only in term of their local approximators on $\bar{S}_k$. Then, the system dynamics (1) can be represented locally on $\bar{S}_k$ as

$$\begin{aligned} \dot{x} &= f_k(x) + g_k(x)u + \epsilon_k(x,u) \qquad (21) \\ &= \Phi_k^\top \Theta^*_k + \epsilon_k(x,u) \qquad (22) \end{aligned}$$

where $\Phi_k = \begin{bmatrix} \bar{x}_k \\ \bar{x}_k u \end{bmatrix}$, $\Theta^*_k = \begin{bmatrix} \theta^*_{f_k} \\ \theta^*_{g_k} \end{bmatrix}$; $\epsilon_k(x,u) = \epsilon_{f_k}(x) + \epsilon_{g_k}(x)u$. Note that Θ^*_k is well defined for each k because $\bar{S}_k$ is compact and functions f and g are smooth on $\bar{S}_k$.

The notation of this paper will continue to use the general linear in the approximator notation of eqn. (17). For examples, as done by the authors of [1], [6], we will select $\bar{x}_k = \begin{bmatrix} x - c_k \\ 1 \end{bmatrix}$ with c_k being the center of the $\bar{S}_k$. Therefore, f_k and g_k are optimal local affine approximations to f and g on $\bar{S}_k$.

4) Function Approximators: Since we have assumed that f and g are unknown, the parameters vector Θ^*_k is unknown for each k. The control law will therefore be written using approximated functions defined locally by (20) on $\bar{S}_k$ and globally on $\mathcal{D}$ as

$$\hat{f}(x) = \sum_k \bar{\omega}_k(x) \hat{f}_k(x) \qquad (23)$$

$$\hat{g}(x) = \sum_k \bar{\omega}_k(x) \hat{g}_k(x). \qquad (24)$$

The controller will be adaptive in the sense that the local parameter vectors $\Theta_k = \left[\theta_{f_k}^\top, \theta_{g_k}^\top\right]^\top$ will be adjusted to improve the function approximation accuracy while the controller is in operation. For analysis of the convergence of the parameter estimates, we define for $j = 1, \ldots, N$ the parameter error vectors as $\tilde{\theta}_{f_j} = \theta_{f_j} - \theta^*_{f_j}$, $\tilde{\theta}_{g_j} = \theta_{g_j} - \theta^*_{g_j}$, and $\tilde{\Theta}_j = \begin{bmatrix} \tilde{\theta}_{f_j} \\ \tilde{\theta}_{g_j} \end{bmatrix}$.

III. Control Design

In this paper, we propose a switching control design utilizing local affine approximators. The locally affine models will facilitate the extension to non-affine systems in Section VI.

Define local control laws for $k = 1, \cdots, N$ according to

$$u_k = \frac{1}{\hat{g}_k} \left(-\hat{f}_k + \dot{x}_d - L\tilde{x} \right), \text{ for } x \in \bar{S}_k \qquad (25)$$

where $\tilde{x} = x - x_d$ is the tracking error and $L > 0$ is a control gain. For $x, x_d \in \mathcal{D}$, each u_k is bounded by b_u, i.e., $|u_k| \le b_u$, where

$$b_u = \frac{1}{g_l} \left(|\dot{x}_d| + 2\rho(D)L + \left(1 + \rho(\mathcal{D})^2\right) \max_k(\|\theta^*_{f_k}\|) \right).$$

where $\rho(\mathcal{D}) = \max_{x_1, x_2 \in \mathcal{D}} \|x_1 - x_2\|$ is the diameter of set $\mathcal{D}$.

The control signal u is defined as a convex linear combination of the u_k plus a term ν_D designed to force initial conditions outside of $\mathcal{D}$ to enter $\mathcal{D}$ in finite time and to not allow the state to leave $\mathcal{D}$ once the state is in $\mathcal{D}$:

$$u = \sum_{k=1}^N \alpha_k u_k + \nu_D. \qquad (26)$$

where the coefficients of the convex combination will change as a function of x and x_d, but will always satisfy

$$\sum_{k=1}^N \alpha_k = 1 \quad \text{and} \quad \alpha_k = \begin{cases} \ge 0 & \text{if } k \in I(x), \\ 0 & \text{otherwise.} \end{cases} \qquad (27)$$

Many alternative definitions of the α_k are possible, as long as the constraints of eqn. (27) is satisfied. Letting m denote the number of elements in $I(x)$, an example definition is

$$\alpha_k = \begin{cases} 1/m & \text{if } k \in I(x), \\ 0 & \text{otherwise.} \end{cases}$$

Note that with the definition of the control signal as eqn. (26), for $x, x_d \in \mathcal{D}$, u is also bounded such that $|u| \le b_u$.

We must now consider two situations: initial conditions in $\mathcal{D}$ and initial conditions in $\Re - \mathcal{D}$. Subsection III-A will design the sliding mode term ν_D and present analysis to show that initial conditions outside of $\mathcal{D}$ will return to $\mathcal{D}$ in finite time. In addition, it will show that initial conditions within $\mathcal{D}$ cannot leave $\mathcal{D}$. Subsection III-B will derive the tracking error dynamic equations that are applicable within $\mathcal{D}$.

A. Sliding Mode Outside $\mathcal{D}$

The ν_D term in (26) denotes a sliding mode control signal with time varying gain

$$\nu_D = \begin{cases} 0, & \text{for } x(t) \in \mathcal{D} \\ \frac{1}{g_l}\left(\dot{x}_d - r(t) sign(\tilde{x})\right), & \text{for } x(t) \notin \mathcal{D}. \end{cases} \tag{28}$$

To define the time-varying gain, $r(t)$, we need the following assumption.

Assumption 3.1: Assume that we know the upper bounds on functions $|f(x)|$ and $g(x)$ such that

$$|f(x)| < (1+|x|)\bar{f}, \qquad g_l < g(x) < (1+|x|)\bar{g}$$

where $\bar{f}$ and $\bar{g}$ are known constants.
Note that if constants $\bar{f}$ and $\bar{g}$ satisfying this assumption are not known, then they could be estimated using the methods suggested in [5], [8]. We do not present such an adaptive bounding approach herein as it would distract from the main topic of the paper.

When x is outside the region $\mathcal{D}$, the previously defined locally linear approximator in (24) will yield $\hat{g} = 0$. Using Assumption 3.1 it can be shown that

$$\frac{g_l}{\bar{g}(1+|x|)} < \frac{g_l}{g(x)} < 1.$$

The sliding gain $r(t)$ is selected as

$$r(t) = \bar{f}(1+|x|) + \left(\frac{(1+|x|)\bar{g}}{g_l} - 1\right)|\dot{x}_d| + \eta,$$

for some $\eta > 0$. When $x \notin \mathcal{D}$, all ω_k are zero; therefore, $I(x)$ is an empty set and all α_k are zero. The closed loop system dynamics outside $\mathcal{D}$ are

$$\begin{aligned} \dot{\tilde{x}} &= f(x) + g(x)u - \dot{x}_d \\ &\leq |f(x)| - \dot{x}_d + \frac{g(x)}{g_l}\left(\dot{x}_d - r(t) sign(\tilde{x})\right) \\ &< \left((1+|x|)\bar{f} + \left(\frac{(1+|x|)\bar{g}}{g_l} - 1\right)|\dot{x}_d|\right) \\ &\quad - \frac{g(x)}{g_l} r(t) sign(\tilde{x}). \end{aligned} \tag{29}$$

Consider the Lyapunov function as $\mathcal{V}(t) = \frac{1}{2}\tilde{x}^2$, the derivative of $\mathcal{V}(t)$ can be simplified as

$$\begin{aligned} \frac{d}{dt}\mathcal{V} &= \tilde{x}\dot{\tilde{x}} < -\frac{g(x)}{g_l} r(t)|\tilde{x}| \\ &\quad + |\tilde{x}|\left((1+|x|)\bar{f} + \left(\frac{(1+|x|)\bar{g}}{g_l} - 1\right)|\dot{x}_d|\right) \end{aligned}$$

Since

$$r(t) > \frac{g_l}{g(x)}\left((1+|x|)\bar{f} + \left(\frac{(1+|x|)\bar{g}}{g_l} - 1\right)|\dot{x}_d| + \eta\right),$$

then

$$\frac{d}{dt}\mathcal{V} < -\eta|\tilde{x}| < -\eta\gamma \tag{30}$$

by eqn. (2) for $x \in \Re - \mathcal{D}$. Therefore, when $x(t_1)$ is outside the region $\mathcal{D}$, the sliding control will ensure that x returns to within $\mathcal{D}$ in finite time. To see why this must be, for purposes of contradiction, assume that $x(t) \notin \mathcal{D}$ for all $t \geq t_1 > 0$. Let $t_2 > t_1$. Under the assumption that x does not return to $\mathcal{D}$ in finite time the variable t_2 can be infinitely large. Also, for all $t_2 > t_1$, by the definition of γ in eqn. (2), we have that $\mathcal{V}(t_2) \geq \frac{\gamma^2}{2}$; however, integration of eqn. (30) yields

$$\begin{aligned} \mathcal{V}(t_2) - \mathcal{V}(t_1) &\leq -\eta\gamma(t_2 - t_1) \\ \mathcal{V}(t_2) &\leq \mathcal{V}(t_1) - \eta\gamma(t_2 - t_1). \end{aligned}$$

This yields a contradiction for

$$t_2 \geq t_1 + \frac{\mathcal{V}(t_1) - \frac{\gamma^2}{2}}{\eta\gamma}. \tag{31}$$

Once $x \in \mathcal{D}$, the sliding mode term will not allow x to leave $\mathcal{D}$.

Note that eqn. (28) is continuous on $\Re - \mathcal{D}$, but discontinuous at the boundary of $\mathcal{D}$. The discontinuity at the boundary of $\mathcal{D}$ could be smoothed, within the γ region of the boundary defined in eqn. (2), to result in a continuous sliding mode term. We do not pursue this herein. The reader interested in this approach should see [11]. The remainder of this article will only be concerned with $x \in \mathcal{D}$.

B. Tracking Error within $\mathcal{D}$

For $x \in \mathcal{D}$, $\nu_D = 0$ in the control law of eqn. (26). The resulting closed-loop tracking error dynamics satisfy

$$\dot{\tilde{x}} = -L\tilde{x} - \sum_{k=1}^{N} \alpha_k \left(\Phi_k^\top \tilde{\Theta}_k - \epsilon_k\right). \tag{32}$$

This is shown in the following by considering the two possible cases.

1) When $I(x)$ contains a single integer j, then $\alpha_j = 1$ and $\alpha_k = 0$ for $k \neq j$. In this case, $\bar{\omega}_j(x)$ is the only normalized weighting function attaining the maximum value and x is on the interior of G_j (i.e., $x \in G_j - \partial G_j \subset \bar{S}_j$). Applying u of (25) and (26) to the system dynamics on $\bar{S}_j$ as defined by eqn. (21), the tracking error dynamics are

$$\begin{aligned} \dot{\tilde{x}} &= -L\tilde{x} + (f_j - \hat{f}_j) + (g_j - \hat{g}_j)u + \epsilon_j \\ &= -L\tilde{x} - \bar{x}_j^\top \tilde{\theta}_{f_j} - \bar{x}_j^\top \tilde{\theta}_{g_j} u + \epsilon_j \\ &= -L\tilde{x} - \Phi_j^\top \tilde{\Theta}_j + \epsilon_j \\ &= -L\tilde{x} - \sum_{k=1}^{N} \alpha_k \left(\Phi_k^\top \tilde{\Theta}_k - \epsilon_k\right). \end{aligned}$$

2) When $I(x)$ contains a set of integers, with the choice of u in (25) and (26), the closed loop system dynamics are

$$\begin{aligned}\dot{x} &= f + g\sum_{k=1}^{N}\alpha_k u_k \\ &= \sum_{k=1}^{N}\alpha_k\left(f + g u_k\right) \\ &= \sum_{k\in I(x)}\alpha_k\left(f + g u_k\right).\end{aligned}$$

For each $k \in I(x)$, $x \in \bar{S}_k$. Therefore, eqn. (21) is applicable, which allows

$$\begin{aligned}\dot{x} &= \sum_{k\in I(x)}\alpha_k\Big(\hat{f}_k + \hat{g}_k u_k \\ &\qquad + (f_k - \hat{f}_k) + (g_k - \hat{g}_k)u_k + \epsilon_k\Big) \\ \dot{x} &= \sum_{k\in I(x)}\alpha_k\left(\dot{x}_d - L\tilde{x} - \Phi_k^\top\tilde{\Theta}_k + \epsilon_k\right) \\ \dot{\tilde{x}} &= -L\tilde{x} - \sum_{k\in I(x)}\alpha_k\left(\Phi_k^\top\tilde{\Theta}_k - \epsilon_k\right) \\ \dot{\tilde{x}} &= -L\tilde{x} - \sum_{k=1}^{N}\alpha_k\left(\Phi_k^\top\tilde{\Theta}_k - \epsilon_k\right).\end{aligned}$$

Note that each controller of (25) is continuous, but the switching controller of (26) is discontinuous. Thus, we have designed a switching controller that yields a set of closed loop differential equations with right hand side that is discontinuous on the set $M = \bigcup_{k=1}^{N}\partial G_k$, which has zero measure.

For a general system $\dot{x} = g(t,x)$ where $g(t,x) = g^k(t,x)$ for $x \in \bar{G}_k, k = 1,\cdots,N$, existence of solutions in the sense of Filippov [3] can be shown such that the solution $x(t)$ satisfies the differential inclusion

$$\dot{x} \in P(x,t), \text{ where} \tag{33}$$

$$P(x) = \left\{v(x,t) | v = \sum_{k=1}^{N}\beta_k g^k\right\}. \tag{34}$$

for some $\beta_1,\cdots,\beta_N$ satisfying

$$\sum_{k=1}^{N}\beta_k = 1 \quad\text{and}\quad \beta_k = \begin{cases}\geq 0 & \text{if } x\in\bar{G}_k,\\ 0 & \text{otherwise.}\end{cases} \tag{35}$$

When the control signal is selected as proposed in eqn. (26), the right-hand side of the system dynamic equation and $u(x,t)$ are continuous on the interior of G_k, but the control signal is discontinuous on the *switching boundary* M. Therefore, a set $K[u]$ can be constructed as

$$K[u] = \begin{cases}\{u_k\}, & \text{if } I(x) = \{k\}\\ \{u | u = \sum_{k\in I(x)}\beta_k u_k\}, & \text{if } \mathcal{N}(I(x)) > 1\end{cases} \tag{36}$$

where $\mathcal{N}(I(x))$ is the number of elements in set $I(x)$. With β defined as above. Note that the control signal of eqn. (26) is in $K[u]$. Therefore,

$$\begin{aligned}\dot{x} &\in P(x), \text{ where} &(37)\\ P(x) &= f(x) + g(x)K[u] &(38)\end{aligned}$$

With $P(x)$ defined in (38), the set-valued map $x \to P(x)$ is nonempty, compact and convex $\forall x \in \mathcal{D}$. Since $P(x)$ is constructed as the set of all possible convex linear combinations of u_k, by applying Lemma 3 (p. 67) of [3], $P(x)$ is upper semi-continuous on $\mathcal{D}$. Therefore, we have proved the existence of the Filippov solution to (1). Stability of the Filippov solution will be considered in Section V, after the parameter adaptation laws are defined in Section IV.

Eqn. (36) states that: for $x \in \mathcal{D} - M$, set $K[u]$ contains a single point $u_k(x,t)$; for $x \in M$, $K[u]$ contains the linear convex combinations of all limit values $u_k(x,t), k \in I(x)$. Note that eqns. (26–27) and (36) are not the same. For implementation purposes, eqns. (26–27) define a particular convex combination of the local controllers. For analysis purposes, eqn. (36) defines the set of all possible convex combinations of the local controllers.

IV. Parameter Adaptation

The approximator parameters Θ_k will be adapted in a composite fashion [13] using prediction error methods on $\bar{S}_k$ and tracking error on $\bar{G}_k$. As the state moves through $\mathcal{D}$, prediction error based adaptation on $\bar{S}_k - \bar{G}_k$ allows the approximator to be tuned prior to the $k-th$ controller being used on $\bar{G}_k$. For $x \in \bar{G}_k$, composite adaptation uses both tracking and prediction error to enhance the rate of convergence.

A. Prediction Error

Assume that the state $x(t)$ (but not the state derivative) is measured and available for use in the control and adaptation laws. This extends the approaches presented in [6], [7] which utilized the state derivative in the implementation of the parameter adaptation laws. To derive the prediction error based adaptation law, we manipulate the model in eqn. (22) as follows [4]:

$$\begin{aligned}\dot{x} &= \Phi_k^\top\Theta_k^* + a_f x - a_f x + \epsilon_k \\ \dot{x} + a_f x &= a_f x + \Phi_k^\top\Theta_k^* + \epsilon_k \\ x &= x_f + \Phi_{f_k}^\top\Theta_k^* + \epsilon_k' &(39)\end{aligned}$$

where $x_f(t) = \frac{a_f}{s+a_f}[x(t)]$, Φ_{f_k} is the filtered regressor defined as $\Phi_{f_k}(t) = \frac{1}{s+a_f}[\Phi_k(x(t))]$, $\epsilon_k'(t) = \frac{1}{s+a_f}[\epsilon_k(x(t))]$. The notation $y_f(t) = H(s)[y(t)]$ means that y_f is the signal at the output of the filter with transfer function $H(s)$ with the signal $y(t)$ as the filter input. The model representation of eqn. (39) is convenient for parameter estimation.

Let $y(t) = x(t) - x_f(t) = \Phi_{f_k}^\top\Theta_k^* + \epsilon_k'$. Since the model parameters Θ_k^* are not known, we define a local estimate $\hat{y}_k$ as

$$\hat{y}_k = \Phi_{f_k}^\top\Theta_k$$

where $\Theta_k = [\theta_{f_k}^\top, \theta_{g_k}^\top]^\top$. Next, we define the prediction error of the k-th local model as

$$e_{pk}(t) = \begin{cases} \hat{y}_k - y(t) & \text{for } x \in \bar{S}_k \\ 0 & x \in \mathcal{D} - \bar{S}_k. \end{cases}$$

For analysis purposes we will use the representation

$$e_{pk}(t) = \Phi_{f_k}^\top \tilde{\Theta}_k - \epsilon'_k, \text{ for } x \in \bar{S}_k$$

with $\tilde{\Theta}_k = \Theta_k - \Theta_k^*$.

To derive the LWL parameter adaptation laws based on the prediction errors, we minimize the individual weighted squared prediction error criterion for each local model,

$$J_k(\Theta_k) = \int_0^t \frac{\omega_k(x(\tau))\left[y - \hat{y}_k(\Theta_k, \Phi_{f_k})\right]^2}{exp\left(\int_\tau^t \lambda \omega_k(x(r))dr\right)} d\tau \tag{40}$$

where we eliminate the explicit τ dependence of variables to make the equation fit on the line. Setting the gradient equal to zero and solving for $\dot{\Theta}_k$ yields

$$\dot{\Theta}_k = -P_k \omega_k(x(t)) e_{pk} \Phi_{f_k}. \tag{41}$$

with P_k computed as the solution of

$$\dot{P}_k = \omega_k(x(t))(\lambda P_k - P_k \Phi_{f_k} \Phi_{f_k}^\top P_k) \tag{42}$$

where $\lambda > 0$ is the forgetting factor. Note that both adaptation and forgetting of Θ_k are localized to $x(t) \in \bar{S}_k$. The derivation is not included herein due to space limitations. The derivation of a similar adaptation law is included in [7].

B. Composite Adaptation Law

For any $x(t) \in \mathcal{D}$, the composite adaptation law to include both tracking and prediction error is

$$\dot{\Theta}_k(t) \quad = \quad P_k(\alpha_k \tilde{x} \Phi_k - \omega_k e_{pk} \Phi_{f_k}) \tag{43}$$

with $k = 1, \cdots, N$. To ensure controllability of the estimated model in accordance with Assumption 2.1, the update law for θ_{g_k} in eqn. (43) needs to be modified using, for example, a projection modification [4] to ensure that $\hat{g}_k$ is bounded away from zero.

For $x \in \bar{G}_k$, then $k \in I(x)$ and $\alpha_k(t) \geq 0$. Therefore, the parameter update for Θ_k is computed using prediction error and a fraction of the tracking error equal to $\alpha_k(t)$. For $x \in \bar{S}_k - \bar{G}_k$, then $k \notin I(x)$ and $\alpha_k = 0$. Therefore, the parameter update simplifies to the prediction error form $\dot{\Theta}_k(t) = -P_k \omega_k e_{pk} \Phi_{f_k}$. For $x \in \mathcal{D} - \bar{S}_k$ (and for $x \notin \mathcal{D}$), the parameter update for Θ_k is zero. For $x \notin \mathcal{D}$ all parameter adaptation is turned off.

V. Stability Analysis

In this section, we present the stability analysis for the closed loop control system of eqns. (25–26) and (42–43), which does have a discontinuous control signal. First, a smooth Lyapunov function is defined, then the chain rule and stability theorem for such systems (e.g., Theorem 2.2 Theorem 3.1 in [12]) are employed to complete the analysis.

Theorem 5.1: The system described by eqn. (1) with control law given by eqns. (25–26) and parameter adaptation laws given by eqns. (42–43) have the following properties. When $\bar{\epsilon}_f = 0$ and $\bar{\epsilon}_g = 0$:

1) $\tilde{x},\ \tilde{\Theta}_k,\ \Theta_k,\ e_{pk} \in \mathcal{L}_\infty$;
2) $\tilde{x} \to 0$ as $t \to \infty$;
3) $\tilde{x} \in \mathcal{L}_2$;

In the following, we prove the above theorem. We recognize that $\bar{\epsilon}_f = 0$ and $\bar{\epsilon}_g = 0$ is an ideal situation that typically does not hold in practice. In the normal case where these approximation errors are nonzero, but bounded on $\mathcal{D}$ convergence to zero is replaced by convergence in the mean squared sense. We do not include that proof herein, due to page length constraints. In addition, the following proof is only concerned with $x \in \mathcal{D}$. The case of $x \notin \mathcal{D}$ was discussed in Section III-A.

Proof: Define the Lyapunov function

$$\mathcal{V}(\tilde{x}, \tilde{\Theta}_1, \cdots, \tilde{\Theta}_N) = \frac{1}{2}\tilde{x}^2 + \frac{1}{2}\sum_{k=1}^N \tilde{\Theta}_k^\top P_k^{-1} \tilde{\Theta}_k. \tag{44}$$

For notational brevity, we define $\chi = [\tilde{x}, \tilde{\Theta}_1^\top, \cdots, \tilde{\Theta}_N^\top]^\top$. Note that the matrices $P_k^{-1}(t)$ are each time dependent. It can be shown that $P_k^{-1}(t)$ is (at least) positive semidefinite and bounded for all $t \geq 0$ by directly solving for $P_k^{-1}(t)$ from eqn. (42) (see page 374 in [13]).

First, we consider the solution to $\dot{\chi} = \Lambda(\chi, t)$, where

$$\Lambda(\chi, t) = \Lambda_k = \left[\begin{array}{c} -L\tilde{x} - \Phi_k^\top \tilde{\Theta}_k + \epsilon_k \\ P_1(-\omega_1 e_{p1} \Phi_{f_1}) \\ \vdots \\ P_k(\tilde{x}\Phi_k - \omega_k e_{pk} \Phi_{f_k}) \\ \vdots \\ P_N(-\omega_N e_{pN} \Phi_{f_N}) \end{array}\right], \tag{45}$$

for $x \in G_k - \partial G_k$. The function $\Lambda(\chi, t)$ is continuous on the interior of each G_k, but discontinuous on the switching boundary M. For $x \in M$, by application of eqn. (36)

$$K[\Lambda](\chi, t) = \{\Lambda(\chi, t) | \Lambda = \sum_{k \in I(x)} \beta_k \Lambda_k\} \tag{46}$$

with $\beta_1, \cdots, \beta_N$ satisfying condition (35). By inspection of eqns. (32) and (43), we see the differential inclusion $\dot{\chi} \in K[\Lambda](\chi, t)$ is satisfied; therefore, $\chi(t)$ is a Filippov solution to $\dot{\chi} = \Lambda(\chi, t)$. This shows existence of at least one solution. The following text will show that any Filippov solution to this system with $\beta_1, \cdots, \beta_N$ satisfying condition (35) has certain stability properties.

According to the chain rule defined in [12], $\frac{d}{dt}\mathcal{V}$ can be expressed in terms of Clarke's generalized gradient $\partial\mathcal{V}(\chi, t)$ and the convex closure $K[\Lambda](\chi, t)$. Then, almost everywhere

$$\frac{d}{dt}\mathcal{V} \in \dot{\tilde{\mathcal{V}}} = \bigcap_{\xi \in \partial\mathcal{V}(\chi,t)} \xi^\top \left[\begin{array}{c} K[\Lambda](\chi, t) \\ 1 \end{array}\right]. \tag{47}$$

Since $\mathcal{V}$ is smooth in $\tilde{x}, \tilde{\Theta}_1, \cdots, \tilde{\Theta}_N$, the set $\partial \mathcal{V}(\chi, t)$ contains only one element $\nabla \mathcal{V}$ defined as

$$\nabla \mathcal{V} = \begin{bmatrix} \tilde{x} \\ P_1^{-1}\tilde{\Theta}_1 \\ \vdots \\ P_N^{-1}\tilde{\Theta}_N \\ \frac{1}{2}\sum_{k=1}^{N} \tilde{\Theta}_k^\top \dot{P}_k^{-1}\tilde{\Theta}_k \end{bmatrix}, \tag{48}$$

which is the gradient of $\mathcal{V}$ in the normal sense at (χ, t).

Therefore, the set $\dot{\tilde{\mathcal{V}}}$ can be expressed as

$$\dot{\tilde{\mathcal{V}}} = \nabla \mathcal{V}^\top \begin{bmatrix} -L\tilde{x} - \sum_{k=1}^{N} \beta_k \left(\Phi_k^\top \tilde{\Theta}_k - \epsilon_k\right) \\ P_1(\beta_1 \tilde{x}\Phi_1 - \omega_1 e_{p1}\Phi_{f_1}) \\ \vdots \\ P_N(\beta_N \tilde{x}\Phi_N - \omega_k e_{pN}\Phi_{f_N}) \\ 1 \end{bmatrix}. \tag{49}$$

After algebraic manipulations, eqn. (49) becomes

$$\begin{aligned} \dot{\tilde{\mathcal{V}}} &= \tilde{x}\left[-L\tilde{x} - \sum_{k=1}^{N}\beta_k\left(\Phi_k^\top\tilde{\Theta}_k - \epsilon_k\right)\right] \\ &\quad + \sum_{k=1}^{N}\tilde{\Theta}_k^\top P_k^{-1}P_k(\beta_k\tilde{x}\Phi_k - \omega_k e_{pk}\Phi_{f_k}) \\ &\quad + \frac{1}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top\left(-\lambda P_k^{-1} + \Phi_{f_k}\Phi_{f_k}^\top\right)\tilde{\Theta}_k \\ &= -L\tilde{x}^2 + \tilde{x}\sum_{k=1}^{N}\beta_k\epsilon_k - \sum_{k=1}^{N}\tilde{\Theta}_k^\top\omega_k e_{pk}\Phi_{f_k} \\ &\quad - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k + \frac{1}{2}\sum_{k=1}^{N}\omega_k\left(\Phi_{f_k}^\top\tilde{\Theta}_k\right)^2 \end{aligned}$$

Note that $\Phi_{f_k}^\top\tilde{\Theta}_k = e_{pk} + \epsilon'_k$,

$$\begin{aligned} \dot{\tilde{\mathcal{V}}} &= -L\tilde{x}^2 + \tilde{x}\sum_{k=1}^{N}\beta_k\epsilon_k - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k \\ &\quad + \sum_{k=1}^{N}\omega_k\left(-e_{pk}(e_{pk}+\epsilon'_k) + \frac{1}{2}(e_{pk}+\epsilon'_k)^2\right) \\ &\le -L\tilde{x}^2 + \tilde{x}\sum_{k=1}^{N}\beta_k\epsilon_k - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k \\ &\quad - \frac{1}{2}\sum_{k=1}^{N}\omega_k e_{pk}^2 + \frac{1}{2}\sum_{k=1}^{N}\omega_k{\epsilon'_k}^2. \end{aligned}$$

If we use the identity (which is true for any $\kappa \neq 0$)

$$\tilde{x}\epsilon_k = -\left(\frac{\kappa}{2}\tilde{x} - \frac{1}{\kappa}\epsilon_k\right)^2 + \frac{\kappa^2}{4}\tilde{x}^2 + \left(\frac{1}{\kappa}\right)^2\epsilon_k^2$$

with $\frac{\kappa^2}{4} = \frac{L}{2}$, then set $\dot{\tilde{\mathcal{V}}}$ becomes

$$\begin{aligned} \dot{\tilde{\mathcal{V}}} &\le -L\tilde{x}^2 + \sum_{k=1}^{N}\beta_k\left(\frac{L}{2}\tilde{x}^2 + \frac{1}{2L}\epsilon_k^2\right) + \frac{1}{2}\sum_{k=1}^{N}\omega_k{\epsilon'_k}^2 \\ &\quad - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k - \frac{1}{2}\sum_{k=1}^{N}\omega_k e_{pk}^2. \end{aligned}$$

If we let $\bar{\epsilon} = \bar{\epsilon}_f + \bar{\epsilon}_g b_u$, then for any k, $\epsilon_k^2 \le \bar{\epsilon}^2$ and thus

$$\begin{aligned} \dot{\tilde{\mathcal{V}}} &\le -\frac{L}{2}\tilde{x}^2 - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k - \frac{1}{2}\sum_{k=1}^{N}\omega_k e_{pk}^2 \\ &\quad + \left(\frac{1}{2L}\bar{\epsilon}^2 + \frac{1}{2}\sum_{k=1}^{N}{\epsilon'_k}^2\right). \end{aligned} \tag{50}$$

This shows that for any element $\frac{d}{dt}\mathcal{V}$ of the set $\dot{\tilde{\mathcal{V}}}$, we have $\frac{d}{dt}\mathcal{V} \le 0$ almost everywhere whenever $L\tilde{x}^2 > \frac{1}{L}\bar{\epsilon}^2 + \sum_{k=1}^{N}{\epsilon'_k}^2$. Therefore, we obtain the stability results using the Lyapunov theorems for set valued maps (see [12]). The stability results are concluded as follows:

If we assume perfect approximation within each local region (i.e., $\epsilon_{f_k} = \epsilon_{g_k} = 0$, $\bar{\epsilon}_f = \bar{\epsilon}_g = 0$), then eqn. (50) simplifies to

$$\dot{\tilde{\mathcal{V}}}(\chi, t) \le -\frac{L}{2}\tilde{x}^2 - \frac{\lambda}{2}\sum_{k=1}^{N}\omega_k\tilde{\Theta}_k^\top P_k^{-1}\tilde{\Theta}_k - \frac{1}{2}\sum_{k=1}^{N}\omega_k e_{pk}^2.$$

Therefore, the set $\dot{\tilde{\mathcal{V}}}(\chi, t) \le 0$. A minor extension to Theorem 3.1 in [12] implies that the solution $\chi = 0$ is stable. This directly yields $\tilde{x},\ \tilde{\Theta}_k \in \mathcal{L}_\infty$ and then $\Theta_k,\ e_{pk} \in \mathcal{L}_\infty$. Since $\dot{\tilde{x}}$ is bounded, Barbalat's lemma implies that $\tilde{x}$ converges to zero as $t \to \infty$. Finally, the above analysis implies that $\tilde{x} \in \mathcal{L}_2$, since after integration, we obtain

$$\tilde{\mathcal{V}}(0) \ \ge \ \int_0^\infty \frac{L}{2}\tilde{x}^2(\tau)\ d\tau.$$

Note that $\tilde{\mathcal{V}}(0)$ is a set of finite, non-negative elements. For any element $\mathcal{V}(0) \in \tilde{\mathcal{V}}(0)$, we have that $\mathcal{V}(0) \ge \frac{L}{2}\int_0^\infty \tilde{x}^2(\tau)\ d\tau$ which implies the $\mathcal{L}_2$ property of $\tilde{x}$. ■

In most applications, perfect approximation is not possible; therefore, when approximation errors exist, we are interested in deriving bounds on $\tilde{x}$ and developing methods to reduce these bounds. Usually, high gain control is not desirable because it will increase the bandwidth of the control system and possibly excite unmodeled dynamics. Starting from (50), in the presence of approximation error, it can be shown that $\tilde{x}$ is on the order of $\bar{\epsilon}^2 + \sum_{k=1}^{N}{\epsilon'_k}^2$ in mean squared sense (m.s.s.), the m.s.s bound on $\tilde{x}$ can be reduced by decreasing $\bar{\epsilon}_f$ and $\bar{\epsilon}_g$ (i.e., by enhancing the structure of the approximator).

VI. Extension to Non-affine System

In practice, many nonlinear systems may not be representable in affine form. Motivated by such applications, we will next consider the extension of the approach of this article to a class of non-affine nonlinear systems.

Consider a system of the form

$$\dot{x} = h(x, u) \tag{51}$$

where $h(x, u)$ is a non-affine unknown function of the control signal $u(t)$. Similar to Assumption 2.1, the following assumption is required to avoid singularity during the controller operation.

Assumption 6.1: There exists positive constant h_l, such that $h_u = \frac{\partial}{\partial u} h(x,u) > h_l$ for all $(x,u) \in \mathcal{D}$.

For the non-affine case, the domain of operation $\mathcal{D}$ and the local regions of support S_k and G_k become 2-dimensional sets. Therefore, for $(x,u) \in \bar{S}_k$, we define local function approximator as $h_k(x,u) = \Phi_k^\top(x,u)\Theta_k^*$ where $\Phi_k = [1 \; x - c_{k,1} \; u - c_{k,2}]^\top$ and Θ_k^* denote the optimal parameter estimate for $(x,u) \in \bar{S}_k$ in the sense that

$$\Theta_k^* = \arg\min_{\Theta_k} \left(\int_{\mathcal{D}} \omega_k(x,u) |h(x,u) - \hat{h}_k(x,u)|^2 dx du \right)$$

with $\hat{h}_k(x,u)$ being the estimate of $h_k(x,u)$ defined as $\hat{h}_k(x,u) = \Phi_k^\top \Theta_k$. where Θ_k will be adjusted on-line to improve function approximation accuracy.

The system dynamic eqn. (51) can be represented locally on $\bar{S}_k$ as

$$\dot{x} = \Theta_{k,1}^* + \Theta_{k,2}^*(x - c_{k,1}) + \Theta_{k,3}^*(u - c_{k,2}) + \epsilon_k(x,u)$$

where $\epsilon_k(x,u)$ is the local approximation error on $\bar{S}_k$. Therefore, even though the actual dynamics of eqn. (51) are not affine in the control signal on $\mathcal{D}$, each local model is affine in the control on $\bar{S}_k$. The local control law is defined using the estimate of Θ_k^* as

$$u_k = \frac{(-\Theta_{k,1} - \Theta_{k,2}(x - c_{k,1}) + \dot{x}_d - L\tilde{x})}{\Theta_{k,3}} + c_{k,2} \quad (52)$$

where the projection modification is used to maintain $\Theta_{k,3} > g_l$. If we choose the definition of the α_k as

1) When $I(x)$ contains a single integer j, then $\alpha_j = 1$ and $\alpha_k = 0$ for $k \neq j$.
2) When $I(x)$ contains a set of integers,

$$\alpha_k = \begin{cases} 1 & \text{if } k \in I(x) \text{ and } \alpha_k(x(t_-)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $x(t_-)$ is the state at the time instant right before t. The α_k is defined such that at any time only one local control law is included in the control signal definition. Then, with the control law given by eqns. (52) and (26), we can obtain the similar closed-loop tracking error dynamics as (32). Together with the parameter adaptation defined by equation (43), we can show the boundedness of x, u, and all parameter estimates Θ_k for the non-affine nonlinear system (51). The proof is similar to the proof of Theorem 5.1 and is not given herein.

VII. Conclusions

This article has considered the design and analysis of a stable locally weighted learning (LWL) framework applicable to systems with unknown nonlinear dynamics where the control signal does not need to appear in an affine fashion. The system does still need to satisfy a controllability condition locally over a compact operating region $\mathcal{D}$. The presented approach can use tracking error, prediction error, or a composite of these errors for parameter adaptation. The control system utilizes a switching approach between locally defined controllers as a function of the state (and control). Lyapunov stability analysis is provided to guarantee the stability and performance.

In our further work, an illustrative example of the proposed method will be given. The extension to a class of higher-order Single-Input-Single-output (SISO) systems and Multiple-Input-Multiple-Output (MIMO) systems is now under study.

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References

[1] C G. Atkeson, A W. Moore and S. Schaal, “Locally weighted learning for control,” *Artificial Intelligence Review*, 1996.

[2] J. Y. Choi, J. A. Farrell, “Nonlinear adaptive control using networks of piecewise linear approximators,” *IEEE Trans. on Neural Networks*, Vol. 11(2): pp. 390-401, 2000.

[3] A. F. Filippov, “Differential equations with discontinuous righthand sides,” *Kluwer Academic Publishers*, 1988.

[4] P. A. Ioannou and J. Sun, “Robust adaptive control,” *Prentice Hall*, 1996.

[5] J.-P. Jiang and L. Praly, “Design of Robust Adaptive Controllers for Nonlinear Systems with Dynamic Uncertainties,” *Automatica*, Vol. 34 (7): pp. 825-840, 1998.

[6] J. Nakanishi, J. A. Farrell, and S. Schaal, “A locally weighted learning composite controller with structure adaptation,” *IEEE/RSJ International Conference on Intelligence Robots and Systems*, pp. 882-889, 2002.

[7] J. Nakanishi, J. A. Farrell, and S. Schaal, “Composite Adaptive Control with Locally Weighted Statistical Learning,” Neural Networks, Vol. 18 (1), pp. 71-90, January 2005.

[8] M. Polycarpou, “Stable adaptive neural control scheme for nonlinear systems,” *IEEE Transactions on Automatic Control*, Vol. 41(3): pp. 447-451, 1996.

[9] M. Polycarpou, M. Mears, “Stable adaptive tracking of uncertian systems using nonlinearly parameterized on-line approximators,” *International journal of control*, Vol. 70(3): pp. 363-384, May 1998.

[10] N. Sadege, “A perceptron network for functional identification and control of nonlinear systems,” *IEEE Trans. on Neural Networks*, Vol. 4: pp. 982-988, Nov. 1993.

[11] R. M. Sanner, J.-J. E. Slotine, “Gaussian networks for direct adaptive control,” *IEEE Trans. on Neural Networks*, Vol. 3: pp. 837-863, Nov. 1992.

[12] D. Sheviez and B. Paden, “Lyapunov stability theory of nonsmooth systems,” *IEEE Trans. Automatic Control*, Vol. 39(9): pp. 1910-1914, 1994.

[13] J.-J. E. Slotine and W. Li, “Applied nonlinear control,” *Prentice-Hall, Inc.*, 1991.

[14] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Trans. Systems, Man, and Cybernetics*, Vol. SMC-15(1): pp. 116-132, 1985.