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Robust sensor location optimization in distributed parameter systems using functional observers

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Index Terms— **Distributed Parameter Systems, Sensor Placement, Functional Observer, Transport Processes, Computational Optimization.**

Abstract— **The focus of this work is to provide an insight into the judicious positioning of either a sensor device or a collocated actuator/sensor pair in SISO transport systems represented by parabolic differential equations. The optimal, with respect to a given measure, placement of a sensor device has a significant effect on the overall performance of a controller with a considerable contribution in energy reduction. This is more evident in the case of a functional observer, i.e. an observer that estimates *not* the entire state, *but* a weighted product of the state. When a functional observer is used to estimate the (inner) product of the state with a feedback gain, the computational demand is significantly reduced since now only a scalar quantity is estimated as opposed to a high-dimensional observer required in an observer-based controller. The efficiency and performance of the functional observer is additionally enhanced when the sensor location is embedded into the control design. By incorporating the effects of exogenous inputs that enter the system via a given distribution vector, a sensor location-parameterized measure is considered and static optimization allows one to optimize both the sensor location and the performance of the resulting functional observer-based controller. A case study of a diffusion process is presented where the performance-enhancing capabilities of the proposed location optimization and control scheme is evaluated through detailed simulation studies.**

I. INTRODUCTION

In the last three decades there has been a significant activity in the problem of actuator and sensor placement in distributed parameter systems. Indeed, the importance of actuator/sensor selection and placement in the overall system performance has been recognized as an important design component in many systems (see the survey papers by van de Wal and de Jager [1] and the book by Uciński [2]).

It is widely accepted that an optimal actuator/sensor placement according to a set of prespecified closed loop performance optimality criteria, results in minimal energy use while important performance objectives can be simultaneously attained. However, the traditional approach to the actuator/sensor placement problem has been the selection of locations based on open loop considerations that ensure that necessary controllability/ observability criteria are met; e.g. improving controllability and observability using controllability and observability indices. This approach relies on a conceptual "decoupling" of the actuator/sensor placement problem from the feedback controller synthesis one.

Quite recently, research efforts focused on the problem of integrating the aforementioned design stages into a coherent and unified scheme that simultaneously addresses the actuator placement policy problem and the controller synthesis one, see for example [3], [4], [5], [6]. In the above approaches, the actuating and sensing devices are placed on locations that minimize a certain functional describing closed loop performance optimality criteria (usually quadratic).

Closer to the current work, the case of sensor placement based on a functional observer requirement has not been addressed for the case of distributed parameter systems other than a preliminary work in [7].

A functional observer may be used to estimate *not* the entire state, *but* rather an inner product of the state weighted by the feedback gain (kernel). Instead of estimating the entire state using a full-order observer and subsequently replacing the estimated state into the control signal computation, one may simply estimate the portion of the state that is required by the control design. The reduction in the computational load to simulate the state of the observer can immediately be seen. In a SISO system one has to simulate a scalar filter as opposed to a full-order state estimator. Adding an element of performance improvement beyond computational savings and ease of real time implementation, one may embed the sensor location problem into the control system which utilizes the output of the functional observer as the control input. The outcome of this optimization will provide the "optimal" sensor location, since it will be chosen from all candidate sensor locations that render the functional observer feasible and minimize an energy norm associated with the resulting closed loop system. Additionally, one may incorporate the issue of actuator placement for the case of a collocated actuator/sensor pair into the problem of functional observer-based controller.

By taking advantage of extensive works on observer design for finite dimensional systems [8], [9], [10], the proposed algorithm extends the works from the finite dimensional literature to a class of infinite dimensional systems governed by transport processes. While earlier works [11], [12], dealt specifically with functional observers for diffusion equations, which are similar in spirit to the problem under consideration, we follow the observer design from the finite dimensional results. Using a variant of the procedure in [13], [14], a performance measure parameterized by the actuator location is minimized in order to provide the best possible actuator location. Once the actuator location and its associated feedback gain are computed for the finite representation of the infinite dimensional system, the functional observer which estimates the product of the state with the feedback gain is parameterized by the sensor locations. Using a similar

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procedure as in [11], the resulting closed loop system is optimized with respect to the sensor locations, but unlike the preliminary work [7], the spatial distribution of the exogenous signal is incorporated into the sensor placement and controller design optimization.

The problem formulation along with the motivation for sensor optimization of functional observers is presented next. The sensor optimization algorithm is summarized in Section III along with the special case of a collocated actuator/ sensor pair. The results of a numerical study are presented in Section IV with conclusions following in Section V.

II. MOTIVATION AND PROBLEM FORMULATION

To demonstrate the proposed integrated sensor location selection and functional observer-based compensator design, we consider the 1-D diffusion equation which models many engineering applications such as conduction in solids, simplified flow problems (linearized Burgers equation), etc. The dynamics of such processes are described by

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &= \frac{\partial}{\partial \xi}\left(\kappa(\xi)\frac{\partial \zeta}{\partial \xi}\right) + a(\xi)\zeta + b(\xi)u + d(\xi)\nu \\ \zeta(t,0) &= \zeta(t,\ell) = 0, \end{aligned} \tag{1}$$

where $\zeta(t,\xi)$ denotes the state of the system and $u(t)$ the control signal. The spatial distribution of the control input is given by $b(\xi)$ and the operator $\mathscr{A}\phi = (\kappa\phi_\xi)_\xi + a\phi$ is assumed to be (strongly) elliptic [13] with $\kappa(\xi) \ge \kappa_0 > 0$ and $0 < a_l \le a(\xi) \le a_u < \infty$ for all $\xi \in [0,\ell]$. The function $d(\xi)$ denotes the spatial distribution of the exogenous input and $\nu(t)$ represents its square integrable temporal component.

In order to improve the system performance, it is desired to simultaneously optimize the control design and the sensor/actuator location. This then leads to an integrated design whereby one penalizes location parameterized performance measures with the ultimate goal of finding the "best" actuator/sensor location that yields the minimum of the performance index over a set of admissible locations, while at the same time providing some level of robustness.

Common approaches on control design for infinite dimensional systems employ a Linear Quadratic Regulator control problem to design a feedback controller which under certain conditions may be expressed in terms of the L^2 inner product of a feedback kernel and the state of the system

$$u(t) = \int_0^\ell k(\xi)\zeta(t,\xi)\,d\xi. \tag{2}$$

In the absence of the full state measurements $\zeta(t,\xi)$, one is required to include a state estimate $\widehat{\zeta}(t,\xi)$ using Kalman/Luenburger filters and then replace $\zeta(t,\xi)$ in (2) by its estimated value to arrive at the control input

$$u(t) = \int_0^\ell k(\xi)\widehat{\zeta}(t,\xi)\,d\xi. \tag{3}$$

In order to analyze the system and calculate its finite dimensional approximation that would allow one to compute the control signal, the system (1) is placed in an abstract setting

$$\dot{x}(t) = \mathscr{A}x(t) + \mathscr{B}u(t) + \mathscr{D}\nu(t) \tag{4}$$

where $x(t) = \zeta(t,\cdot)$ is the state of the infinite dimensional system having a state space $X = L_2(0,\ell)$ with inner product and norm denoted by $\langle\cdot,\cdot\rangle_X$ and $|\cdot|_X$ respectively, $\mathscr{A}$ denotes the elliptic operator defined above and the input operator is

$$\langle \mathscr{B}u, \phi\rangle_X = \int_0^\ell b(\xi)\phi(\xi)\,d\xi\, u(t), \quad \phi \in X. \tag{5}$$

Similarly, the operator associated with the spatial distribution of the exogenous input is given by

$$\langle \mathscr{D}\nu, \phi\rangle_X = \int_0^\ell d(\xi)\phi(\xi)\,d\xi\, \nu(t), \quad \phi \in X.$$

The control law corresponding to (2) is now given by

$$u(t) = \mathscr{K}x(t), \tag{6}$$

where $\mathscr{K}$ is the feedback operator (*cf.* (2), (3)):

$$\mathscr{K}\phi = \int_0^\ell k(\xi)\phi(\xi)\,d\xi. \tag{7}$$

Using established results [14], [15], a state feedback controller that minimizes the associated LQR functional

$$J(x_0;u) = \int_{t_0}^\infty \left[\langle x, Qx\rangle_X + ru^2\right] dt,$$

can be synthesized, where $r > 0$ is a suitably chosen "weight-factor" and Q is a coercive operator. The cost functional above is finite for a square integrable control input since the diffusion system in question is optimizable as a consequence of its exponential stabilizability [14]. In this case one solves the LQR/$\mathscr{H}^2$ Operator Algebraic Riccati Equation (OARE)

$$\begin{aligned} &\langle \mathscr{A}\phi, \mathscr{P}\psi\rangle_X + \langle \phi, \mathscr{P}\mathscr{A}\psi\rangle_X + \langle Q\phi, \psi\rangle_X \\ &\qquad + \langle \mathscr{P}\mathscr{B}r^{-1}\mathscr{B}^*\mathscr{P}\phi, \psi\rangle_X = 0, \end{aligned}$$

for $\phi, \psi \in D\text{om}(\mathscr{A})$. The optimal control signal can be proven to be [14] $u(t) = -r^{-1}\mathscr{B}^*\mathscr{P}x(t) = \mathscr{K}x(t)$, and the resulting optimal value of the cost functional is given by $J^*(x_0;u) = \langle x(t_0), \mathscr{P}x(t_0)\rangle_X$. For a given operator Q and a fixed value of r, one may further enhance the closed loop performance by finding an optimal location ξ of the actuating device, in the sense of minimizing $J^*(x_0;u)$. Usually, there is a finite set $\Xi = \{\xi_1, \xi_2, \ldots \xi_N\}$ of candidate actuator positions each of which renders the ξ-parameterized pair $(\mathscr{A}, \mathscr{B}(\xi_i))$ approximately controllable $\forall i = 1, \ldots, N$, and hence one may optimize the cost value $J^*(x_0, u)$ over this set of actuator locations [15]. This amounts to solving the OARE for each element in Ξ and setting up the now location-parameterized optimal costs $\langle x(t_0), \mathscr{P}(\xi)x(t_0)\rangle_X$. The optimal actuator location is given via $\xi^{opt} = \arg\min_{\xi\in\Xi} \langle x(t_0), \mathscr{P}(\xi)x(t_0)\rangle_X$.

However, full state information is seldom feasible and hence partial state information is given by the expression

$$y(t) = \int_0^\ell c(\xi)z(t,\xi)\,d\xi = \mathscr{C}x(t), \tag{8}$$

where $\mathscr{C}$ is the operator associated with the observation distribution function[1] $c(\xi)$. Using the input and output signals, one may design a state estimator for (1) in abstract form

$$\dot{\widehat{x}}(t) = (\mathscr{A} - \mathscr{L}\mathscr{C})\widehat{x}(t) + \mathscr{L}y(t) + \mathscr{B}u(t), \tag{9}$$

[1]implicitly assuming that the output operator $\mathscr{C}$ admits the kernel representation $\mathscr{C}\phi = \int_0^\ell c(\xi)\phi(\xi)d\xi$.

where the filter gain $\mathscr{L}$ is chosen appropriately, [14]. The control law now takes the form (*cf.* (3))

$$u(t) = \mathscr{K}\widehat{x}(t). \tag{10}$$

Implicitly assumed, is the *approximate observability* [14] of the pair $(\mathscr{A},\mathscr{C})$. Associated with the proposed sensor location optimization, we define the *set of admissible* sensor locations

$$\mathscr{O} = \left\{ \xi \in [0,\ell] : (\mathscr{A},\mathscr{C}(\xi)) \text{ is approx. observable} \right\},$$

where $\mathscr{C}(\xi)$ denotes the *location-parameterized* observation operator. The control (10) requires simulating the observer (9) and then utilize this estimate in (3) or in its abstract representation (10). In practice, one uses a finite dimensional representation of (1) or (4)

$$\dot{x}^n(t) = Ax^n(t) + Bu^n(t) + D\nu(t), \quad y^n(t) = Cx^n(t), \tag{11}$$

where A,B,D,C are the matrix representations of the operators $\mathscr{A},\mathscr{B},\mathscr{D},\mathscr{C}$ respectively, and $x^n(t)$ the finite dimensional representation of the state $x(t)$, in order to design an observer-based controller

$$\begin{cases} u^n(t) = K\widehat{x}^n(t), \\ \dfrac{d}{dt}\widehat{x}^n(t) = (A-LC)\widehat{x}^n(t) + Ly^n(t) + Bu^n(t). \end{cases} \tag{12}$$

The implementation of the above controller (12) in the infinite dimensional system (1) requires the on-line solution to the observer state, which for a large approximation index n, imposes a heavy computational load. To address this concern of the computational burden, we consider a functional observer-based compensator and its associated sensor location selection, which constitute the main theme and contribution of this work.

III. Sensor Location Optimization of Functional Observers

Instead of estimating the state x (or its finite dimensional approximation x^n) and then use that estimate in the control expression (3) or (10), we consider a functional observer where we now build an estimate of the product of K and x in the term $z(t) = Kx(t)$ and use as the control signal the estimate $\widehat{z}$ with $u = \widehat{z}$, see [11] for a procedure on finite dimensional systems. For the case of a SISO system this amounts to generating a scalar signal (filter) as opposed to an n-dimensional estimate of the state and performing multiplication by an n-dimensional vector in order to produce the control signal in each time instance.

For brevity, we now omit the superscript n with the understanding that it is the finite dimensional representation of the infinite dimensional state that we consider.

A. Functional observer for fixed actuator and sensor locations

The use of a functional observer as a means to estimate a feedback product $Kx(t)$ and subsequently used as the control signal is summarized. It is assumed that one has chosen a priori both the actuator and sensor locations.

The equations describing the proposed functional observer-based controller are found as follows:

$$\begin{cases} \dot{w}(t) = Nw(t) + Jy(t) + Hu(t) \\ \widehat{z}(t) = w(t) + Ey(t), \\ u(t) = \widehat{z}(t), \end{cases} \tag{13}$$

where the functional observer matrices are given by [9] and summarized in the algorithm below:

Algorithm 1:

1) first define $\overline{A} = A(I-K^\dagger K)$ and $\overline{C} = C(I-K^\dagger K)$,
2) and then set

$$\Sigma = \begin{bmatrix} C\overline{A} \\ \overline{C} \end{bmatrix}, \quad F = KAK^\dagger - K\overline{A}\Sigma^\dagger \begin{bmatrix} CAK^\dagger \\ CK^\dagger \end{bmatrix},$$

$$G = (I - \Sigma\Sigma^\dagger) \begin{bmatrix} CAK^\dagger \\ CK^\dagger \end{bmatrix},$$

3) then use pole placing techniques for Z in $N = F - ZG$,
4) set $\begin{bmatrix} E & L \end{bmatrix} = K\overline{A}\Sigma^\dagger + Z(I-\Sigma\Sigma^\dagger)$,
5) set $J = L + NE$, and finally
6) solve the matrix equation $PA - NP - JC = 0$ for P, to get $H = PB$.

The matrix $K^\dagger$ above denotes the Moore-Penrore generalized inverse of the matrix K. The control signal supplied to the system is simply the output of the functional observer (13), namely the estimate $\widehat{Kx}$ of Kx. The combined plant and functional observer is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + D\nu(t), \quad y(t) = Cx(t), \\ \dot{w}(t) &= Nw(t) + Jy(t) + Hu(t), \\ u(t) &= w(t) + Ey(t). \end{aligned} \tag{14}$$

When the above are combined, one then arrives at

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} &= \begin{bmatrix} A+BEC & B \\ (J+HE)C & N+H \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ &+ \begin{bmatrix} D \\ 0 \end{bmatrix} \nu = A_0x_0 + D_0\nu, \end{aligned} \tag{15}$$

where $x_0 \triangleq \begin{bmatrix} x^T & w \end{bmatrix}^T$. A more convenient form of the above system which may be used for well posedness and stability analysis utilizes the error $e \triangleq \widehat{z} - z$ and is given by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} &= \begin{bmatrix} A+BK & B \\ 0 & N \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} - \begin{bmatrix} -D \\ PD \end{bmatrix} \nu \\ &= A_1x_1 - D_1\nu, \quad x_1 \triangleq \begin{bmatrix} x^T & e \end{bmatrix}^T. \end{aligned}$$

One can establish the stability of the above system using the fact that both $A+BK$ (being the stable finite dimensional approximation of the generator of an exponentially stable C_0 semigroup) and N are stable and thus A_1 is stable as well. The fact that the exogenous signal is square integrable allows one to conclude asymptotic stability of the closed loop system.

B. Sensor location optimization using functional observers with a fixed optimal actuator location

Here, it is assumed that the actuator is optimally placed by first assuming full state availability and optimizing an associated LQR or $\mathscr{H}^2$ performance measure with respect to the candidate actuator locations. This then makes the optimization problem a two-stage optimization process in which the actuator and sensor locations are optimized separately. Basically, it is assumed that the actuator location is fixed and chosen a priori using separate optimality criteria and that only the sensor location is intended to be optimized. The same is used when one has no flexibility in the actuator placement and only the sensor location can be optimized.

In order to optimize the system with respect to the sensor locations, one parameterizes the output matrix by the sensor locations $C(\theta)$ as derived, for example, from the expression

$$y(t) = \int_0^{\ell} \delta(\xi - \theta)\zeta(t,\xi)\, d\xi = \zeta(t,\theta), \qquad (16)$$

for the case of pointwise measurements. It is therefore assumed that there exists a finite set $\Theta \subset \mathscr{O}$ defined by

$$\Theta = \left\{\theta_1, \theta_2, \ldots, \theta_m\right\} = \left\{\theta \in \mathscr{O} : (C(\theta), A) \text{ observable pair and (13) feasible}\right\}.$$

Remark 3.1: The reason for considering the finite set Θ instead of the set $\mathscr{O}$ for sensor location optimization is twofold: first, due to practical design considerations, one seldom has the freedom to place sensors in more than a finite number of candidate locations; second, one avoids considering a continuum of sensor locations as it would lead to possibly infeasible optimization and hence by considering a finite set Θ, one reduces to simple static optimization techniques that represent considerable computation savings.

In order to find the optimal sensor location of the closed loop system (15), one considers the combined closed loop system parameterized by the sensor location

$$\frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A + BE(\theta)C(\theta) & B \\ (J(\theta) + H(\theta)E(\theta))C(\theta) & N(\theta) + H(\theta) \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} \nu = A_0(\theta)x_0 + D_0\nu. \qquad (17)$$

Alternative to the above, is the following closed loop system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} &= \begin{bmatrix} A + BK & B \\ 0 & N(\theta) \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} - \begin{bmatrix} -D \\ P(\theta)D \end{bmatrix}\nu \\ &= A_1(\theta)x_1 - D_1(\theta)\nu, \end{aligned}$$

which may also be used for sensor location optimization.

One may then minimize the effects of the exogenous signal ν on the augmented state x_0 by optimizing the $\mathscr{H}^2$ norm from ν to x_0. Thus, the optimal sensor location can be found by minimizing $T_{x_0\nu}(s,\theta)$ with respect to $\theta \in \Theta_s$. Towards that end, one simply solves the θ-parameterized Lyapunov (observability Gramian) equation

$$A_0^T(\theta)\Sigma_0(\theta) + \Sigma_0(\theta)A_0(\theta) = -I, \quad \theta \in \Theta_s. \qquad (18)$$

The above are now presented in the main result.

Lemma 3.1: Consider the finite dimensional representation (11) of the infinite dimensional system (1) or (4), along with the finite dimensional representation of the feedback control (6). Further assume that a finite number of sensor locations from the set $\mathscr{O}$ exist such that the functional observer (13) is feasible and which ensures that the matrix $A_0(\theta)$ in (17) is Hurwitz for all $\theta \in \Theta_s$. Then, the optimal sensor location is found via the expression

$$\begin{aligned} \theta^{opt} &= \arg\min_{\theta\in\Theta_s} \|T_{x_0,\nu}(s;\theta)\|_2 \\ &= \arg\min_{\theta\in\Theta_s} \sqrt{\text{trace}[D_0^T\Sigma_0(\theta)D_0]}, \end{aligned} \qquad (19)$$

and which results in a stable closed loop system having an optimally placed sensor.

Remark 3.2: Alternative to the system (17), the Gramian (18) and the optimization (19), one may consider

$$A_1^T(\theta)\Sigma_1(\theta) + \Sigma_1(\theta)A_1(\theta) = -I, \quad \theta \in \Theta_s,$$

and use the following for sensor location optimization

$$\theta^{opt} = \arg\min_{\theta\in\Theta_s} \sqrt{\text{trace}[D_1^T(\theta)\Sigma_1(\theta)D_1(\theta)]}.$$

Sketch of Proof of Lemma 3.1. Using the fact that for all the sensor locations in Θ, the functional observer (13) is rendered feasible, and that the resulting sensor location-parameterized matrix $A_0(\theta)$ is Hurwitz when restricted to $\theta \in \Theta_s \subset \Theta$, one then arrives at a stable closed loop system. This is because the matrix $A_0(\theta)$ is stable and the exogenous signal in (17) is square integrable. Static optimization of the cost (19) over all θ in Θ_s would then provide the (stable) functional observer that minimizes the effects of the exogenous input signal on the entire augmented state, see also [11] for detailed stability arguments of a similarly proposed functional observer. □

Remark 3.3: It should be pointed out that in the finite dimensional case, the conditions for the existence and stability of the functional observer (13), as taken from [9], are

$$\text{rank}\begin{bmatrix} KA \\ CA \\ C \\ K \end{bmatrix} = \text{rank}\begin{bmatrix} CA \\ C \\ K \end{bmatrix}, \quad \text{and}$$

$$\text{rank}\begin{bmatrix} sK - KA \\ CA \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} CA \\ C \\ K \end{bmatrix},$$

for all $s \in \mathbb{C}$, $Re(s) \geq 0$. The above conditions then define the class of admissible gain feedback gains K and the set of candidate sensor positions. Similar conditions may be obtained from equation (2.20) of [11], which uses the finite dimensional approximation (truncation) of the zero-state response of diffusion equations with boundary measurements and controls.

C. Collocated actuator/sensor location optimization using functional observers

In the event that it is known and dictated a priori that the actuator/sensor configuration must be collocated, which for the case of a pointwise device translates to $\mathscr{B}u(t) = \delta(\xi - \theta)u(t)$ and $\mathscr{C}x(t) = \int_0^L \delta(\xi - \theta)\zeta(t,\xi)\, d\xi = z(\theta,t)$, one may integrate the actuator/sensor location optimization into the functional observer in a similar fashion as above. Using the above, one has the following $\mathscr{C} = \mathscr{B}^*$. In this case one considers (17) with B replaced with $C^T(\theta)$

$$\frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A+C^T(\theta)E(\theta)C(\theta) & C^T(\theta) \\ (J(\theta)+H(\theta)E(\theta))C(\theta) & N(\theta)+H(\theta) \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} v = A_0(\theta)x_0 + D_1 v, \tag{20}$$

and with the feedback gain K in (6) replaced by a location-parameterized gain $K(\theta) = -r^{-1}C(\theta)\Pi(\theta)$ which is found by solving the Algebraic Riccati equation

$$A^T\Pi(\theta)+\Pi(\theta)A+Q+\Pi(\theta)C^T(\theta)r^{-1}C(\theta)\Pi(\theta) = 0. \tag{21}$$

An algorithm for the integrated functional observer-based feedback with an optimal actuator/sensor location follows.

Algorithm 2:

(1) Define the set of admissible actuator/sensor locations Θ_s which implicitly assume the feasibility of the functional observer (13).
(2) for each $\theta \in \Theta_s$ solve for the Riccati equation $\Pi(\theta)$ (21) and set up the corresponding θ-parameterized feedback gains $K(\theta) = -r^{-1}C(\theta)\Pi(\theta)$.
(3) for each $\theta \in \Theta_s$, use the feedback gain $K(\theta)$ from step 2 and solve for the θ-parameterized matrices $N(\theta), P(\theta), E(\theta), H(\theta), J(\theta)$ of the functional observer in (13) using Algorithm 1.
(4) set up the matrix $A(\theta)$ in (20) for each $\theta \in \Theta_s$.
(5) for each $\theta \in \Theta_s$, find the associated $\mathscr{H}^2$ norm from v to x_0 by solving the θ-parameterized Gramian $A_0^T(\theta)\Sigma_0(\theta)+\Sigma_0(\theta)A_0(\theta) = -I$.
(6) the optimal actuator/sensor location is found via

$$\theta^{opt} = \arg\min_{\theta\in\Theta_s}\sqrt{\text{trace}\left[D_0^T\Sigma_0(\theta)D_0\right]}. \tag{22}$$

IV. NUMERICAL STUDIES

For our study, we consider (1) with

$$\kappa(\xi) = 0.05\Big(1+0.3\sin(2\pi\xi/\ell)((\sin^3(\xi^3)+\sin^3((\ell-\xi)^3)\Big),$$

$a = 0$, $\zeta(0,\xi) = 5\sin(\pi\xi)$, $\ell = 1.6$. Using $n = 60$ linear elements, the finite dimensional representation of (1) was simulated for $t = 20$ seconds.

First, by considering full state availability and using an LQR cost functional with $Q = I$ and $r = 1$, parameterized by the admissible actuator locations, an optimal actuator location was found at $\xi = 0.508\ell$. The optimization of the pointwise sensor location in (19) predicted a location at $\xi = 0.967\ell$. Similarly the optimization of a collocated actuator/sensor pair via (22) in *Algorithm 2* predicted the optimal actuator/sensor location $\xi = 0.4415\ell$. For comparison, a sensor placed at the non-optimal location $\xi = 0.996\ell$ was used to simulate a controller based on a functional observer. This functional observer-based controller (FOC) used the same actuator location of $\xi = 0.996\ell$. Finally, a full-order observer-based controller (LQG compensator) using a sensor placed at $\xi = 0.996\ell$ and a collocated actuator (placed at $\xi = 0.996\ell$) was included in the simulation study in order to give insight on the performance expectations and computational benefits/disadvantages of a high dimensional compensator versus that of a low dimensional functional observer-based controller.

The evolution of the closed loop $L_2(0,\ell)$ state norm is depicted in Figure 1. One may observe that the optimal collocated sensor/actuator pair combined with the functional-observer (dashed) may perform better than the case of a non-optimally placed actuator/sensor (dotted) that employs a full-order compensator (LQG controller). Even in the case where the performance is comparable, one must not discount the computational savings resulting from the implementation of a functional-observer based controller; 60-dimensional LQG compensator versus a 1-dimensional controller.

In addition to the above, the frequency response (magnitude in dB) of the closed loop transfer function $T_{x,v}(s)$ system for three cases is depicted in Figure 2: functional observer with optimal sensor (solid), optimal collocated actuator/sensor with functional observer (dashed) and non-optimal sensor with functional observer (dotted). In the low frequency range, one may observe that the functional observer with a non-optimal sensor may not be able to attenuate disturbances as well as an optimally placed sensor. This points to the fact that while a functional observer-based controller might be attractive from a computational/implementational point of view, it nonetheless makes it imperative to also place the actuator/sensor at an optimal location in order to introduce an element of performance improvement comparable to that of a full order observer-based controller.

This importance of the optimal sensor location can also be observed in Table 1 in which the L_2 norm, given by $\|z\|_2^2 = \int_0^{12}\left(\int_0^\ell \zeta^2(t,\xi)\,d\xi\right)dt$, is presented for various cases. When an optimally placed sensor is combined with a functional observer, it performs equally well as a non-optimally placed sensor having a full order observer-based feedback. Additionally, an optimally placed collocated actuator/sensor pair with a functional observer-based controller appears to perform better. While not reported here due to space limitations, when an optimally placed sensor is used in a full-order observer based feedback, it performs (as expected) better than a functional observer based feedback with an optimally placed sensor. The advantage of the functional observer though, lies in the significant computational savings: *simulating a single scalar equation* (13) *versus an n-dimensional system for the full-order observer-based feedback* (12).

V. CONCLUSIONS

In this work, the sensor placement problem is combined with the functional observer problem and applied to a 1-D transport system governed by a parabolic differential equation. First, it is assumed that full state is available to build an $\mathscr{H}^2$-based full state feedback controller. By parameterizing the optimal value of the $\mathscr{H}^2$ control functional with

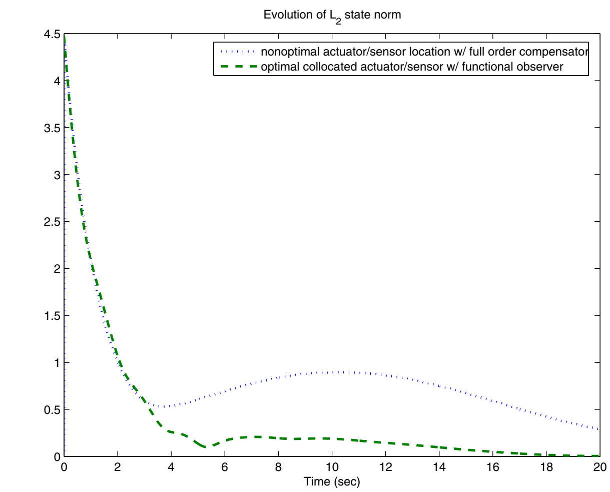

Fig. 1. Evolution of state L_2 norm using full-state feedback with a non-optimal sensor (dotted) and a functional observer-based feedback with optimal sensor (dashed).

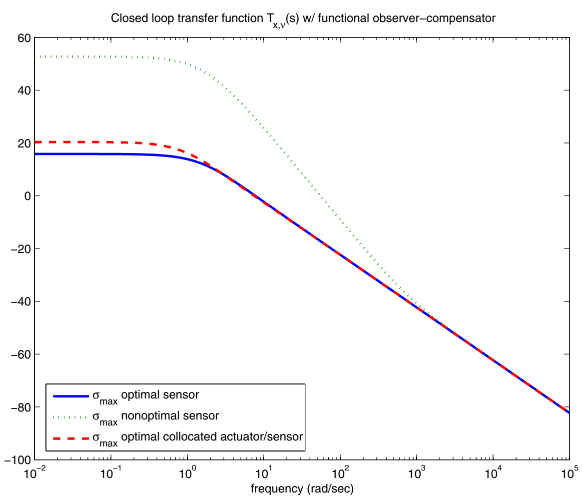

Fig. 2. Magnitude plot of the closed loop transfer function $T_{xv}(s)$ for the optimal and non-optimal sensor functional observer-based feedback.

respect to the candidate actuator locations, and performing a simple static optimization, the optimal actuator location was calculated. This resulted in the best performance with the smallest $\mathscr{H}^2$ cost for all candidate actuator locations. Second, by building a functional observer that estimates the weighted sum of the states, an estimate of the control input was then resulted as the outcome of this observer. By parameterizing now the closed loop system, consisting of the plant and the functional observer, by the candidate sensor locations, the optimal sensor location was found as the one that minimized the effects of the exogenous input signal on the entire state. This was possible by solving the associated location-parameterized Lyapunov equation and (statically) minimizing with respect to the candidate sensor locations. The case of a collocated actuator/sensor pair was considered in the context of a joint optimal location and a functional observer-based controller. The procedure was similar in spirit

TABLE I

TABLE 1. $L_2([0,20];L_2(0,\ell))$ NORMS FOR DIFFERENT CASES.

case	L_2 norm
non-opt. collocated actuator/sensor w/ FOC	5.9367
opt. sensor, fixed actuator w/ FOC	4.8813
non-opt. collocated actuator/sensor w/ LQG	4.1053
optimal collocated actuator/sensor w/ FOC	2.7806

as the one for the sensor location optimization. The major advantage of the proposed work lies in the considerable computational savings resulting from the estimate of a single scalar quantity (filter) than estimating the entire state and then multiplying it by the feedback gain.

REFERENCES

[1] M. van de Wal and B. de Jager, "A review of methods for input/output selection," *Automatica*, vol. 37, no. 4, pp. 487–510, 2001.

[2] D. Uciński, *Optimal Measurement Methods for Distributed Parameter System Identification*. Boca Raton: CDC Press, 2004.

[3] C. Antoniades and P. D. Christofides, "Computation of optimal actuator locations for nonlinear controllers in transport-reaction processes," *Computers and Chemical Engineering*, vol. 24, pp. 577–583, 2000.

[4] ——, "Integrating nonlinear output feedback control and optimal actuator/sensor placement for transport-reaction processes," *Chemical Engineering Science*, vol. 56, pp. 4517–4535, 2001.

[5] J. Borggaard, "A continuous control design method," in *Proceedings of the 15th International Symposium on Mathematical Theory of Networks and Systems*, University of Notre Dame, August 12-16 2002.

[6] M. A. Demetriou, "Numerical investigation on optimal actuator/sensor location of parabolic pde's," in *Proceedings of the 1999 Americal Control Conference*, San Diego, June, 2-4 1999.

[7] ——, "Optimization of sensor locations in transport systems using functional observers," in *Proceedings of the 9th IEEE International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, August 25-28 2003.

[8] C.-T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart and Winston, Inc., 1984.

[9] M. Darouach, "Existence and Design of Functional Observers for Linear Systems," *IEEE Trans. Automatic Control*, vol. 45, no. 5, pp. 940–943, 2000.

[10] R. D. Gupta, F. Fairman, and T. Hinamoto, "A direct procedure for the design of single functional observers," *IEEE Trans. Circuits and Systems*, vol. 28, no. 4, pp. 294–300, 1981.

[11] M. Aldeen and H. Trinh, "Reduced-order linear functional observer for linear systems," *IEE Proceedings*, pp. 399–405, 1999.

[12] R. Okabayashi and K. Furuta, "Arbitrary pole assignment using dynamic compensators based on linear function observers," in *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, Florida, December 1998, pp. 1734–1739.

[13] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*. Berlin Heidelberg New York: Springer Verlag, 2000, vol. 2: Functional and Variational Methods.

[14] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, ser. Texts in Applied Mathematics, Vol. 21. Berlin: Springer-Verlag, 1995.

[15] S. Omatu and J. H. Seinfeld, *Distributed Parameter Systems: Theory and Applications*. New York: Oxford University press, 1989.