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Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling.

N. Farhi, M. Goursat & J.P. Quadrat

Abstract— **The fundamental diagram gives the relation between the flow and the density of vehicles for the car traffic on a road. We extend this diagram to the case of two circular roads with an intersection. From a Petri net model we derive the dynamics in terms of the composition of a standard linear operator and a minplus operator which determines uniquely the system trajectory. We show, experimentally, that there exists an average flow which becomes asymptotically independent of the initial position of the vehicles when the size of the system goes to infinity. We study the corresponding fundamental diagram and analyze the different phases that can appear.**

I. Introduction

The fundamental diagram gives the relation between the flow and the vehicle density for the car traffic on a road. It has been observed empirically and derived theoretically in the case of a unique road or a regular system of roads see for example [5], [15], [2], [17], [16] and the surveys [7], [4]. We study here the fundamental diagram for a system of two circular roads without overtaking and a unique crossing.

Microscopic traffic systems have been studied in statistical physics as particular classes of cellular automata. In the cellular automaton classification [4], the model used here is of Biham-Middleton-Levine [3] (BML) type (used to describe simplified regular towns). But contrary to BML, we accept the possibility of turning like in [11], [12], [13] but, here, with deterministic turn (if we give a number to each car entering in the crossing, the odd cars go south and the even cars go to west) and making a special attention to the case of only two roads and a crossing. This last case has been studied in detail in [8], [9], [10] without the possibility of turning with a stochastic modeling. We follow the point of view discussed in [16] where the exclusion process is given in terms of timed Petri nets using the minplus algebra to write the corresponding dynamics. The model considered here is the simplest one with turning possibility. It is also the simplest traffic model which is not an event graph.

From a Petri net model we derive a well posed dynamic obtained by composing a standard linear operator with a minplus operator. In the preliminary case of only a circular road with a retarder, adapting [16], the fundamental diagram is obtained immediately from the graphic interpretation of the eigenvalue of the minplus linear dynamics of the corresponding event graph. In the complete case of the two roads and a crossing, we show, experimentally, that there exists an average flow which becomes independent of the initial vehicle positions when the size of the system goes to infinity.

INRIA-Rocquencourt, B.P. 105, Le Chesnay Cedex (France)
Jean-Pierre.Quadrat@inria.fr

We show experimentally that the corresponding fluid Petri net presents the same properties and has the same asymptotic diagram. We recover, on the computed fundamental diagram, the presence of several traffic phases known in the cellular automata literature (in the case of stochastic models).

The influence of a traffic light control on this diagram is also given. It shows the potential gain obtained by setting traffic lights with or without feedback on the states of the roads.

II. Minplus Algebra

In this section we present the main definitions and properties of the minplus algebra. The reader is referred to [1] for an in-depth treatment of the subject.

The structure $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ is defined by the set $\mathbb{R} \cup \{+\infty\}$ endowed with the operations $\min$ (denoted by $\oplus$, called minplus *sum*) and $+$ (denoted by $\otimes$, called minplus *product*). The element $\varepsilon = +\infty$ is the *zero* element, it satisfies $\varepsilon \oplus x = x$ and is *absorbing* $\varepsilon \otimes x = \varepsilon$. The element $e = 0$ is the *unity*, it satisfies $e \otimes x = x$. The main difference with respect to the conventional algebra is the idempotency of the addition $x \oplus x = x$ and the fact that addition cannot be *simplified*, that is: $a \oplus b = c \oplus b \not\Rightarrow a = c$. It is called *minplus algebra*.

This minplus structure on scalars induces an idempotent semiring structure on $m \times m$ square matrices with the element-wise minimum and matrix product defined by $(A \otimes B)_{ik} = \min_j (A_{ij} + B_{jk})$, where the zero and unit matrices are still denoted by ε and e. We associate to a square matrix A, a precedence graph $\mathcal{G}(A)$ where the nodes correspond to the columns (or the rows) of the matrix A and the arcs to the nonzero entries (the *weight* of the arc (i, j) being the non zero entry A_{ji}). We define $|p|_w$ the weight of a path p in $\mathcal{G}(A)$ as the minplus product of the weights of the arcs composing the path (that is the standard sum of weights). The arc number of the path p is denoted $|p|_l$. We will use the three fundamental results (see [1]).

Theorem 1: Given A a $m \times m$ minplus matrix, if the weights of all the circuits of $\mathcal{G}(A)$ are positive, the equation $X = A \otimes X \oplus B$ admits a unique solution: $X = A^* \otimes B$ where

$$A^* = \bigoplus_{n=0}^{\infty} A^n = \bigoplus_{n=0}^{m-1} A^n \ .$$

Theorem 2: If the graph $\mathcal{G}(A)$ associated with the minplus matrix A is strongly connected, the matrix A admits a unique *eigenvalue* $\lambda \in \mathbb{R}_{\min}$:

$$\exists\ X \in \mathbb{R}_{\min}^N : A \otimes X = \lambda \otimes X \text{ with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l} \ , \quad (1)$$

where $\mathcal{C}$ is the set of circuits of $\mathcal{G}(A)$.

Theorem 3: The minplus linear dynamic system associated with the minplus matrix A, with strongly connected graph $\mathcal{G}(A)$, defined by: $X_{n+1} = A \otimes X_n$ is *asymptotically periodic*:

$$\exists T, K, \lambda : \forall k \geq K : A^{k+T} = \lambda^T \otimes A^k \ .$$

III. Petri Net Dynamics

A Petri net $\mathcal{N}$ is a graph with two set of nodes: the *transitions* $\mathcal{Q}$ (with Q elements) and the *places* $\mathcal{P}$ (with P elements) and two sorts of arcs, the *synchronization arcs* (from a place to a transition) and the *production arcs* (from a transition to a place).

A minplus $P \times Q$ matrix D, called *synchronization matrix* is associated to the synchronization arcs. It is defined by $D_{pq} = a_p$ if there exists an arc from the place $p \in \mathcal{P}$ to the transition $q \in \mathcal{Q}$ and $D_{pq} = \varepsilon$ elsewhere, where a_p is the *initial marking* and is represented graphically by the tokens in the places. To count the tokens in the places we also use a standard matrix associated with D denoted χ_D (with the same size as D) defined by $\chi_{D_{pq}} = 1$ if $D_{pq} \neq \varepsilon$ and equal to $\chi_{D_{pq}} = 0$ elsewhere.

A standard algebra $Q \times P$ matrix $H(\delta)$, called *production matrix* is associated with the production arcs. Its pq entry is the delay operator $H_{qp}(\delta) = m_{qp}\delta^{\tau_p}$ (where m_{qp} is the multiplicity of the arc and τ_p the minimal sojourn time in the place p (represented by sticks in the place) if there exists an arc from q to p and 0 elsewhere. A delay operator acts on series, it is defined by $m\delta^\tau : (X_n)_{n\in\mathbb{N}} \mapsto m(X_{n-\tau})_{(n-\tau)\in\mathbb{N}}$. By extension $H(1)$ is a matrix with real entries m_{qp}.

Therefore a Petri net is characterized by the quadruple :

$$(\mathcal{P}, \mathcal{Q}, H(\delta), D) \ .$$

It is a dynamic system in which the token evolution is partially defined by the transition firings saying that a transition can fire as soon as all its upstream places contain at least a token having stayed a time larger than the place sojourn time. When a transition fires, it generates a number of tokens in each downstream place equal to the arc multiplicity of the arc joining the transition to the place.

In the case of a *deterministic* Petri net, where all the places have only one arc downstream, the dynamic is well defined, that is, there is no token consuming conflict between the downstream transitions. In the non deterministic case, we have to precise the rules which resolve the conflicts by, for example, giving priorities to the consuming transitions or by imposing ratios to be respected. As soon as this rules are added, the initial nondeterministic Petri becomes a deterministic one.

Theorem 4: Denoting $X = (X^q)_{q\in\mathcal{Q}}$ the vector of sequences $X^q = (X^q_n)_{n\in\mathbb{N}}$ such that the entry X^q_n, is the firing number of the transition q up to time n, we have[1]:

$$X = \big(XH(\delta)\big) \otimes D \ .$$

[1] If f is a nonlinear function $X = (X\delta)f$ means $(X_{n+1} = f(X_n))_{n\in\mathbb{N}}$.

Proof: By definition of the firing instants, the Petri net is such that at each instant, at least, one place upstream of each transition, is empty. The series representing the numbers of tokens in a place as function of time being: $a + X(\delta)H(\delta) - X(\delta)$ (where a is the vector of the initial markings) and each minimum of the token numbers in the places upwards each transition being 0 (by definition of firing) we have:

$$\big(XH(\delta) - X\chi_D^t\big) \otimes D = 0 \ ,$$

which gives the announced result, using the fact that, the Petri net being deterministic, from each place leaves exactly one arc. □

The nondeterministic Petri nets do not have a well defined dynamics but some constraints on the token dynamics are imposed by the nets. We can define dynamics invariants satisfied by all the possible dynamics. Given a sequence of firings σ, we denote by X^σ the line vector of the transition firing numbers and by a^σ the line vector of the token numbers in the places after this firing sequence. We have:

$$a^\sigma = a + X^\sigma(H(1) - \chi_D^t) \ .$$

From this equation the following result is clear.

Theorem 5: For all column vector ρ satisfying

$$(H(1) - \chi_D^t)\rho = 0$$

we have

$$a^\sigma\rho = a\rho, \ \ \forall\sigma \ .$$

The numbers $a\rho$ are the *right-invariants* of the Petri net.

In the case of *event graphs*, particular deterministic Petri nets where all the multiplicity m_{qp} are equal to 1 and all the places have exactly one arc upstream, the dynamics is linear in the minplus sense. It is:

$$X = X \otimes A(\delta),$$

with $A(\delta)_{qq'} = a_p \otimes \delta^p$ with p the unique place connected to the transitions q and q'.

A corollary (see [1]) of Theorem 2 is:

Theorem 6: For a strongly connected event graph the throughput defined by $\lambda = \lim_n X^q_n/n$ is independent of q and is equal to:

$$\lambda = \min_{c\in\mathcal{C}} \frac{|c|_a}{|c|_t},$$

where $\mathcal{C}$ denotes the set of circuits of the event graph, $|c|_a$ the total token number in the circuit c and $|c|_t$ the total stick number in the circuit c.

IV. The Fundamental Traffic Diagram for a Circular Road with a Retarder

It is easy to derive a quite realistic fundamental traffic diagram from simple dynamical minplus linear models. For pedagogical reason only the most simple diagram will be given here. To see more realistic one, in the simpler case where there is not a retarder, see [16]. For another analytic result see Derida [5].

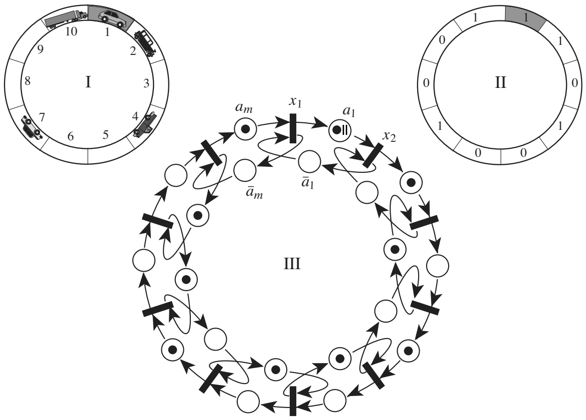

Fig. 1. A circular road.

A. Exclusion process modeling for the road with a retarder

Adapting [2] to the case of a road with a retarder, let us consider a circular road with places occupied or not by a car symbolized by a 1. Let us suppose that there is a retarder in position 1 where we have to stay during at least two time steps. The dynamics is defined by applying at each time step, simultaneously in all positions (at the exception of position 1), the rule $10 \to 01$. In position 1 and time n we can apply the same rule only if there was a vehicle at time $n-2$. For the example given in Figure 1 Part II, we give, in Table I, three simulations of the system with three different vehicle densities corresponding to the three phases (that will be discussed later) of the system. For such a system we can

n	$1/3 < d/(1+1/m)$ $d < 2/3$	$d/(1+1/m) \leq 1/3$	$d \geq 2/3$
1	1010100101	1000100100	0111011011
2	1001010011	1000010010	1110110110
3	0100101011	0100001001	1101101101
4	1010010110	1010000100	1011011011
5	1001001101	1001000010	0110110111
6	0100101011	0100100001	1101101110

TABLE I

TRAFFIC SIMULATION FOR THE THREE POSSIBLE PHASES.

call density d the number of cars p divided by the number of places m that is $d = p/m$. We call flow

$$f = \frac{m}{m+1} \frac{\lim_{n\to\infty} P_n/n}{p} d = \frac{1}{m+1} \lim_{n\to\infty} P_n/n$$

where P_n is the total number of car displacements until instant n. The fundamental traffic diagram gives the relation between f and d.

If $p/(m+1) \leq 1/3$, after a transient period of time all the cars are enough separated to go forward without interaction with the other cars. Then $f = p/(m+1)$ that can be written as function of the density as $f = d/(1+1/m)$ which is $\simeq d$ as soon as $1/m$ becomes small, that is when the size of the system becomes large.

In the contrary, if the density is larger than $2/3$, all the free places are enough separated, after a finite amount of time, to go backwards freely; that means that: always $m-p$ cars can go forward. Then, the relation between the flow and the density becomes $f = (m-p)/m = 1-d$.

The third case is when $d/(1+1/m) > 1/3$ and $d < 2/3$ then the retarder is fully busy that is all the three steps a car leaves the retarder and the flow is $1/3$.

These results can be observed numerically and will be proved easily using event graph modeling in the next subsection.

B. Event graph modeling

Using the notation of Section III, the dynamics of the event graph associated with the traffic on a circular road with a retarder and described by Figure 1 Part III[2], is defined by the matrix:

$$A(\delta) = \delta \begin{bmatrix} \varepsilon & \delta a_1 & \varepsilon & . & . & \varepsilon & \bar{a}_m \\ \bar{a}_1 & \varepsilon & a_2 & \varepsilon & . & . & \varepsilon \\ . & . & . & . & . & . & . \\ \varepsilon & . & . & \varepsilon & \bar{a}_{m-2} & \varepsilon & a_{m-1} \\ a_m & \varepsilon & . & . & \varepsilon & \bar{a}_{m-1} & \varepsilon \end{bmatrix}$$

where the initial car positions are given by the booleans a_q which are equal to 1 when the cell is filled by a car and 0 in the other case and where we use also the notation $\bar{a} = 1-a$.

The numbers of cars entered in section q of the road before time n being denoted X_n^q, we have:

$$X_{n+1}^q = \min\{a_{q-1} + X_n^{q-1}, \bar{a}_q + X_n^{q+1}\}, \ \forall q \neq 2 \ ,$$
$$X_{n+1}^2 = \min\{a_1 + X_{n-1}^1, \bar{a}_2 + X_n^3\} \ ,$$

where the operations on the index q are done modulo m.

A corollary of the Theorem 6 gives the fundamental diagram.

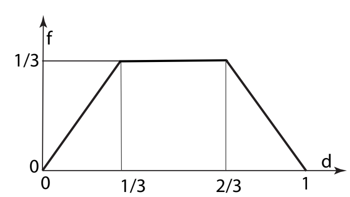

Fig. 2. Fundamental traffic law for one circular road with a retarder.

Theorem 7: The fundamental diagram, giving the average flow f as a function of the vehicle density d, for a circular road with a retarder is:

$$f(d) = \min(d/(1+1/m), 1-d, 1/3),$$

where m is the size of the road counted in vehicle places.

Proof: We see on Figure 1 Part III that there are four types of circuits :

- the exterior of average weight $p/(m+1)$,
- the interior circuit of average weight $(m-p)/m$,
- circuits, corresponding to going forward some steps and going backward the same number of steps without using the retarder, of average weight $1/2$,

[2]In this picture in all the places, at the exception of place a_1, there is one stick not shown.

- circuits, corresponding to going forward some steps and going backward the same number of steps, of average using the retarder, with smallest average weight equal to $1/3$.

Therefore we have $f = \min(p/(m+1), (m-p)/m, 1/3)$, which gives the result. ■

V. Two circular roads with one intersection

We consider a system composed of two unique-direction circular roads with a unique intersection shown in Figure 3.

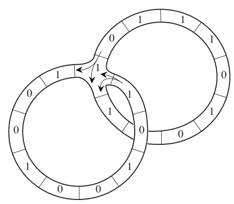

Fig. 3. Two circular roads with a unique crossing.

We suppose that the cars cannot overtake and we model their moving in the same way as in the previous section. The cars leaving the intersection take the two downstream roads with the same proportion. We show the existence of a fundamental diagram for this new case and discuss its properties showing the existence of several traffic phases.

A. Minplus Petri net modeling

We can define completely the system in terms of a deterministic Petri net given in Figure 4 in the case of two roads with the same sizes.

As in the previous section, each road is cut up in ν [resp. ν'] sections able to contain one car. To each section q is associated the couple a_q and $\bar{a}_q$ of the Petri net places. If $a_q = 1$ then $\bar{a}_q = 0$ and the section is occupied by a car. If $a_q = 0$ then $\bar{a}_q = 1$ and the section is free. The transition q correspond to the input of the section q. The crossing is considered as a section with two inputs ant two outputs.

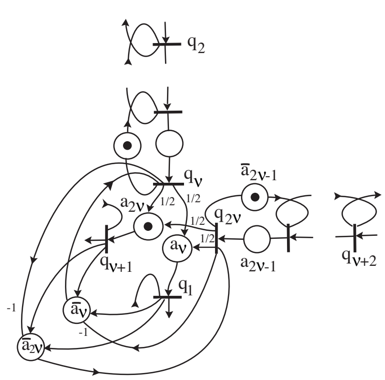

Fig. 4. The complete Petri net for the intersection.

If X^q_n denotes the number of firings of the transition q until time n, using the minplus notations, the dynamic of the system is defined, in the case where $\nu = \nu'$, by:

$$X^q/\delta = a_{q-1}X^{q-1} \oplus \bar{a}_q X^{q+1}, \tag{2}$$

$$q \in \{2, \dots, \nu-1, \nu+2, \dots, 2\nu-1\}, \tag{3}$$

$$X^\nu/\delta = \bar{a}_\nu X^1 X^{\nu+1}/X^{2\nu} \oplus a_{\nu-1}X^{\nu-1}, \tag{4}$$

$$X^{2\nu}/\delta = \bar{a}_{2\nu} X^1 X^{\nu+1}/(X^\nu/\delta) \oplus a_{2\nu-1}X^{2\nu-1}, \tag{5}$$

$$X^1/\delta = a_\nu \left\lfloor \sqrt{1X^\nu X^{2\nu}} \right\rfloor \oplus \bar{a}_1 X^2, \tag{6}$$

$$X^{\nu+1}/\delta = a_{2\nu} \left\lfloor \sqrt{X^\nu X^{2\nu}} \right\rfloor \oplus \bar{a}^{\nu+1} X^{\nu+2}, \tag{7}$$

where the minplus division "/" is the standard subtraction, taking the square root in minplus algebra means take the half part, $\lfloor . \rfloor$ denotes the rounding down operator, and δ the back time shifting operator acting on time sequences. For example

$$X^1/\delta = a_\nu \left\lfloor \sqrt{1X^\nu X^{2\nu}} \right\rfloor \oplus \bar{a}_1 X^2$$

means in standard algebra

$$X^1_{n+1} = \min\left\{ a_\nu + \left\lfloor \frac{1 + X^\nu_n + X^{2\nu}_n}{2} \right\rfloor, \bar{a}_1 + X^2_n \right\}.$$

This system of equations is implicit but triangular and therefore defines uniquely the trajectory of the system.

Neglecting the rounding in (6) and (7), corresponds to fluid Petri nets where it is not necessary to have integer numbers of tokens in upstream places of a transition to start the firings[3]. In this case the system can be written in matrix form using both standard algebra and minplus algebra.

$$X = (XH(\delta)) \otimes D,$$

with $H(\delta)$ matrix $2\nu \times 4\nu$ defined by:

$$H(\delta) = \delta \begin{bmatrix} F & E_{\nu\nu}/2 & P^{-1} & E_{1\nu} - E_{\nu\nu}/\delta \\ E_{\nu\nu}/2 & F & E_{1\nu} - E_{\nu\nu} & P^{-1} \end{bmatrix}$$

$D = A \otimes J$ with J a $4\nu \times 2\nu$ matrix par defined by:

$$J = \begin{bmatrix} P & \varepsilon \\ \varepsilon & P \\ I & \varepsilon \\ \varepsilon & I \end{bmatrix}$$

and $A = \mathrm{diag}(a, \bar{a})$ a $4\nu \times 4\nu$ diagonal matrix where

- ε denotes the zero element of the minplus algebra that is $+\infty$,
- I the $\nu \times \nu$ minplus identity matrix,
- P the $\nu \times \nu$ matrix associated with the permutation $(2, \cdots, \nu, 1)$,
- E_{ij} the $\nu \times \nu$ matrix having only one non null element in position ij equal to the unit element,
- $F = I_d - E_{\nu\nu}/2$ with I_d the standard algebra identity matrix.

[3] As soon as there are real token numbers in each upstream places the minimum of this number is consumed in each place.

B. The global fundamental diagram

The simulation can be easily done using the maxplus toolbox of Scilab [18]. In Figure 5, we show the relation between the average density and the average flow in the case of one crossing. Analyzing the results, it appears that the maximal flow is equal to the half of the maximal flow of a unique road case. Indeed the crossing has to serve two streets, one West-East and one South-North with the same capacities. The crossing is the bottleneck of the system and determines its average speed. The maximum flow corresponds to the optimal level of congestion that is the saturation of the crossing.

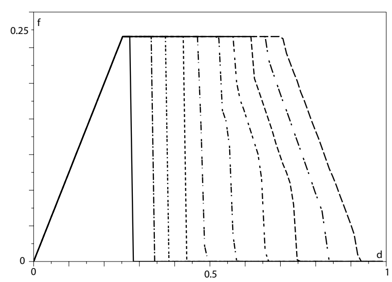

Fig. 5. Fundamental diagram for two circle-streets and a crossing with priority to the right (the relative size of the roads varies from 1 to 10).

Moreover, we observe on these diagrams three phases:

- The *low density phase* where a periodic regime appears with the average of the total population in each road. On each road the vehicles go freely not bothered by the others. At each time step, when a car reaches the intersection, the intersection is free. The system is similar to an ideal gas where there is no interaction between the molecules.

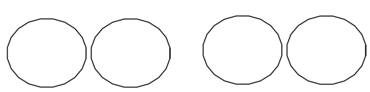

Fig. 6. Initial position and asymptotic periodic position of vehicles in the low density case.

- The *middle density phase* where a periodic regime appears with different populations on each road. The size of the population on the road with priority stays constant when we change the total number of vehicles (staying in this phase); new vehicles will be absorbed by the road without priority. The intersection reaches its maximal regime (one time step free one time step full). We can think to a system with a gas in the priority road and a melange of liquid and gas in the other road (lower temperature). If we add molecules they condense into liquid in the road without priority.
- The *high density phase* there is a blocked regime. There are still free places in the system, but a vehicle still wants enter in the road without priority which is full.

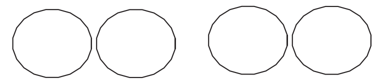

Fig. 7. Initial position and asymptotic periodic position of vehicles in the middle density case.

If the vehicles could go forward in this last road, some vehicles would be able to leave it, but they cannot go forward because this road is full. For two roads with the same size, this very abrupt blocking appears at density 1/2. At this density, everything works well but if a car is added, this car slows down the average speed of the road without priority. Then, there is a leak of the population from the priority road to the other. The road without priority fills up, becomes full and the last car in the priority road wants to enter in the other one and the system blocks.

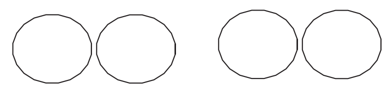

Fig. 8. Initial position and asymptotic blocked position of vehicles in the high density case.

C. The fundamental diagram of each road

In Figure 9 we show the flow density relation for the road without priority. We see that its shape is very closed to the system with a unique road and a retarder.

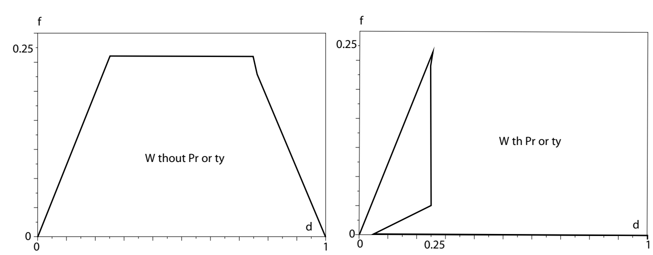

Fig. 9. Fundamental diagram for the two streets (without and with priority) for the system with two circle-streets and a crossing.

The priority road has a fundamental diagram Figure 9 completely different which shows that its density does not increase when we add cars but stay in middle global density phase. On this diagram we see that the phase transition to the blocking phase is very abrupt with a car pumping from the priority road towards the other road.

D. Comparison of intersection policies

To avoid this blocking phenomena, light control can be added at the intersection. In future work, we intend to optimize the light control by computing feedback on the jam level of each road. For the time being we give the fundamental diagram for the three following policies:

- right priority,
- half time red, half time green,
- alternating the priority: during T time steps right priority, then during the T next time steps we use left priority (this case corresponds to a policeman making the circulation).

We compute the global fundamental diagram for this three cases and show them in the same Figure 10 (the two roads have the same size).

The comparison of the three diagrams shows that the light control degrades a little the circulation in the low density case but improves a lot in the high density blocking case. The third policy is the best and could be implemented by a simple feedback. In a real town a global feedback would be much more complicated to obtain see [6].

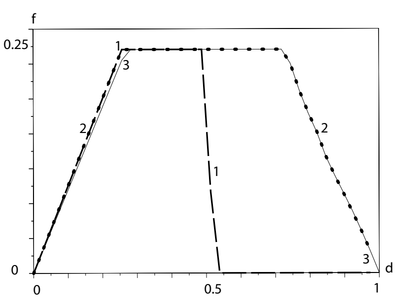

Fig. 10. Comparison of the fundamental diagrams for the complete system (two roads of same size) for different control policies (no light control and right priority (1), light control with half time green half time red (3), without light control and alternate priority(2)).

E. Mathematical justification of the fundamental diagram

We have obtained experimentally the fundamental diagram. It is interesting to try to justify mathematically its existence. To justify its existence we need two things:

- The existence of an average flow: that is the limit of X_n/n exists when goes to infinity.
- This independence of average flow f with the initial vehicle positions.

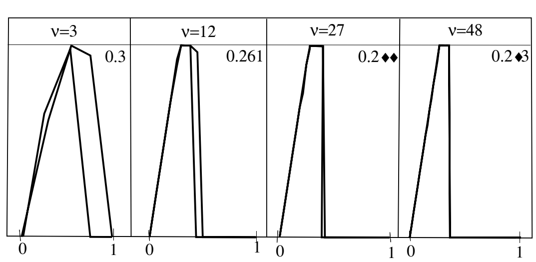

Fig. 11. Asymptotic independence of the fundamental diagram with the initial vehicle positions (the diagram for several initial positions are plotted on the same picture for increasing sizes of the system).

These two affirmations have not still be proved. The second is false in general. But it is true that the asymptotic flow (where the asymptotic is done on the size of the roads without changing their respective size) does not depend of initial position of the vehicles. Moreover the fluid and the discrete Petri nets models have the same asymptotic diagram. This result is clear experimentally, see Figure 11, but has to be proved mathematically.

VI. Conclusion

The traffic model proposed here, using simple Petri nets and minplus algebra, seems to contain the essence of the traffic problem. From this model, we can compute the fundamental traffic diagram linking the average flow and the car density. Even on the very simple system studied, here consisting of two circular roads and an intersection, several phases appear in the fundamental diagram.

To define these fundamental diagrams, the existence of an average flow independent of the initial vehicle position is necessary. Future mathematical analysis will attempt to justify the existence of this average flow variable shown here experimentally and its independence with the initial vehicle positions.

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